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06 Elementary Solutions to the Wave Equation

Charles G. Torre

Department of Physics, Utah State University, charles.torre@usu.edu

6. Elementary Solutions to the Wave Equation.

Before systematically exploring the wave equation, it is good to pause and contemplate some simple solutions. We are looking for a function of 2 variables, $q = q(x,t)$, whose second x derivatives and second t derivatives are proportional. You can probably guess such functions with a little thought. But our derivation of the equation from the model of a chain of oscillators gives a strong hint. The normal modes for a chain of oscillators look like products of sinusoidal functions of j (now replaced by x) and sinusoidal functions of t. If we try such a combination, say,

$$q(x,t) = A\sin(Bx)\cos(Ct), \tag{6.1}$$

where A, B, and C are real constants, we find that the wave equation is satisfied provided

$$\frac{C^2}{B^2} = v^2, \tag{6.2}$$

which is one restriction on the constants B and C. Note that A is unrestricted by the wave equation. (*Exercise:* what property of the wave equation guarantees that A is unrestricted?) Thus we can let A be any constant, we can let C be any constant, and then set $B = C/v$ to get a solution. It is customary to set

$$C \equiv \omega = \frac{2\pi v}{\lambda} \quad \text{and} \quad B = \frac{2\pi}{\lambda}, \tag{6.3}$$

Then our solution takes the form

$$q(x,t) = A\sin(\frac{2\pi x}{\lambda})\cos(\omega t). \tag{6.4}$$

We have obtained a *standing wave* with *wavelength* λ and *angular frequency* ω. In detail, at each instant of time, the displacement from equilibrium is a sinusoidal function of x with wavelength λ, and at any given value of x the displacement is a sinusoidal function of time with angular frequency ω.

Do not confuse the angular frequency of a wave with the natural angular frequency of an oscillator. Unfortunately, it is a standard notation to use ω for both. They are not the same quantity – we are finished with oscillators at this point.

As you probably know, we often define the *wave number* k by

$$k := \frac{2\pi}{\lambda}. \tag{6.5}$$

Just as the wavelength measures the length of one period of the wave, the wave number gives the number of wavelengths per unit length. The standing wave solution thus has the alternate form:

$$q(x,t) = A\sin(kx)\cos(\omega t). \tag{6.6}$$

When you check that the standing wave (6.4) (or (6.6)) does indeed satisfy the wave equation, you will immediately see we can add a constant to the argument of either sine or cosine function and we still will have a solution.* By adding a constant to ωt, say,

$$\omega t \to \omega t + c,$$

we are, in effect, shifting the origin of time (*exercise*). This has the effect of shifting the initial conditions to an earlier time if $c > 0$ (*exercise*). By adding a constant to $2\pi x/\lambda$ we are making a rigid displacement of the wave in the negative x directions. In particular, we are shifting the locations of the *nodes*, *i.e.,* the locations of the points of zero displacement. Of course, we are also shifting the locations of the anti-nodes, *i.e.,* the locations of maximum displacement. You can also check that we can interchange the role of sine and cosine in the above solution, or we can have both functions as sines or both as cosines (exercises). And, because the wave equation is linear and homogeneous, any linear combination of these possibilities will solve the wave equation.

A particularly interesting linear combination is of the form

$$\begin{aligned} q(x,t) &= A\cos(\frac{2\pi x}{\lambda})\cos(\omega t) + A\sin(\frac{2\pi x}{\lambda})\sin(\omega t) \\ &= A\cos[\frac{2\pi}{\lambda}(x - vt)]. \end{aligned} \tag{6.7}$$

This is an example of a *traveling wave*. If you take a photograph of the system at $t = 0$, you will see a nice sinusoidal displacement with maximum displacement at $x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, \ldots$, and zero displacement at $x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \ldots$. At some later time t a photograph will reveal the same sinusoidal displacement pattern, but shifted to the right by an amount vt (*exercise*). Thus the constant v in the wave equation (5.11) is the speed of the traveling wave.

A complex version of the traveling wave will be handy later. It takes the form

$$q(x,t) = Ae^{i(kx \pm \omega t)}, \tag{6.8}$$

where $k = \frac{2\pi}{\lambda}$ and $\omega = |k|v$. You can easily see that the real and imaginary parts of this complex-valued q will satisfy the wave equation, as does the complex conjugate of q.

* For now we are imposing no boundary or initial conditions. We are simply studying solutions of the differential equation.