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1-1-2012

05 The Continuum Limit and the Wave Equation

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Recommended Citation

Torre, Charles G., "05 The Continuum Limit and the Wave Equation" (2012). *Foundations of Wave Phenomena.* Book 18.
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5. The Continuum Limit and the Wave Equation.

Our example of a chain of oscillators is nice because it is easy to visualize such a system, namely, a chain of masses connected by springs. But the ideas of our example are far more useful than might appear from this one simple mechanical model. Indeed, many materials (including solids, liquids and gases) have some aspects of their physical response to (usually small) perturbations behaving just as if they were a bunch of coupled oscillators — at least to a first approximation. In a sense we will explore later, even the electromagnetic field behaves this way! This "harmonic oscillator" response to perturbations leads — in a continuum model — to the appearance of wave phenomena in the traditional sense. We caught a glimpse of this when we examined the normal modes for a chain of oscillators with various boundary conditions. Because the harmonic approximation is often a good first approximation to the behavior of systems near equilibrium, you can see why wave phenomena are so ubiquitous. The key difference between a wave in some medium and the examples of §4 is that wave phenomena are typically associated with propagation media (stone, water, air, etc.) which are modeled as *continuous* rather than *discrete*. As mentioned earlier, our chain of oscillators in §4 can be viewed as a discrete model of a continuous (one-dimensional) material. We now want to introduce a phenomenological description of the material in which we ignore the atomic discreteness of matter.

The basic physical idea is reasonably simple. Often times we are interested in certain macroscopic properties of the material (*e.g.,* the behavior of a plucked guitar string as a function of time and space) and we want to ignore most of the details of the microscopic make-up of the material since they should be irrelevant for the most part. So long as the length scales associated with the macroscopic behavior of the material (*e.g.,* wavelengths) are much larger than the length scales associated with the microscopic structure (*e.g.,* the interparticle spacing) we can approximate the behavior of the material by taking a limit in which we set the interparticle spacing to zero while letting the number of oscillators become arbitrarily large ("approach infinity").

We will have to exercise a little care in this limiting process. Here "care" means that we keep fixed some macroscopic quantities characterizing the material in which the waves are propagating. Some good examples of the materials to keep in mind are: sound waves in an "elastic solid", *e.g.,* in a metal rod; a vibrating string or rope under tension; sound waves in a gas. Each of these materials will have certain physical parameters which are relevant to the propagation of the wave and which are macroscopic reflections of the effective oscillator parameters which model the microscopic behavior of the material. For example each of the three illustrations just mentioned will be characterized (in part) by their mass density.

Let us emphasize that a continuum approximation, by its very nature, will not have universal validity. For example, if we consider wave phenomena in which the wavelengths are comparable to (or smaller than) the interparticle spacing, then we don't expect our

model will accurately model what is actually happening physically.

5.1 Derivation of the Wave Equation

As in §4, we suppose that the equilibrium separation of the oscillators is d and we label the equilibrium position of the oscillators by $x = jd$. We can then denote by $q(x,t)$ the displacement of the jth oscillator from its equilibrium position at time t. Our use of the symbol x, usually reserved for a continuous variable, anticipates our implementation of the strategy wherein the interparticle spacing is so small (compared to the typical sizes of macroscopic phenomena) that we can model the particles as forming a continuous mass distribution. We rearrange the equations of motion (4.1) into the form (*exercise*)

$$\frac{d^2q(x,t)}{dt^2} = -\tilde{\omega}^2 d\left[\frac{1}{d}\{q(x,t) - q(x-d,t)\}\right] + \tilde{\omega}^2 d\left[\frac{1}{d}\{q(x+d,t) - q(x,t)\}\right]. \tag{5.1}$$

We now study the right-hand side of this equation in the limit where d is very small. In this case we can view $q(x,t)$ as a continuous function of x to a better and better approximation, and we have that† (exercises)

$$\frac{1}{d}\{q(x,t) - q(x-d,t)\} \approx \left(\frac{\partial q}{\partial x}\right)_{x=jd-d/2}, \tag{5.2}$$

and

$$\frac{1}{d}\{q(x+d,t) - q(x,t)\} \approx \left(\frac{\partial q}{\partial x}\right)_{x=jd+d/2}. \tag{5.3}$$

In the same manner, the difference of these terms defines, in the limit as $d \to 0$, the second derivative of q times d:

$$\left(\frac{\partial q}{\partial x}\right)_{x=jd+d/2} - \left(\frac{\partial q}{\partial x}\right)_{x=jd-d/2} \approx d\left(\frac{\partial^2 q(x,t)}{\partial x^2}\right)_{x=jd}. \tag{5.4}$$

We can therefore write the equation of motion in this approximation as:

$$\frac{\partial^2 q(x,t)}{\partial t^2} = \tilde{\omega}^2 d^2 \frac{\partial^2 q(x,t)}{\partial x^2}. \tag{5.5}$$

This is already a wave equation, but to get our final form of it we need to consider the limit as $d \to 0$. We do this as follows.

First recall that

$$\tilde{\omega} = \sqrt{\frac{k}{m}}, \tag{5.6}$$

† Yes, those definitions for derivatives as limits of diffferences that you learned in calculus class really do come in handy after all!

We express the spring constant k — which represents "microscopic" information — as

$$k = \frac{\kappa}{d}, \tag{5.7}$$

where the physical interpretation of the "macroscopic" constant κ depends upon what we are modeling. For transverse† vibrations of a string, κ will represent the tension on the string. For the longitudinal† vibrations of an elastic medium (*e.g.,* sound waves in a solid), κ will represent the Young's modulus, which determines the stiffness of the material making up the medium. For compression (sound) waves in air, κ will be the elastic modulus. The quantity κ is one of two macroscopic quantities that are held fixed when taking the continuum limit. We now have (*exercise*)

$$\tilde{\omega}^2 d^2 = \frac{\kappa d}{m}. \tag{5.8}$$

Next we express the mass of the oscillators — another microscopic quantity – as

$$m = \mu d, \tag{5.9}$$

where μ, which is a macroscopic quantity, represents the mass per unit length of the continuum medium. The mass per unit length is the other macroscopic quantity that is held fixed in the continuum limit. We now have

$$\tilde{\omega}^2 d^2 = \frac{\kappa}{\mu}. \tag{5.10}$$

The continuum limit has the microscopic parameters behaving as $d \to 0$, $\tilde{\omega} \to \infty$ (also $m \to 0$, $k \to \infty$), with the macroscopic parameters κ and μ — characterizing the continuous material in question — held fixed. Setting $v^2 := \frac{\kappa}{\mu}$,* (5.5) becomes

$$\frac{\partial^2 q(x,t)}{\partial t^2} = v^2 \frac{\partial^2 q(x,t)}{\partial x^2}. \tag{5.11}$$

This is the one-dimensional *wave equation*, which is a fundamental example of a *partial differential equation.* The equation (5.11), and its generalizations, will be the subject of all of our attention from now on.

Note that we are now using *partial* time derivatives, *i.e.,* we hold x fixed when we vary t to take the time derivative and we hold t fixed when we compute derivatives with

† Recall that "transverse waves" have a displacement which is orthogonal to the direction of propagation of the displacement, while "longitudnal waves" have a displacement which is parallel to the direction of propagation.

* The notation "$a := b$" indicates that we are making a definition, namely, "a is defined to be the quantity b". Thus we can distinguish between equalities that we should be able to deduce from some other facts (which use "="), and equalities true merely by definition (which use ":="). The notation "$a := b$" is close to, but not quite the same as, $a \equiv b$, which means "a is identically equal to b", as in $5 \equiv 5$.

respect to x. Don't forget: $x = jd$ formerly labeled equilibrium positions of the jth mass and $q_j(t)$ denoted the displacement of that mass from its equilibrium position at time t. In the continuum approximation, the chain of oscillators is represented by a line of fixed mass density. Points on the line are labeled x and the displacement "from equilibrium" of a point at x on the line at time t is denoted by $q(x,t)$. The second time derivative of $q(x,t)$ in the wave equation is just the acceleration that features in Newton's second law. The second spatial derivative of $q(x,t)$ is the continuum limit of the harmonic "nearest neighbor" interaction.

If the continuum is meant to describe an elastic medium undergoing longitudinal vibrations (sound waves) then the displacement $q(x,t)$ represents a compression or rarefication of the elastic medium the sound is traveling in at the point x and time t, that is, q represents a longitudinal density wave. If the continuum is meant to represent a vibrating string under tension, then $q(x,t)$ represents the deflection of the string at (x,t) from its equilibrium position, that is, q represents a transverse displacement wave.

The parameter v that appears in (5.11) is easily seen to have units of speed (*exercise*). We shall see that v characterizes the speed of the waves that satisfy (5.11). By the way, an easy way to remember how the velocity factor enters the wave equation is to use dimensional analysis: the $\frac{1}{v^2}$ is needed to balance the units (*exercise*).

To summarize: *the 1-dimensional wave equation describes a continuum of matter in which displacements of infinitesimal elements of mass experience a Hooke's law restoring force.*

5.2. Boundary Conditions

In our discussion of the chain of oscillators we considered various boundary conditions. Since the wave equation can be viewed as a limiting case of the chain of oscillators, there are corresponding boundary conditions here as well. Let us briefly describe them here.

Of course, the case of no boundary conditions has for the most part been dispensed with in the previous paragraphs — no boundary conditions were imposed there. Typically, one will ignore boundary conditions if one is not near a boundary and one is considering features of the wave which are much smaller than the spatial domain of the problem. One will then speak of, *e.g.,* waves on a "long" string. The usual mathematical model for such a situation is to suppose that the spatial domain x is all of the real numbers, $-\infty < x < \infty$.

Let us consider our original fixed-wall boundary conditions. These are sometimes called *Dirichlet conditions* in this context. Here the spatial domain of interest is finite, say, $0 \le x \le L$, and the displacement $q = q(x,t)$ vanishes at the boundaries for all time:

$$q(0,t) = 0 = q(L,t). \tag{5.12}$$

This boundary condition models a string under tension with fixed ends (*e.g.,* a guitar string). It also models sound waves in air in a finite, closed region (in a one-dimensional approximation — *e.g.,* a long closed pipe.)

A vibrating metal rod (whose spatial cross section is small compared to its length L) which is clamped at one end or a vibrating column of air in a pipe which is open at one end could be modeled with just the boundary conditions

$$q(0,t) = 0.$$

Periodic boundary conditions can be handled similarly, on a finite interval $0 \leq x \leq L$ we insist that

$$q(0,t) = q(L,t). \tag{5.13}$$

This condition could be used to model a vibrating loop of string, or sound in a closed circular pipe.