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Darboux Integrability - A Brief Historial Survey

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A Brief Historical Survey

Symmetry in Variational Problems and Differential Equations

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History Past

The classical method of Monge. Given

$$F(x, y, u, u_x, u_y) = 0 \quad \text{add} \quad G(x, y, u, u_x, u_y) = 0$$

so that $u_x = A(x, y, u)$, $u_y = B(x, y, u)$ is consistent [Jacobi, Lie].

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The generalization of the Monge method to 2nd order PDE leads to the methods of Ampère and Darboux (and [Cartan, 1910])

$$F(x, y, u, p, q, r, s, t) = 0 \quad \text{add} \quad G(x, y, u, p, q) = 0$$
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$$F(x, y, u, p, q, r, s, t) = 0 \quad \text{add} \quad G(\text{3rd ord}) = 0$$

The compatible equation $G = 0$ is called an intermediate integral (order, complete, general ...)

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- Bäcklund transformations for the classical DI systems of Goursat were studied [Clelland and Ivey].
- DI systems always appear in geometric studies of PDE and in equivalence problems [eg. D. The].
- Many papers in the theoretical physics literature (σ-models) on integrable systems unwittingly arrive at DI systems.

Vessiot's Fundamental Discovery

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But the groups arising in Vessiot's approach are *not* symmetry groups in the usual sense.

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• proves that the Vessiot group is an invariant of any DI system.

• uses the Vessiot group to construct a non-linear superposition formula for any DI system.

• gives a completely algorithmic integration procedure, much better that the classical one.

Example

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$$J^3(\mathbf{R}, \mathbf{R})[x, X, X', X'', X'''] \quad \text{and} \quad J^3(\mathbf{R}, \mathbf{R})(y, Y, Y', Y'', Y''')$$

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- Look to the diagonal action of $SL(2)$ with infinitesimal generators

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• Calculate the reduced differential system $(J^3 \times J^3)/SL(2)$, that is, calculate joint differential invariants.

• In the context of this simple example, the lowest order joint differential invariant gives the general solution.

$$u = log\frac{2X'Y'}{(X-Y)^2}$$

Intermediate Integrals and Differential Invariants

Every intermediate integral for any DI system is in fact a differential invariant for the Vessiot group action.

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Every intermediate integral for any DI system is in fact a differential invariant for the Vessiot group action.

All the classical work of Goursat on studying intermediate integrals is in fact (essentially) covered by Lie's work on differential invariants and invariant differential operators [Olver].

Classification

From this new viewpoint:

There are as many DI EDS as there are symmetry groups of differential equations!

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There are as many DI EDS as there are symmetry groups of differential equations!

BUT, only certain symmetry groups of very special DE will lead to DI EDS representing a desired type of equation.

Primitive and Imprimitive Actions

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Primitive and Imprimitive Actions

We have calculated all systems of DI equations arising from vector field systems in the plane [Lie, GLKO].

- Vessiot groups with imprimitive actions give "triangularized" DI systems – essentially known examples
- Vessiot groups with primitive actions give genuinely new examples.

Cauchy Problem

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$$u_{xy} = e^u$$

$$3 * u_{xx} u_{yy}^3 + 1 = 0$$

Bäcklund Transformations

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- Many new examples can easily be derived by symmetry reduction.

- The equation

$$u_{xy} = \frac{\sqrt{1-u_x^2}\sqrt{1-u_y^2}}{\sin u}$$

has Vessiot group $SO(3)$. But $SO(3)$ has no real 2 dimensional subalgebras and therefore it does not admit a 1-dimensional Bäcklund transformation to the wave equation.

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- verify Sokolov's classification using group theoretical methods.
- analyze completely the Toda lattice systems (parabolic geometries associated to simple Lie algebras).
- study multi-soliton solutions from our group theoretic non-linear superposition viewpoint.
- decide what to do about 'parabolic' DI systems ([Cartan, 1911])

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