Foundations of Wave Phenomena   Open Textbooks

02 Coupled Oscillators

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2. Two Coupled Oscillators.

Our next step on the road to finding a wave is to consider a more interesting oscillating system: two coupled oscillators. Suppose we have two identical oscillators, both characterized by an angular frequency ω. Let the displacement of each oscillator from equilibrium be q_1 and q_2, respectively. Of course, if the two oscillators are uncoupled, that is, do not interact in any way, each of the displacements satisfies a harmonic oscillator equation

$$\frac{d^2 q_1}{dt^2} = -\omega^2 q_1, \tag{2.1}$$

$$\frac{d^2 q_2}{dt^2} = -\omega^2 q_2. \tag{2.2}$$

For example, the oscillators could each consist of a mass m connected by a spring (with spring constant $k = m\omega^2$) to a wall (see figure 3a). Now suppose that the masses are joined by a spring characterized by spring constant $\tilde{k}$. With a little thought, you can see that the forces on the masses are such that their equations of motion take the form:

$$\frac{d^2 q_1}{dt^2} + \omega^2 q_1 - \tilde{\omega}^2(q_2 - q_1) = 0, \tag{2.3}$$

$$\frac{d^2 q_2}{dt^2} + \omega^2 q_2 + \tilde{\omega}^2(q_2 - q_1) = 0, \tag{2.4}$$

where $\tilde{\omega} = \sqrt{\frac{\tilde{k}}{m}}$. Notice that while the equations are still linear and homogeneous with constant coefficients they are now *coupled*, that is, the equation for $q_1(t)$ depends on $q_2(t)$ and *vice versa*.

2.1 Normal Modes

The motion of the masses described by (2.3) and (2.4) can be relatively complicated but, remarkably enough, it can be viewed as a superposition of harmonic motions! One quick way to see this is to simply take the sum and difference of (2.3) and (2.4). You will find (*exercise*) that the quantity $q_1(t) + q_2(t)$ satisfies the harmonic oscillator equation with frequency* ω, while the quantity $q_2(t) - q_1(t)$ satisfies the harmonic oscillator equation with frequency $\sqrt{\omega^2 + 2\tilde{\omega}^2}$. One can therefore solve the harmonic oscillator equations with the indicated frequencies for the combinations $q_2(t) \pm q_1(t)$ and then reconstruct the motion of the individual oscillators, $q_1(t)$ and $q_2(t)$.

To see how to do this systematically, let us define new position variables,

$$Q_1 = \frac{1}{2}(q_1 + q_2), \quad Q_2 = \frac{1}{2}(q_2 - q_1). \tag{2.5}$$

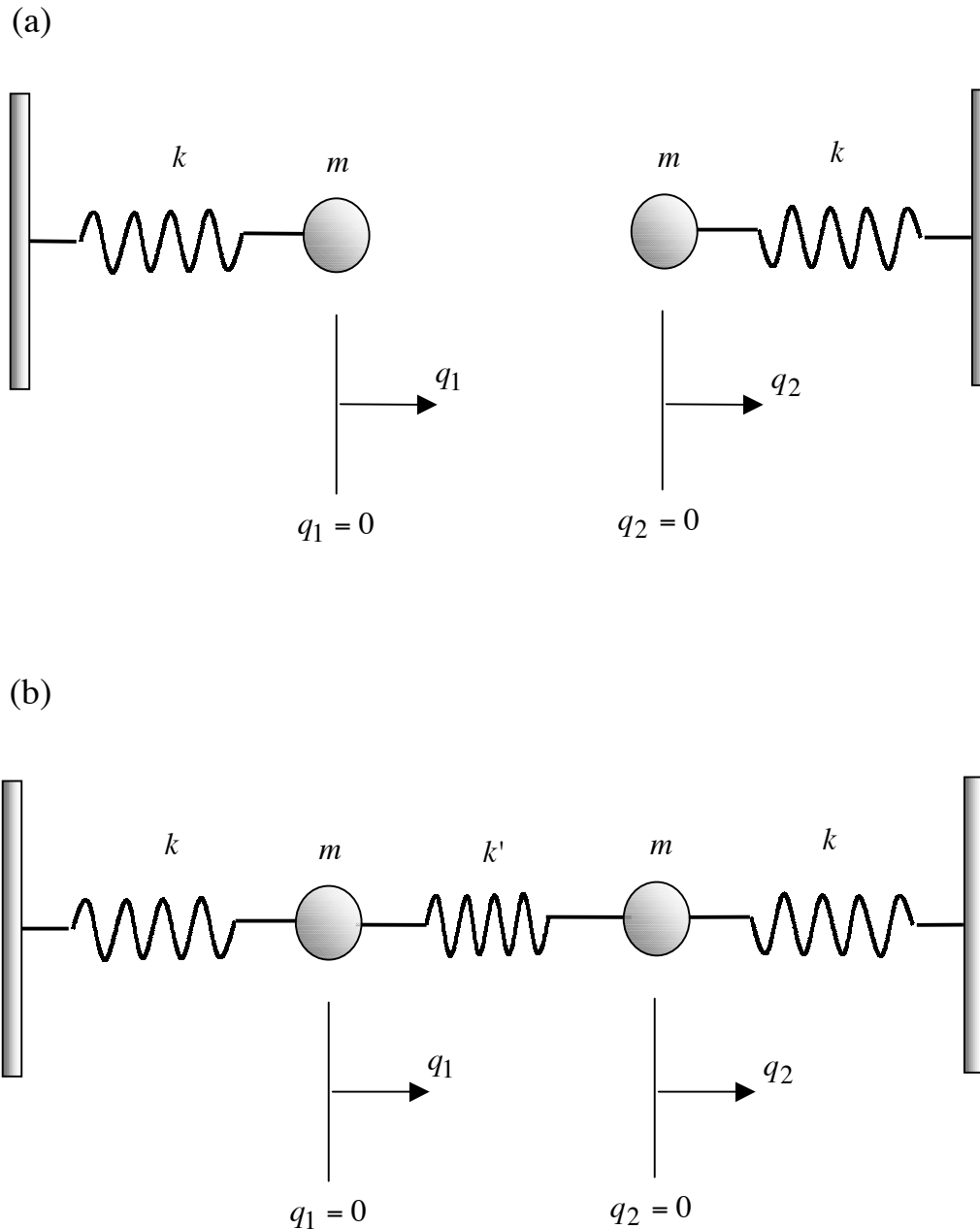* Reminder: the frequencies are always chosen positive.

(a)

(b)

Figure 3. (a) Two uncoupled oscillators. (b) Two coupled oscillators. Coupling occurs through the spring k'.

This defines the Q's in terms of the q's. (The factors of $1/2$ are there for later convenience.) The Q's are examples of *generalized coordinates*. A physical interpretation of these variables is not hard to come by. The variable Q_1 carries information about the center of mass

of the system (Q_1 differs from the center of mass by an additive constant – *exercise*). The variable Q_2 carries information about the relative separation of the two masses. Thus the new variables correspond to describing the motion of a two body system in terms of its relative and center of mass parts. We can invert the definitions of the Q's to solve for the q's (*exercise*):

$$q_1 = Q_1 - Q_2, \quad q_2 = Q_1 + Q_2. \tag{2.6}$$

So, if we can solve the differential equations for the Q's, then we can also get the solutions for the q's from (2.6). To get the equations satisfied by the Q's, we use (2.6) in (2.3)–(2.4); after a little simplification, we find that the Q's satisfy (*exercise*)

$$\frac{d^2 Q_1}{dt^2} = -\omega^2 Q_1 \tag{2.7}$$

$$\frac{d^2 Q_2}{dt^2} = -(\omega^2 + 2\tilde{\omega}^2)Q_2. \tag{2.8}$$

The equations for Q_1 and Q_2 are *decoupled*. Moreover, as expected, Q_1 executes harmonic motion with angular frequency $\Omega_1 = \omega$ and Q_2 executes harmonic motion with frequency $\Omega_2 = \sqrt{\omega^2 + 2\tilde{\omega}^2}$.

The generalized coordinates Q_1 and Q_2, each of which satisfy uncoupled harmonic oscillator equations of motion, are called *normal coordinates*. The solutions in which

$$(1) \qquad Q_1 = Q_1(t), \quad Q_2 = 0$$

and

$$(2) \qquad Q_1 = 0, \quad Q_2 = Q_2(t)$$

are called the two *normal modes* of vibration. The corresponding frequencies Ω_1 and Ω_2 are called the *resonant frequencies*, or the *natural frequencies*, or the *characteristic frequencies* of the normal modes (1) and (2) of vibration. Note that case (1) corresponds to a solution to the equations of motion given by (*exercise*)

$$q_1(t) = q_2(t) = Re(A_1 e^{i\Omega_1 t}), \tag{2.9}$$

while case (2) corresponds to a solution of the form (*exercise*)

$$q_1(t) = -q_2(t) = Re(A_2 e^{i\Omega_2 t}). \tag{2.10}$$

Here A_1 and A_2 are any complex numbers.

We can now write down the general solution for the motion of the system. In normal coordinates the general solution takes the form:

$$Q_1 = Re(A_1 e^{i\,\Omega_1 t}), \quad Q_2 = Re(A_2 e^{i\,\Omega_2 t}), \tag{2.11}$$

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where A_1 and A_2 are any complex numbers. Using the change of variables formula (2.6) we can write the general solution in terms of the original variables (*exercise*):

$$q_1 = Re\left(A_1 e^{i\,\Omega_1 t} - A_2 e^{i\,\Omega_2 t}\right)$$
$$q_2 = Re\left(A_1 e^{i\,\Omega_1 t} + A_2 e^{i\,\Omega_2 t}\right) \tag{2.12}$$

From (2.9) and (2.10) you can see that the complex numbers A_1 and A_2 control the amplitude and phase of the normal modes of vibration which are being superposed to get the solution for q_1 and q_2. Note that we have (2 complex) = (4 real) numbers at our disposal in (2.12), namely, A_1 and A_2. From a mathematical point of view this is exactly what is needed to give the general solution to a pair of linear, second-order ordinary differential equations. Physically, we need 4 real numbers to specify the initial positions and velocities of the 2 oscillators. Indeed, each solution of the coupled oscillator equations (2.3)–(2.4) is uniquely determined by the choice of A_1 and A_2, which can be specified by giving the initial conditions (*exercise*, see also §2.2). See figure 4 for some graphical descriptions of the general solution.

Let us summarize the results of this section thus far. The normal modes of vibration correspond to having either $Q_1(t)$ or $Q_2(t)$ zero for all time. These modes of vibration are harmonic motions at the frequencies Ω_1 and Ω_2, respectively. In general, the motion of the system is a superposition of the normal modes of vibration. The particular superposition that arises is determined by initial conditions.

Once again the basic results we have encountered for the system of two coupled oscillators can be given a vector space interpretation (see Appendix B for information about vector spaces). In this interpretation the underlying set of objects is the set of all possible solutions $(q_1(t), q_2(t))$ to the equations (2.3), (2.4). Let us write these solutions as a column vector of functions:

$$\mathbf{q}(t) = \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix}. \tag{2.13}$$

Since the equations for $\mathbf{q}$ are linear and homogeneous we can add solutions with numerical factors to make new solutions. More precisely, if we have 2 solutions $\mathbf{q}$ and $\tilde{\mathbf{q}}$,

$$\mathbf{q}(t) = \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix}, \quad \tilde{\mathbf{q}}(t) = \begin{pmatrix} \tilde{q}_1(t) \\ \tilde{q}_2(t) \end{pmatrix}, \tag{2.14}$$

then for any real constants a and b

$$\hat{\mathbf{q}}(t) = a\mathbf{q}(t) + b\tilde{\mathbf{q}}(t) = \begin{pmatrix} aq_1(t) + b\tilde{q}_1(t) \\ aq_2(t) + b\tilde{q}_2(t) \end{pmatrix} \tag{2.15}$$

is also a solution to the coupled equations. We define the vector space scalar multiplication operation and addition operation as indicated in (2.15). Note that we are taking advantage

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of the familiar vector space structure admitted by the set of column vectors. From the explicit form (2.12) of the solutions it follows that the following is a *basis* for the real vector space of solutions to the coupled oscillator equations:

$$\mathbf{b}_1(t) = \cos(\Omega_1 t) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{b}_2(t) = \sin(\Omega_1 t) \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$\mathbf{b}_3(t) = \cos(\Omega_2 t) \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \mathbf{b}_4(t) = \sin(\Omega_2 t) \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{2.16}$$

(You will be asked to show this in the Problems.) In other words, every solution to the coupled equations can be uniquely expressed as a superposition

$$\mathbf{q}(t) = a_1 \mathbf{b}_1(t) + a_2 \mathbf{b}_2(t) + a_3 \mathbf{b}_3(t) + a_4 \mathbf{b}_4(t). \tag{2.17}$$

Note that $\mathbf{b}_1$ and $\mathbf{b}_2$ give the part of the solution oscillating at frequency Ω_1. You can check that this part of the basis defines the part of the solution corresponding to the normal mode labeled by Q_1. Similarly, $\mathbf{b}_3$ and $\mathbf{b}_4$ give the part of the motion coming from normal mode labeled by Q_2. Because there are 4 basis vectors, the vector space of solutions is 4-dimensional. As we have already mentioned, this should match your physical intuition: it should take four real numbers – *e.g.,* 2 initial positions and 2 initial velocities – to specify a solution.

It is possible to generalize the scalar product (1.19) to our current example, in which case the basis (2.16) is orthonormal. If you are interested, you might try to work out the details as a nice *exercise*. We won't do it here.

2.2 Physical Meaning of the Normal Modes

The normal mode of vibration corresponding to $Q_1 = Q_1(t)$, $Q_2 = 0$ is a motion of the system in which the displacement of each oscillator is equal and *in phase* (*exercise*).*
In other words, the masses oscillate together (with a constant separation) at an angular frequency of $\Omega_1 = \omega$. Recall that the normal coordinate Q_1 represented, essentially, the center of mass of the system. The normal mode of vibration $Q_1(t)$ is a harmonic oscillation of the center of mass with frequency ω. Because the two oscillators keep the same relative distance, there is no compression of the spring which couples the oscillators and so it is easy to see why the frequency of this normal mode is controlled by ω alone. One way to

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* The use of the term "phase" in this context refers to the phase of the cosine functions that describe the displacement of each of the two masses in this normal mode. To say that the two masses are "in phase" is to say that the arguments of the cosines are the same for all time. Physically, the two masses are always in the same part of their sinusoidal oscillation.

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"excite" the normal mode associated with Q_1 is to start the system off at $t = 0$ such that the masses have the same displacement and same velocity:

$$q_1(0) = q_2(0) \quad \text{and} \quad v_1(0) = v_2(0). \tag{2.18}$$

This forces $A_2 = 0$ in (2.12), *i.e.*, $Q_2(t) = 0$ (*exercise*).

If we start the system out so that

$$q_1(0) = -q_2(0) \quad \text{and} \quad v_1(0) = -v_2(0), \tag{2.19}$$

then this forces $A_1 = 0$ in (2.12), so that $Q_1(t) = 0$ (*exercise*), and we get the other normal mode of vibration. Note that these initial conditions amount to displacing each mass in the opposite direction by the same amount and giving each mass a velocity which is the same in magnitude but oppositely directed.† In this mode the particles oscillate oppositely, or completely *out of phase* (*i.e.*, the phases of the cosine functions that describe the oscillations of each mass differ by π radians). This is consistent with the interpretation of Q_2 as the relative position of the particles. Clearly the spring which couples the particles (the one characterized by $\tilde{\omega}$) is going to play a role here – this spring is going to be compressed or stretched – which is why the (higher) frequency of oscillation of this mode, Ω_2, involves $\tilde{\omega}$. From (2.12), all other motions of the system are superpositions of these two basic kinds of motion and are obtained by using initial conditions other than (2.18) or (2.19). This is the meaning of the basis (2.16) and the form (2.17) of the general solution.

At this point you cannot be blamed if you feel that, aside from its interest as a step on the road to better understanding waves, the system of coupled oscillators is not particularly relevant in physics. After all, how useful can a system be that consists of a couple of masses connected by springs? Actually, the mathematics used in this section (and generalizations thereof) can be fruitfully applied to vibrational motions of a variety of systems. Molecular vibrations represent a particularly important application. For example, one can use the above normal mode analysis to find the possible vibrational motion of a triatomic molecule, such as ozone (O_3). The vibrational motion of such a molecule can be excited by an oscillating electric field (*e.g.,* an electromagnetic wave), hence normal mode calculations are common in optical spectroscopy. Simple variations on these calculations occur when the masses are not equal (*e.g., CO_2*), when the molecule is not linear (*e.g., NO_2*), or when there are more atoms in the molecule (*e.g.,* methane (CH_4) or ammonia (NH_3)).

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† Note that in both (2.18) and in (2.19) one can have vanishing initial displacements or velocities.

(a)

(b)
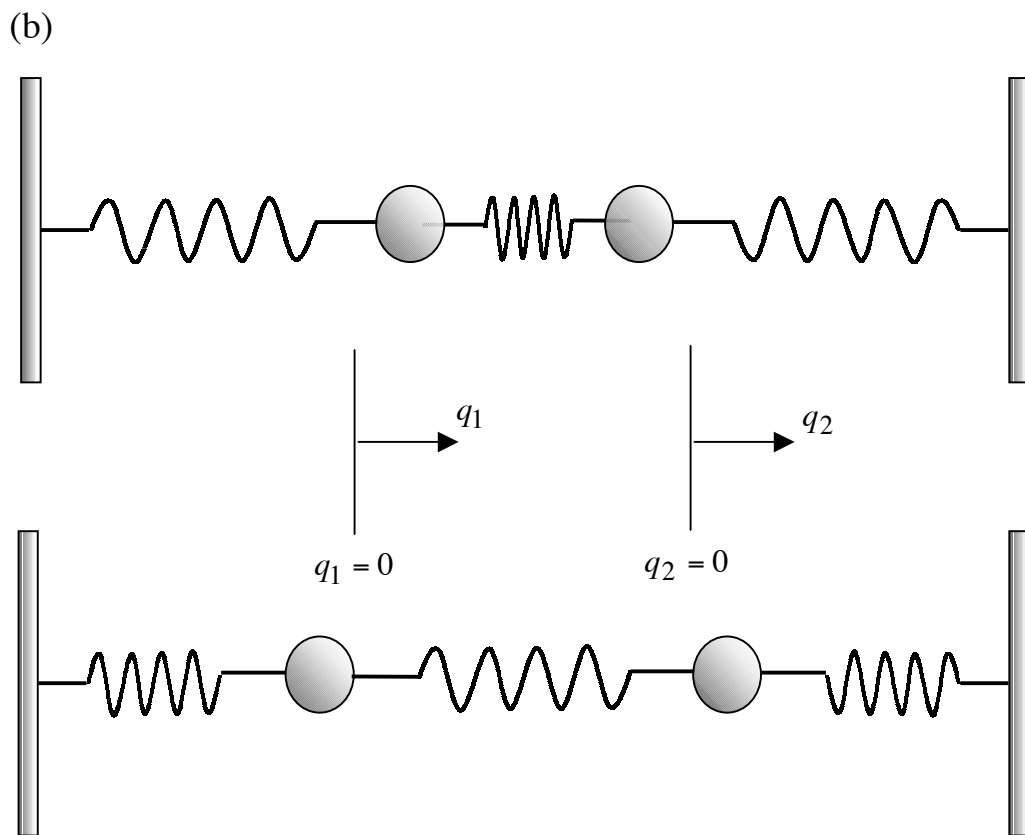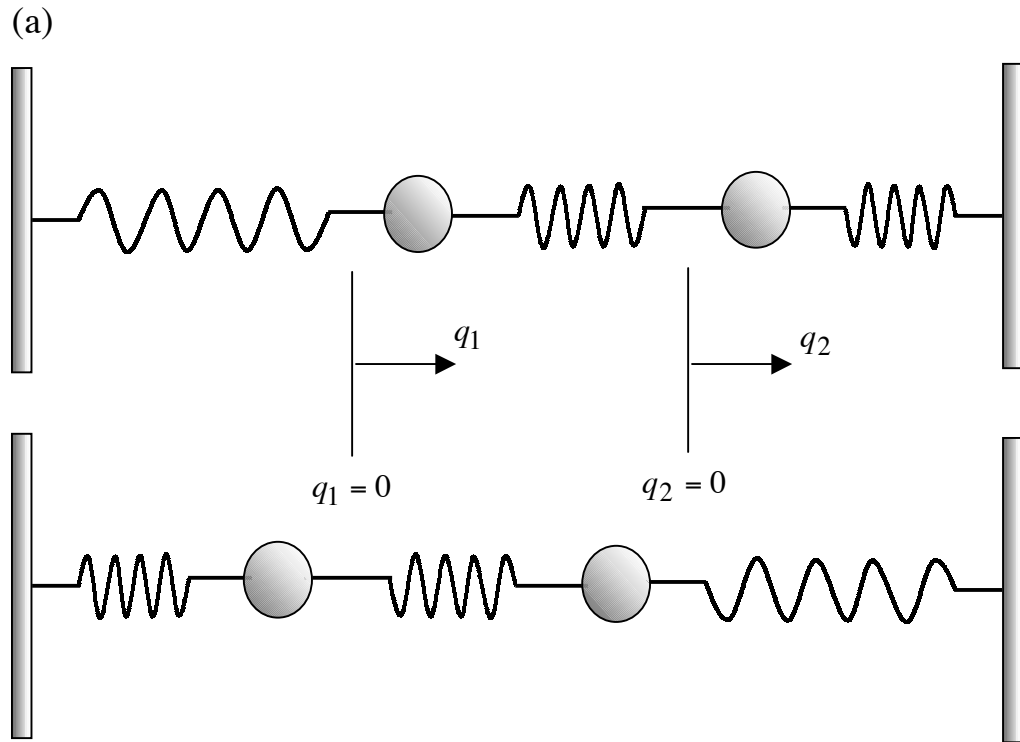
Figure 5. Illustration of the normal modes for two coupled oscillators. (a) Symmetric mode where $q_1 = q_2$. (b) Antisymmetric mode where $q_1 = -q_2$.
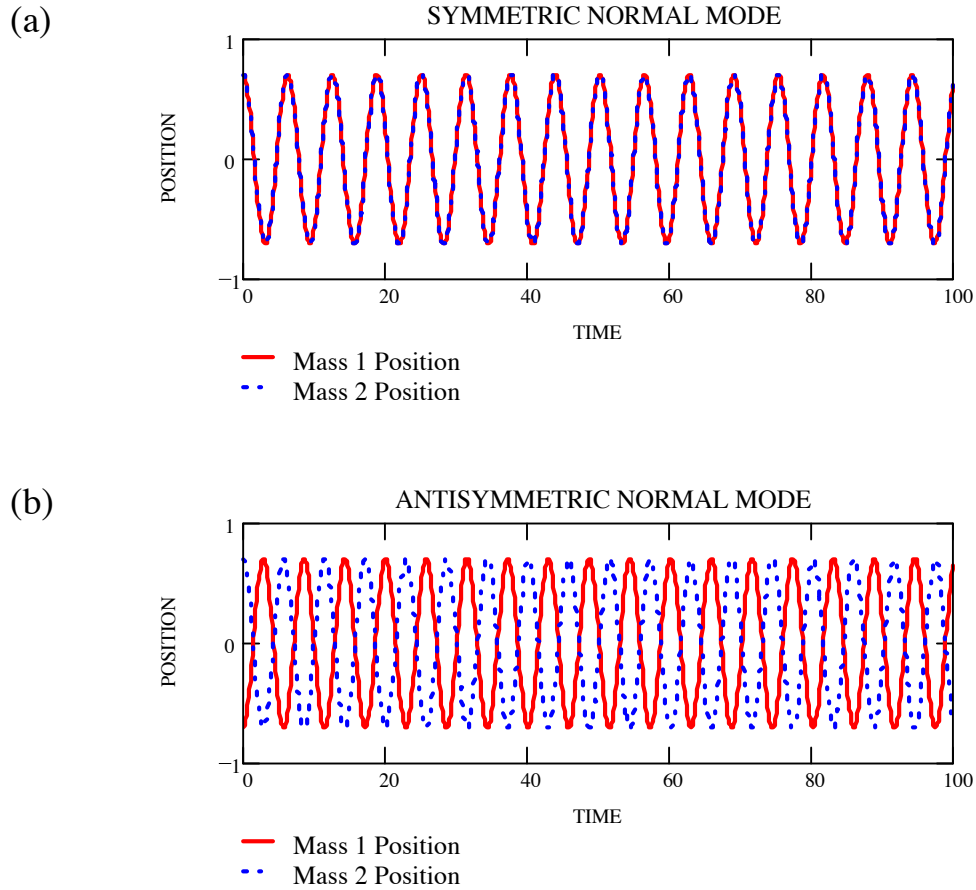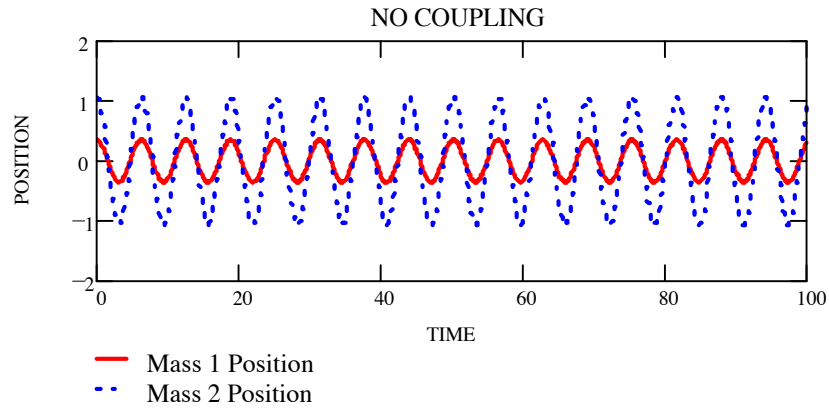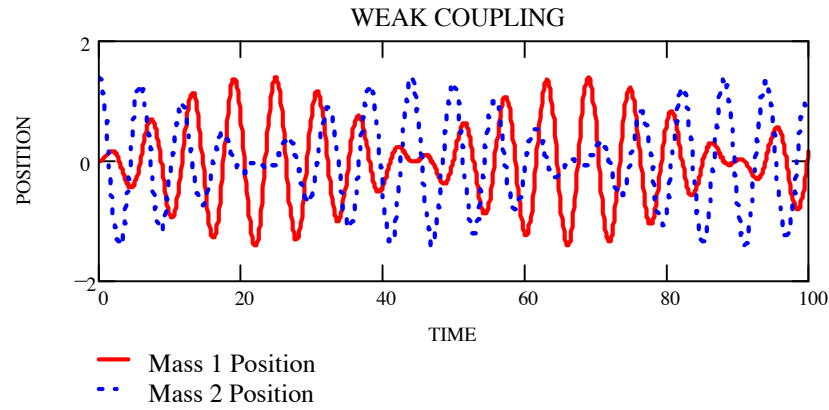
(a)

(b)

Figure 6. Time dependence of coupled oscillator positions q_1 and q_2 for (a) oscillation in the symmetric normal mode ($q_1 = q_2$), and (b) oscillation in antisymmetric normal mode ($q_1 = -q_2$). For all graphs $M = 1, k = 1$. For (a) $A_1 = 1$, $A_2 = 0$. For (b) $A_1 = 0$, $A_2 = 1$.
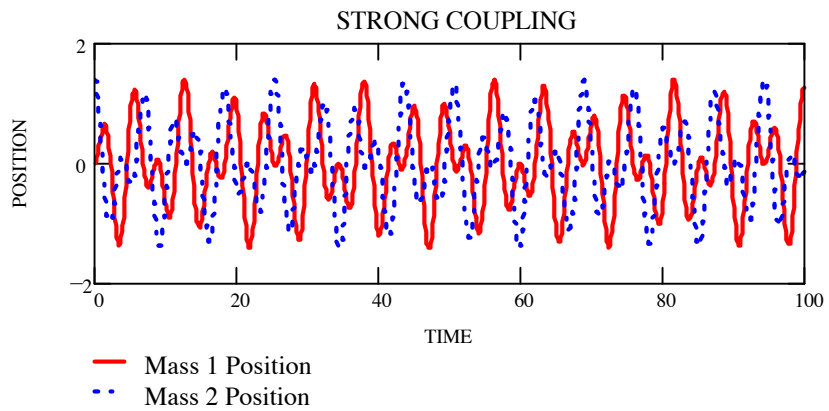
(a)

(b)

(c)

Figure 4. Time dependence of coupled oscillator positions q_1 and q_2 for (a) no coupling ($k' = 0$), (b) weak coupling ($k' = 0.1$), and (c) strong coupling ($k' = 1$). For all graphs $M = 1, k = 1$. For (a) $A_1 = 1$, $A_2 = 0.5$. For (b) and (c) $A_1 = A_2 = 1$.