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03 How To Find Normal Modes

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3. How to find normal modes.

How do we find the normal modes and resonant frequencies without making a clever guess? Well, you can get a more complete explanation in an upper-level mechanics course, but the gist of the trick involves a little linear algebra. The idea is the same for any number of coupled oscillators, but let us stick to our example of two oscillators.

To begin, we again assemble the 2 coordinates, q_i, $i = 1, 2$, into a column vector $\mathbf{q}$,

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}. \tag{3.1}$$

Let K be the 2×2 symmetric matrix

$$K = \begin{pmatrix} \omega^2 + \tilde{\omega}^2 & -\tilde{\omega}^2 \\ -\tilde{\omega}^2 & \omega^2 + \tilde{\omega}^2 \end{pmatrix}. \tag{3.2}$$

The coupled oscillator equations (2.3), (2.4) can then be written in matrix form as (*exercise*)

$$\frac{d^2\mathbf{q}}{dt^2} = -K\mathbf{q}. \tag{3.3}$$

The fact that the matrix K is not diagonal corresponds to the fact that the equations for $q_i(t)$ are coupled.

Exercise: Check that the matrix form of the uncoupled equations (2.1), (2.2) gives a diagonal matrix K.

You may already know how to find a new basis for the vector space in which the matrix K *is diagonal* – this is the basis provided by the *eigenvectors* of K. So, our strategy for solving (3.3) is to find the *eigenvalues* λ and eigenvectors $\mathbf{e}$ of K. These are the solutions to the equation

$$K\mathbf{e} = \lambda\mathbf{e}, \tag{3.4}$$

where λ is a scalar and $\mathbf{e}$ is a (column) vector. The eigenvalues and eigenvectors are fundamental characteristics of the matrix K. As we shall discuss further below, the matrix K will turn out to be such that its two (possibly equal) eigenvalues, λ_1 and λ_2 are both positive. In addition, it will turn out that the corresponding eigenvectors, $\mathbf{e}_1$ and $\mathbf{e}_2$, are linearly independent.* This means that any column vector $\mathbf{v}$ can be expressed as

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2,$$

* In other words, the eigenvectors form a *basis* for the vector space of 2-component column vectors.

for some real numbers v_1 and v_2, which are the *components* of $\mathbf{v}$ in the basis $(\mathbf{e}_1, \mathbf{e}_2)$. We shall soon see why these properties arise

Given the solutions $(\lambda_1, \mathbf{e}_1)$, $(\lambda_2, \mathbf{e}_2)$ to (3.4), we can build a solution to (3.3) as follows. Write

$$\mathbf{q}(t) = \alpha_1(t)\mathbf{e}_1 + \alpha_2(t)\mathbf{e}_2. \tag{3.5}$$

Exercise: Why can we always do this?

Using

$$\frac{d^2\mathbf{q}}{dt^2} = \frac{d^2\alpha_1}{dt^2}\mathbf{e}_1 + \frac{d^2\alpha_2}{dt^2}\mathbf{e}_2, \tag{3.6}$$

and†

$$\begin{aligned} K\mathbf{q} &= \alpha_1(t)K\mathbf{e}_1 + \alpha_2(t)K\mathbf{e}_2 \\ &= \lambda_1\alpha_1(t)\mathbf{e}_1 + \lambda_2\alpha_2(t)\mathbf{e}_2, \end{aligned} \tag{3.7}$$

you can easily check that (3.5) defines a solution to (3.3) if and only if

$$\left(\frac{d^2\alpha_1}{dt^2} + \lambda_1\alpha_1(t)\right)\mathbf{e}_1 + \left(\frac{d^2\alpha_2}{dt^2} + \lambda_2\alpha_2(t)\right)\mathbf{e}_2 = 0. \tag{3.8}$$

Using the linear independence of the eigenvectors, this means (*exercise*) that α_1 and α_2 each solves the harmonic oscillator equation with frequency $\sqrt{\lambda_1}$ and $\sqrt{\lambda_2}$, respectively:

$$\frac{d^2\alpha_n}{dt^2} = -\lambda_n\alpha_n(t), \quad n = 1, 2. \tag{3.9}$$

The general solution to (3.3) can then be written as (*exercise*)

$$\mathbf{q}(t) = Re(A_1 e^{i\sqrt{\lambda_1}t}\mathbf{e}_1 + A_2 e^{i\sqrt{\lambda_2}t}\mathbf{e}_2), \tag{3.10}$$

where A_1 and A_2 are any complex numbers. Thus, by finding the eigenvalues and eigenvectors we can reduce our problem to two copies of the harmonic oscillator equation, which we already know how to solve.

Now you can see why we needed those properties of the eigenvalues and eigenvectors. Firstly, if the eigenvectors don't form a basis, we can't assume $\mathbf{q}$ takes the form (3.5) nor that (3.8) implies (3.9). It is an important theorem from linear algebra that for any *symmetric* matrix with real entries, such as (3.2), the eigenvectors will form a basis, so this assumption is satisfied in our current example. Secondly, the frequencies $\sqrt{\lambda_n}$ will be real numbers if and only if the eigenvalues λ_n are always positive. While the aforementioned linear algebra theorem guarantees the eigenvalues of a symmetric matrix

† Note that here we use the fact that matrix multiplication is a linear operation.

will be real, it doesn't guarantee that they will be positive. However, as we shall see, for the coupled oscillators the eigenvalues are positive definite, which one should expect on physical grounds. (*Exercise:* How would you interpret the situation in which the eigenvalues are negative?).

Comparing our general solution (3.10) with (2.12) we see that the resonant frequencies ought to be related to the eigenvalues of K via

$$\Omega_i = \sqrt{\lambda_i}, \quad i = 1, 2$$

and the normal modes should correspond to the eigenvectors $\mathbf{e}_i$. Let us work this out in detail.

The eigenvalues of K are obtained by finding the two solutions λ to the equation (3.4). This equation is equivalent to

$$(K - \lambda I)\mathbf{e} = 0,$$

where I is the identity matrix. A standard result from linear algebra is that this equation has a non-trivial solution† $\mathbf{e}$ if and only if λ is a solution of the *characteristic (or secular) equation*:

$$\det[K - \lambda I] = 0.$$

You can easily check that the characteristic equation for (3.2) is

$$\lambda^2 - 2(\omega^2 + \tilde{\omega}^2)\lambda - \tilde{\omega}^4 + (\omega^2 + \tilde{\omega}^2)^2 = 0. \tag{3.11}$$

This is a quadratic equation in λ, which is easily solved to get the two roots (*exercise*)

$$\begin{aligned} \lambda_1 &= \omega^2 \\ \lambda_2 &= \omega^2 + 2\tilde{\omega}^2. \end{aligned} \tag{3.12}$$

Note that we have just recovered the (squares of the) resonant frequencies by finding the eigenvalues of K.

To find the eigenvectors $\mathbf{e}_i$ of K we substitute each of the eigenvalues λ_i, $i = 1, 2$ into the eigenvalue equation (3.4) and solve for the components of the $\mathbf{e}_i$ using standard techniques. As a very nice exercise you should check that the resulting eigenvectors are of the form

$$\begin{aligned} \mathbf{e}_1 &= a \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \mathbf{e}_2 &= b \begin{pmatrix} -1 \\ 1 \end{pmatrix} \end{aligned} \tag{3.13}$$

where a and b are any constants, which can be absorbed into the definition of A_1 and A_2 in (3.10) (*exercise*).

† *Exercise:* what is the trivial solution?

Exercise: Just from the form of (3.4), can you explain why the eigenvectors are only determined up to an overall multiplicative factor?

Using these eigenvectors in (3.10) we recover the expression (2.12) – you really should verify this yourself. In particular, it is the eigenvectors of K that determine the column vectors appearing in (2.16) *(exercise)*.

Note that the eigenvectors are linearly independent as advertised (*exercise*). Indeed, using the usual scalar product on the vector space of column vectors $\mathbf{v}$ and $\mathbf{w}$,

$$(\mathbf{v}, \mathbf{w}) = \mathbf{v}^T \mathbf{w},$$

you can check that $\mathbf{e}_1$ and $\mathbf{e}_2$ are orthogonal (see Problems).

To summarize: The resonant frequencies of a system of coupled oscillators, described by the matrix differential equation

$$\frac{d^2}{dt^2}\mathbf{q} = -K\mathbf{q},$$

are determined by the eigenvalues of the matrix K. The normal modes of vibration are determined by the eigenvectors of K.

4. Linear Chain of Coupled Oscillators.

As an important application and extension of the foregoing ideas, and to obtain a first glimpse of wave phenomena, we consider the following system. Suppose we have N identical particles of mass m in a line, with each particle bound to its neighbors by a Hooke's law force, with "spring constant" k. Let us assume the particles can only be displaced in one-dimension; label the displacement from equilibrium for the j^{th} particle by q_j, $j = 1, 2, \ldots, N$. Let us also assume that particle 1 is attached to particle 2 on the right and a rigid wall on the left, and that particle N is attached to particle $N-1$ on the left and another rigid wall on the right. The equations of motion then take the form (*exercise*):

$$\frac{d^2 q_j}{dt^2} + \omega^2(q_j - q_{j-1}) - \omega^2(q_{j+1} - q_j) = 0, \quad j = 1, 2, \ldots, N. \tag{4.1}$$

For convenience, in this equation and in all that follows we have extended the range of the index j on q_j to include $j = 0$ and $j = N + 1$. You can pretend that there is a particle fixed to each wall with displacements labeled by q_0 and q_{N+1}. Since the walls are rigid, to obtain the correct equations of motion we must set

$$q_0 = 0 = q_{N+1}. \tag{4.2}$$