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01 Harmonic Oscillations

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1. Harmonic Oscillations.

Everyone has seen waves on water, heard sound waves and seen light waves. But, what exactly is a wave? Of course, the goal of this course is to address this question. But for now you can think of a wave as a traveling or oscillatory disturbance in some continuous medium (air, water, the electromagnetic field, etc.). As we shall see, waves can be viewed as a collective effect resulting from a combination of many harmonic oscillations. So, to begin, we review the basics of harmonic motion.

Harmonic motion of some quantity (*e.g.*, displacement, intensity, current, angle, . . .) means the quantity exhibits a sinusoidal time dependence. In particular, this means that the quantity oscillates sinusoidally in value with a frequency which is independent of both amplitude and time. Generically, we will refer to the quantity of interest as the *displacement* and denote it by the symbol q . The value of the displacement at time t is denoted $q(t)$; for harmonic motion it can be written in one of the equivalent forms

$$q(t) = B \sin(\omega t + \phi) \quad (1.1)$$

$$= C \cos(\omega t + \psi) \quad (1.2)$$

$$= D \cos(\omega t) + E \sin(\omega t). \quad (1.3)$$

Here $B, C, D, E, \omega, \phi, \psi$ are all constants. The relationship between the various versions of harmonic motion shown above is obtained using the trigonometric identities

$$\sin(a + b) = \sin a \cos b + \sin b \cos a, \quad (1.4)$$

$$\cos(a + b) = \cos a \cos b - \sin a \sin b. \quad (1.5)$$

You will be asked to explore these relationships in the Problems.

The parameter ω represents the (angular) frequency of the oscillation and is normally determined by the nature of the physical system being considered. So, typically, ω is fixed once and for all. (Recall that the angular frequency is related to the physical frequency f of the motion via $\omega = 2\pi f$.) For example, the time evolution of angular displacement of a pendulum is, for small displacements, harmonic with frequency determined by the length of the pendulum and the acceleration due to gravity. In all that follows we assume that $\omega > 0$; there is no loss of generality in doing so (*exercise*). The other constants B, C, D, E, ϕ, ψ , which represent amplitudes and phases, are normally determined by the initial conditions for the problem at hand. For example, it is not hard to check that $D = q(0)$ and $E = \frac{1}{\omega}v(0)$, where $v(t) = \frac{dq(t)}{dt}$ is the velocity at time t . Given ω , no matter which form of the harmonic motion is used, you can check that one must pick two real numbers to uniquely specify the motion (*exercise*). Alternatively, we say that all harmonic oscillations (at a given angular frequency) can be uniquely labeled by two parameters.

1.1 The Harmonic Oscillator Equation

To say that the displacement q exhibits harmonic motion is equivalent to saying that $q(t)$ satisfies the *harmonic oscillator equation*:

$$\frac{d^2q(t)}{dt^2} = -\omega^2q(t), \quad (1.6)$$

for some real constant $\omega > 0$. This is because each of the functions (1.1)–(1.3) satisfy (1.6) (*exercise*) and, in particular, every solution of (1.6) can be put into the form (1.1)–(1.3). This latter fact is proved using techniques from a basic course in differential equations. (See also the discussion in §1.2, below.)

The frequency ω is the only parameter needed to characterize a harmonic oscillator. Normally the constant ω is fixed once and for all in the specification of the problem at hand. When we speak of the “the” harmonic oscillator equation we will always mean that ω has been given.

It is via the harmonic oscillator equation that one usually arrives at harmonic motion when modeling a physical system. For example, if q is a displacement of a particle with mass m whose dynamical behavior is controlled (at least approximately – see below) by a Hooke’s force law,

$$F = -kq, \quad k = \text{const.}, \quad (1.7)$$

then Newton’s second law reduces to the harmonic oscillator equation with $\omega = \sqrt{\frac{k}{m}}$ (*exercise*). More generally, we have the following situation. Let the *potential energy* function describing the variable q be denoted by $V(q)$. This means that the equation governing q takes the form*

$$m \frac{d^2q(t)}{dt^2} = -V'(q(t)). \quad (1.8)$$

Suppose the function $V(q)$ has a local minimum at the point q_0 , that is, suppose q_0 is a point of stable equilibrium. Let us choose our origin of coordinates at this point, so that $q_0 = 0$. Further, let us choose our reference point of potential energy (by adding a constant to V if necessary) so that $V(0) = 0$. These choices are not necessary; they are for convenience only. In the vicinity of this minimum we can write a Taylor series approximation to $V(q)$ (*exercise*):

$$\begin{aligned} V(q) &\approx V(0) + V'(0)q + \frac{1}{2}V''(0)q^2 \\ &= \frac{1}{2}V''(0)q^2. \end{aligned} \quad (1.9)$$

* Here we use a prime on a function to indicate a derivative with respect to the argument of the function, *e.g.*,

$$f'(x) = \frac{df(x)}{dx}.$$

(If you would like a quick review of Taylor’s theorem and Taylor series, have a look at Appendix A.) The zeroth and first order terms in the Taylor series are absent, respectively, because (i) we have chosen $V(0) = 0$ and (ii) for an equilibrium point $q = 0$ we have $V'(0) = 0$. Because the equilibrium point is a minimum (stable equilibrium) we have that

$$V''(0) > 0. \quad (1.10)$$

Incidentally, the notation $V'(0)$, $V''(0)$, *etc.*, means “take the derivative(s) and evaluate the result at zero”:

$$V'(0) \equiv V'(x) \Big|_{x=0}.$$

If we use the Taylor series approximation (1.9), we can approximate the force on the system near a point of stable equilibrium as that of a harmonic oscillator with “spring constant” (*exercise*)

$$k = V''(0). \quad (1.11)$$

The equation of motion of the system can thus be approximated by

$$m \frac{d^2 q(t)}{dt^2} = -V'(q(t)) \approx -kq(t). \quad (1.12)$$

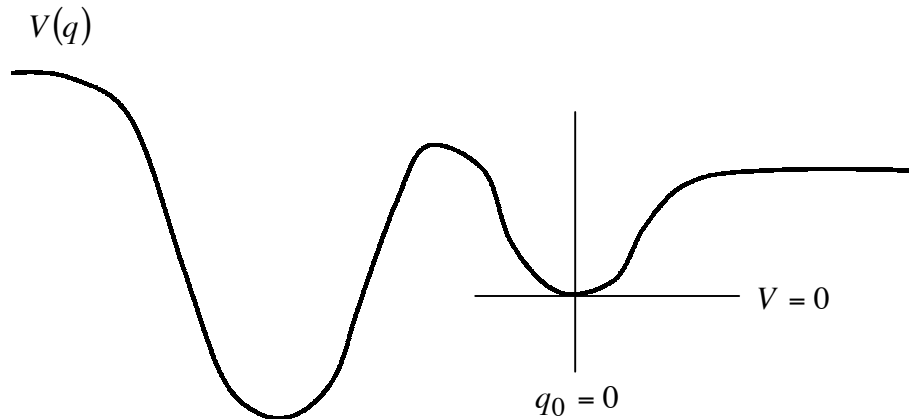


Figure 1. One dimensional potential $V(q)$.

Dividing both sides by m and identifying $\omega^2 = \frac{k}{m}$ and we obtain the harmonic oscillator equation (1.6). The approximation of the potential energy as a quadratic function

in the neighborhood of some local minimum is called the *harmonic approximation*. No matter what the physical system is, or the meaning of the variable $q(t)$, the harmonic approximation leads to harmonic motion as described in (1.1)–(1.3).

The harmonic oscillator equation is a second-order, linear, homogeneous ordinary differential equation with constant coefficients. That’s a lot of terminology. Let’s pause for a moment to explain it. The general form of a linear, second-order ordinary differential equation is

$$a(t)q''(t) + b(t)q'(t) + c(t)q(t) = d(t), \quad (1.13)$$

where the *coefficients* a, b, c some given functions of t . There is one *dependent variable*, q , and one *independent variable*, t . Because there is only one independent variable, all derivatives are *ordinary* derivatives rather than partial derivatives, which is why the equation is called an *ordinary* differential equation. The harmonic oscillator has *constant coefficients* because a, b, c do not in fact depend upon time. The harmonic oscillator equation is called *homogeneous* because $d = 0$, making the equation homogeneous of the first degree in the dependent variable q .† If $d \neq 0$, then the equation is called *inhomogeneous*. As you can see, this equation involves no more than 2 derivatives, hence the designation *second-order*. We say that this equation is *linear* because the differential operator*

$$L = a(t)\frac{d^2}{dt^2} + b(t)\frac{d}{dt} + c(t) \quad (1.14)$$

appearing in equation (1.13) via

$$Lq = d, \quad (1.15)$$

has the following property (exercise)

$$L(rq_1 + sq_2) = rL(q_1) + sL(q_2), \quad (1.16)$$

for any constants r and s .

Indeed, to be more formal about this, following the general discussion in Appendix B, we can view the set of all real-valued functions $q(t)$ as a real vector space. So, in this case the “vectors” are actually functions of t !‡ The addition rule is the usual point-wise addition of functions. The scalar multiplication is the usual multiplication of functions by real numbers. The zero vector is the zero function, which assigns the value zero to all t . The additive inverse of a function $q(t)$ is the function $-q(t)$, and so forth. We can then view L as a linear operator, as defined in Appendix B (exercise).

† A function $f(x)$ is homogeneous of degree p if it satisfies $f(ax) = a^p f(x)$ for any constant a (such that ax is in the domain of f).

* In this context, a differential operator is simply a rule for making a function (or functions) from any given function $q(t)$ and its derivatives.

‡ There are a *lot* of functions, so this vector space is infinite dimensional. We shall make this a little more precise when we discuss Fourier analysis.

The linearity and homogeneity of the harmonic oscillator equation has a very important consequence. It guarantees that given two solutions, $q_1(t)$ and $q_2(t)$, we can take any linear combination of these solutions with constant coefficients to make a new solution $q_3(t)$. In other words, if r and s are constants, and if $q_1(t)$ and $q_2(t)$ are solutions to the harmonic oscillator equation, then

$$q_3(t) = rq_1(t) + sq_2(t) \quad (1.17)$$

also solves that equation.

Exercise: Check that $q_3(t)$ is a solution of (1.6) if $q_1(t)$ and $q_2(t)$ are solutions. Do this explicitly, using the harmonic oscillator equation, and then do it just using homogeneity and the linearity of L .

1.2 The General Solution to the Harmonic Oscillator Equation

The *general solution* to the harmonic oscillator equation is a solution depending upon some freely specifiable constants such that all possible solutions can be obtained by making suitable choices of the constants. Physically, we expect to need two arbitrary constants in the general solution since we must have the freedom to adjust the solution to match any initial conditions, *e.g.*, initial position and velocity. Mathematically this can be shown to be the case thanks to the fact that the differential equation is second order and any solution will involve two constants of integration.

To construct the most general solution it is enough to find two “independent” solutions (which are not related by linear combinations) and take their general linear combination. For example, we can take $q_1(t) = \cos(\omega t)$ and $q_2(t) = \sin(\omega t)$ and the general solution is (*c.f.* (1.3)):

$$q(t) = A \cos(\omega t) + B \sin(\omega t). \quad (1.18)$$

While we won’t prove that all solutions can be put into this form, we can easily show that this solution accommodates any initial conditions. If we choose our initial time to be $t = 0$, then this follows from the fact that (exercise)

$$q(0) = A, \quad \frac{dq(0)}{dt} = B\omega.$$

You can now see that the choice of the constants A and B is equivalent to the choice of initial conditions. Note in particular that solutions are uniquely determined by their initial conditions. Indeed, suppose two solutions $q_1(t)$ and $q_2(t)$ satisfy the harmonic oscillator equation with the same initial conditions, say,

$$q_1(0) = a = q_2(0), \quad \frac{dq_1(0)}{dt} = b = \frac{dq_2(0)}{dt},$$

The difference between these solutions, $q(t) := q_1(t) - q_2(t)$ is also a solution of the harmonic oscillator equation, but now with vanishing initial position and initial velocity. Therefore $q(t) = 0$ and we see that $q_1(t) = q_2(t)$ by virtue of having the same initial conditions.

It is worth noting that the solutions to the oscillator equation form a *vector space*. See Appendix B for the definition of a vector space. The underlying set is the set of solutions to the harmonic oscillator equation. So, here again the “vectors” are functions of t . (But this time it will turn out that the vector space will be finite-dimensional because only the functions which are mapped to zero by the operator L are being considered.) As we have seen, the solutions can certainly be added and multiplied by scalars (*exercise*) to produce new solutions. This is because of the linear, homogeneous nature of the equation. So, in this vector space, “addition” is just the usual addition of functions, and “scalar multiplication” is just the usual multiplication of functions by real numbers. The function $q = 0$ is a solution to the oscillator equation, so it is one of the “vectors”; it plays the role of the zero vector. As discussed in Appendix B, a *basis* for a vector space is a subset of linearly independent elements out of which all other elements can be obtained by linear combination. The number of elements needed to make a basis is the *dimension* of the vector space. Evidently, the functions $q_1(t) = \cos \omega t$ and $q_2(t) = \sin \omega t$ form a basis for the vector space of solutions to the harmonic oscillator equation (*c.f.* equation (1.18)). Indeed, both $q_1(t)$ and $q_2(t)$ are vectors, *i.e.*, solutions of the harmonic oscillator equation, and all solutions can be built from these two solutions by linear combination. Thus the set of solutions to the harmonic oscillator equation constitutes a two-dimensional vector space. You can also understand the appearance of two dimensions from the fact that each solution is uniquely determined by the two initial conditions.

It is possible to equip the vector space of solutions to the harmonic oscillator equation with a scalar product. While we won’t need this result in what follows, we briefly describe it here just to illustrate the concept of a scalar product. Being a scalar product, it should be defined by a rule which takes any two vectors, *i.e.*, solutions of the harmonic oscillator equation, and returns a real number (see Appendix B for the complete definition of a scalar product). Let $q_1(t)$ and $q_2(t)$ be solutions of the harmonic oscillator equation. Their scalar product is defined by

$$(q_1, q_2) = q_1 q_2 + \frac{1}{\omega^2} q_1' q_2'. \quad (1.19)$$

In terms of this scalar product, the basis of solutions provided by $\cos \omega t$ and $\sin \omega t$ is orthonormal (*exercise*).

1.3 Complex Representation of Solutions

In any but the simplest applications of harmonic motion it is very convenient to work with the *complex representation* of harmonic motion, where we use complex numbers to

keep track of the harmonic motion. Before showing how this is done, we'll spend a little time reviewing the basic properties of complex numbers.

Recall that the *imaginary unit* is defined formally via

$$i^2 = -1$$

so that

$$\frac{1}{i} = -i. \quad (1.20)$$

A *complex number* z is an ordered pair of real numbers (x, y) which we write as

$$z = x + iy. \quad (1.21)$$

The variable x is called the *real part* of z , denoted $Re(z)$, and y is called the *imaginary part* of z , denoted by $Im(z)$. We apply the usual rules for addition and multiplication of numbers to z , keeping in mind that $i^2 = -1$. In particular, the sum of two complex numbers, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, is given by

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2).$$

The product of two complex number is given by (*exercise*)

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).$$

Two complex numbers, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, are equal if and only if their real parts are equal and their imaginary parts are equal:

$$z_1 = z_2 \iff x_1 = x_2 \quad \text{and} \quad y_1 = y_2.$$

Given $z = x + iy$, we define the *complex conjugate* $z^* = x - iy$. It is straightforward to check that (*exercise*)

$$Re(z) = x = \frac{1}{2}(z + z^*); \quad Im(z) = y = \frac{1}{2i}(z - z^*). \quad (1.22)$$

Note that (*exercise*)

$$z^2 = x^2 - y^2 + 2ixy, \quad (1.23)$$

so that the square of a complex number z is another complex number.

Exercise: What are the real and imaginary parts of z^2 ?

A complex number is neither positive or negative; these distinctions are only used for real numbers. Note in particular that z^2 , being a complex number, cannot be said to be

positive (unlike the case with real numbers). On the other hand, if we multiply z by its complex conjugate we get a non-negative real number:

$$zz^* \equiv |z|^2 = x^2 + y^2 \geq 0. \quad (1.24)$$

We call the non-negative real number $|z| = \sqrt{zz^*} = \sqrt{x^2 + y^2}$ the *absolute value* of z .

A complex number defines a point in the x - y plane — in this context also called the *complex plane* — and *vice versa*. We can therefore introduce polar coordinates to describe any complex number. So, for any $z = x + iy$, let

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta, \quad (1.25)$$

where $r \geq 0$ and $0 \leq \theta \leq 2\pi$. (Note that, for a fixed value of $r > 0$, $\theta = 0$ and $\theta = 2\pi$ represent the same point. Also, θ is not defined when $r = 0$.) You can check that (*exercise*)

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right). \quad (1.26)$$

It now follows that for any complex number z there is a radius r and an angle θ such that

$$z = r \cos \theta + ir \sin \theta = r(\cos \theta + i \sin \theta). \quad (1.27)$$

A famous result of Euler is that

$$\cos \theta + i \sin \theta = e^{i\theta}, \quad (1.28)$$

(see the Problems for a proof) so we can write

$$z = re^{i\theta}. \quad (1.29)$$

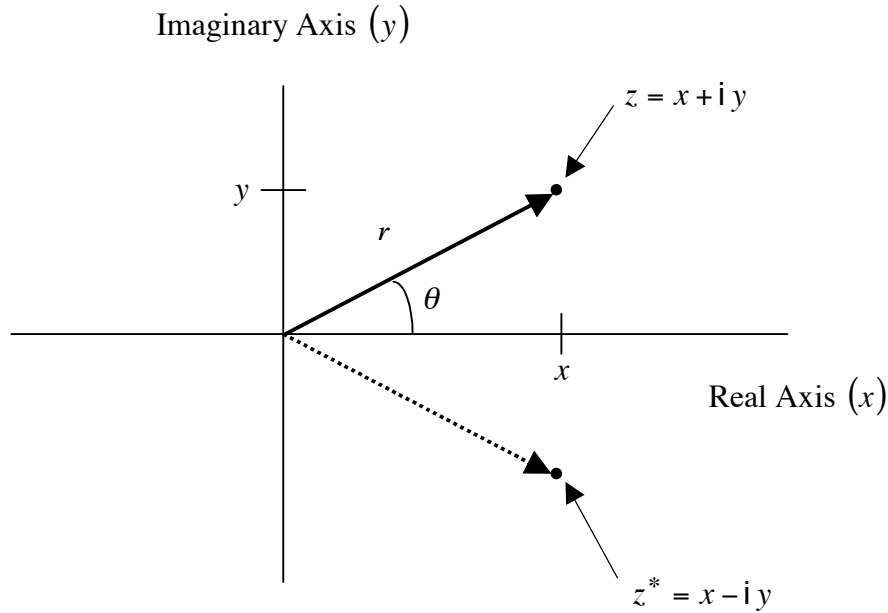


Figure 2. Illustration of a point z in the complex plane and its complex conjugate z^* .

In the polar representation of a complex number z the radius r corresponds to the absolute value of z while the angle θ is known as the *phase* or *argument* of z . Evidently, a complex number is completely characterized by its absolute value and phase.

As an *exercise* you can show that

$$\cos \theta = \operatorname{Re}(e^{i\theta}) = \frac{1}{2}(e^{i\theta} + e^{-i\theta}), \quad \sin \theta = \operatorname{Im}(e^{i\theta}) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}).$$

We also have (*exercise*)

$$z^2 = r^2 e^{2i\theta} \quad \text{and} \quad zz^* = r^2. \quad (1.30)$$

With $r = 1$, these last two relations summarize (1) the double angle formulas for sine and cosine,

$$\sin 2\theta = 2 \cos \theta \sin \theta, \quad \cos 2\theta = \cos^2 \theta - \sin^2 \theta, \quad (1.31)$$

and (2) the familiar identity $\cos^2 \theta + \sin^2 \theta = 1$. More generally, all those trigonometric identities you have encountered from time to time are simple consequences of the Euler formula. As a good *exercise* you should verify (1) and (2) using the complex exponential and Euler's formula.

Complex numbers are well-suited to describing harmonic motion because, as we have just seen, the real and imaginary parts involve cosines and sines. Indeed, it is easy to check

that both $e^{i\omega t}$ and $e^{-i\omega t}$ solve the oscillator equation (1.6) (*exercise*). (Here you should treat i as just another constant.) Hence, for any complex numbers α and β , we can solve the harmonic oscillator equation via

$$q(t) = \alpha e^{i\omega t} + \beta e^{-i\omega t}. \quad (1.32)$$

Note that we have added, or superposed, two solutions of the equation. We shall refer to (1.32) as the *complex solution* since, as it stands, $q(t)$ is not a real number.

Normally, q represents a real-valued quantity (displacement, temperature,...). In this case, we require

$$\text{Im}(q(t)) = 0 \quad (1.33)$$

or (*exercise*)

$$q^*(t) = q(t). \quad (1.34)$$

Substituting our complex form (1.32) for $q(t)$ into (1.34) gives

$$\alpha e^{i\omega t} + \beta e^{-i\omega t} = \alpha^* e^{-i\omega t} + \beta^* e^{i\omega t}. \quad (1.35)$$

In this equation, for each t , the real parts of the left and right-hand sides must be equal as must be the imaginary parts. In a homework problem you will see this implies that $\beta = \alpha^*$. Real solutions of the harmonic oscillator equation can therefore be written as

$$q(t) = \alpha e^{i\omega t} + \alpha^* e^{-i\omega t}, \quad (1.36)$$

where α is any complex number. We shall refer to (1.36) as the *complex form* or *complex representation* of the *real* solution of the harmonic oscillator equation. From (1.36) it is clear that $q(t)$ is a real number because it is a sum of a number and its complex conjugate, *i.e.*, is the real part of a complex number. As it should, the solution depends on two real, freely specifiable constants: the real and imaginary parts of α .

Using Euler's formula, equation (1.36) is equivalent to any of the sinusoidal representations given in (1.1)–(1.3). Let us see how to recover (1.2); the others are obtained in a similar fashion (see the problems). Since α is a complex number, we use the polar representation of the complex number to write

$$\alpha = a e^{i\psi}, \quad (1.37)$$

where a and ψ are both real numbers ($a = a^*$, $\psi = \psi^*$). We then have that

$$\begin{aligned} q(t) &= a e^{i(\omega t + \psi)} + a e^{-i(\omega t + \psi)} \\ &= a [e^{i(\omega t + \psi)} + e^{-i(\omega t + \psi)}] \\ &= 2a \cos(\omega t + \psi), \end{aligned} \quad (1.38)$$

which matches our representation of harmonic motion (1.2) with the identification $2a = C$.

So, we see that any particular harmonic motion is determined by a choice of the complex number α through (1.36). Using (1.36), you can check that $q(0) = 2\text{Re}(\alpha)$ and $v(0) = -2\omega\text{Im}(\alpha)$ (*exercise*). Hence the real and imaginary parts of α encode the initial conditions. From the point of view of the polar representation of α , the absolute value of the complex number α determines the amplitude of the oscillation. The phase of the complex number α determines where the oscillation is in its cycle relative to $t = 0$. Evidently, the amplitude and phase information are equivalent to the initial displacement and initial velocity information.

Note that we can also write the general solution (1.36) as

$$q(t) = 2\text{Re}(\alpha e^{i\omega t}). \quad (1.39)$$

When you see a formula such as this you should remember that it works because (1) $p(t) \equiv \alpha e^{i\omega t}$ is a complex solution to the oscillator equation; (2) if $p(t)$ is any complex solution, then so is $p^*(t)$ (just conjugate the oscillator equation – *exercise*); (3) because the oscillator equation is linear and homogeneous, the superposition of any two (complex) solutions to the harmonic oscillator equation is another (in general, complex) solution (*exercise*). Because the real part of a complex solution $p(t)$ is proportional to $p(t) + p^*(t)$ we get a real solution to the harmonic oscillator equation by taking the real part of a complex solution. As a nice exercise you should check that the same logic shows we can also get a real solution of the harmonic oscillator equation by taking the imaginary part of a complex solution. The relation between these two forms of a real solution is explored in the problems.

We have obtained a useful strategy that we shall take advantage of from time to time: When manipulating solutions to a linear and homogeneous equation we work with complex solutions to simplify various manipulations, and only take the real part when appropriate. This is allowed because any real linear equation which admits a complex solution will also admit the complex conjugate as a solution. By linearity and homogeneity, any linear combination of these two solutions will be a solution, and in particular the real and imaginary parts will be real solutions.