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# The Riemann curvature tensor, its invariants, and their use in the classification of spacetimes

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	A Little Background	The Idea	The Example	Conclusion
The	Riemann curvati	ire tensor	its invariants, a	nd
	their use in the cl	assification	of spacetimes	
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Introd	luction
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- "Spacetime tells matter how to move; matter tells spacetime how to curve" - John Wheeler
- The Einstein field equations:

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R + \Lambda g_{\alpha\beta} = \kappa T_{\alpha\beta}$$

- Solutions are components  $g_{\alpha\beta}$  of metric tensor g
- Equivalence Problem:  $g_1 \equiv g_2$ ?

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### Definition

A metric tensor field g on M is a mapping  $p \mapsto g_p$ , where  $p \in M$ and

$$g_p: T_pM \times T_pM \to \mathbb{R}$$

is a symmetric, non-degenerate, bilinear form on the tangent space to p on M.

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<b>G</b>	iven a basis $\{X_{\alpha}(p)\}$ o	$f T_p M, g(p)$	has components	
		, ,		
	$g(p)_{\alpha\beta}$	$= g_p(X_{\alpha}(p), \lambda)$	$X_{\beta}(p)$	
	<b>0</b> (174p		ρ(, , ,	
📕 In	a neighborhood $U$ o	f a point,		
		σ		
		δαβ		
٦r	• $C^{\infty}$ functions of coordinates of the coordina	dinates		
ai		unates		
Definit	ion			
A space	c <b>etime</b> is a 4-dimensior	nal manifold w	ith metric tensor	
having	signature (3 1) i.e. a	Lorentzian sig	mature	
naving	Signature (0, 1), i.e. a			

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	A Little Dackground	The Idea	
We	refer to $g_{\alpha\beta}$ as the	metric.	

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The metric defines a unique connection

$$\Gamma_{\alpha \ \beta}^{\ \lambda} = \frac{g^{\lambda\mu}}{2} (\partial_{\alpha}g_{\beta\mu} + \partial_{\beta}g_{\alpha\mu} - \partial_{\mu}g_{\alpha\beta})$$

$$= \Gamma_{\beta \alpha}^{\lambda}$$

•  $\Gamma^{\lambda}_{\alpha \ \beta}$  allows us to define a derivative, called the **covariant** derivative

...which returns a new tensor when applied to a given tensor.

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Introduction	A Little Background	The fuea		Conclusion
On	a 4-dimensional manif	old <i>M</i>		
	$X: C^{\infty}(A)$	$(M,\mathbb{R}) o C^{\circ}$	$^\infty(M,\mathbb{R})$	
	$X = X^1(\mathbf{x})\partial_{x^1} + X^2$	$(\mathbf{x})\partial_{x^2} + X$	$^{3}(\mathbf{x})\partial_{x^{3}}+X^{4}(\mathbf{x})\partial_{x^{4}}$	

• The components of the covariant derivative of X are

$$X^{\nu}_{;\alpha} = \partial_{\alpha}X^{\nu} + \Gamma^{\nu}_{\alpha\beta}X^{\beta}$$

 When performing repeated covariant differentiation and taking the difference, we get

$$X^{\nu}_{;\alpha;\beta} - X^{\nu}_{;\beta;\alpha} = (\partial_{\beta} \Gamma^{\nu}_{\mu\ \alpha} - \partial_{\alpha} \Gamma^{\nu}_{\mu\ \beta} + \Gamma^{\nu}_{\lambda\ \beta} \Gamma^{\lambda}_{\mu\ \alpha} - \Gamma^{\nu}_{\lambda\ \alpha} \Gamma^{\lambda}_{\mu\ \beta}) X^{\mu}$$

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# Define

$$R_{\mu \ \alpha\beta}^{\ \nu} = \partial_{\beta} \Gamma_{\mu \ \alpha}^{\ \nu} - \partial_{\alpha} \Gamma_{\mu \ \beta}^{\ \nu} + \Gamma_{\lambda \ \beta}^{\ \nu} \Gamma_{\mu \ \alpha}^{\ \lambda} - \Gamma_{\lambda \ \alpha}^{\ \nu} \Gamma_{\mu \ \beta}^{\ \lambda}$$

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# ■ $R^{\nu}_{\mu \ \alpha\beta}$ are the components of the **Riemann curvature** tensor!

• 
$$R_{\mu \ \alpha\beta}^{\ \nu} \neq 0$$
 indicates **curvature**.

If all 
$$R_{\mu \ \alpha\beta}^{\ \nu} = 0$$
, the spacetime is **flat**.

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The Idea



- The  $R_{\mu \ \alpha\beta}^{\ \nu}$  change when changing coordinates,
- BUT certain contractions and products and combinations of its contractions do NOT!

### Called Riemann invariants

### Example

From 
$$R_{\mu\alpha} := R_{\mu}^{\ \nu}{}_{\alpha\nu}$$
 is created the Ricci scalar  $R := g^{\mu\alpha}R_{\mu\alpha}$ 

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 From J. Carminati and R. Mclenaghan's paper "Algebraic invariants of the Riemann tensor in a four-dimensional Lorentzian space"

$$\begin{split} R &:= g^{ad}g^{bc}R_{abcd}, \\ r_1 &:= \Phi_{AB\dot{A}\dot{B}} \Phi^{AB\dot{A}\dot{B}} = \frac{1}{4}S_a{}^bS_b{}^a, \\ r_2 &:= \Phi_{AB\dot{A}\dot{B}} \Phi^B{}_C{}^b{}_C \Phi^{CA\dot{C}\dot{A}} = -\frac{1}{8}S_a{}^bS_b{}^cS_c{}^a, \\ r_3 &:= \Phi_{AB\dot{A}\dot{B}} \Phi^B{}_C{}^b{}_C \Phi^C{}_D{}^c{}_D \Phi^{DA\dot{D}\dot{A}} = \frac{1}{16}S_a{}^bS_b{}^cS_c{}^dS_d{}^a, \\ w_1 &:= \Psi_{ABCD} \Psi^{ABCD} = \frac{1}{4}\overline{C}_{abcd}\overline{C}{}^{abcd}, \\ w_2 &:= \Psi_{ABCD} \Psi^{CD}{}_{EF} \Psi^{EFAB} = -\frac{1}{8}\overline{C}_{abcd}\overline{C}{}^{cd}{}_{ef}\overline{C}{}^{efab}, \end{split}$$

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cor	ntinued			
	$m_1 := \Psi_{ABCD} \Phi^{CL}$	$P_{CD} \Phi^{ABCD} = \frac{1}{4}\overline{C}$	$S_{acdb}S^{cd}S^{ab}$ ,	
	$m_2 := \Psi_{ABCD} \Phi^{CL}$	$\Phi_{CD}\Psi^{AB}{}_{EF}\Phi^{EFC}$	Ď	
	$= \frac{1}{4}\overline{C}_{acdb}S^{cd}$	$\bar{Z}^{a}{}_{ef}{}^{b}S^{ef}$ ,		
	$m_3 := \Psi^{AB}{}_{CD} \Phi^{CL}$	$\Phi_{AB}\overline{\Psi}^{AB}{}_{CD}\Phi_{AB}{}^{C}$	ĊĎ	
	$= \frac{1}{4} \overline{C}_{acdb} S^{cd}$	$\overset{+}{C}{}^{a}{}_{ef}{}^{b}S^{ef}$ ,		

$$m_4 := \Psi_A^{\ B}{}_{DE} \Phi^{DE}{}_{A}^{\ B} \overline{\Psi}_{\dot{B}}^{\ C}{}_{\dot{DE}} \Phi_B^{\ C\dot{DE}} \Phi_C^{\ A}{}_{\dot{C}}^{\ A}$$

$$= -\frac{1}{8}\overline{C}_{acdb}S^{cd}C^{+b}_{efg}S^{ef}S^{ag},$$

$$m_5 := \Psi^{AB}{}_{CD} \Psi^{CD}{}_{EF} \Phi^{EF}{}_{EF} \overline{\Psi}^{EF}{}_{CD} \Phi_{AB}{}^{CD}$$

$$= \frac{1}{4} \overline{C}_{aefb} \overline{C}^{a}{}_{cd}{}^{b} S^{cd} \overline{C}^{g}{}_{ef}{}^{h} S^{cf},$$

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THE KEY: Obvious when two metrics are NOT equivalent

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- In Einstein Spaces, A. Z. Petrov gave a "complete" classification of spacetimes with symmetry according to group action.
- In Exact Solutions of Einstein's Field
   Equations, Stephani et al compiled many famous spacetimes.
- The Gödel metric SHOULD be in Petrov's classification, as it's a homogeneous space with isometry dimension 5.

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	A Little Background	The Idea	The Example	Conclusion
The Gö	del Metric			
■ g = a <sup>2</sup> €	$= -a^2 dt dt - a^2 e^x dt$ $e^x dz dt - 1/2 a^2 e^{2x} dz$	dz + a² dx dx Iz dz	$+a^2 dy dy -$	
■ g =	$= 1/4 a^2 dr dr + 1/4 a^2$	$a^2 dr ds + 3/4 a$	$a^2 dr du +$	

$$g = \frac{1}{4}a^{2} dr dr + \frac{1}{4}a^{2} dr ds + \frac{3}{4}a^{2} dr du + \frac{1}{4}a^{2} (2e^{s/2+u/2+v/2+r/2} + 3) dr dv + \frac{1}{4}a^{2} ds dr + \frac{1}{4}a^{2} ds ds - \frac{1}{4}a^{2} ds du - \frac{1}{4}a^{2} (1 + 2e^{s/2+u/2+v/2+r/2}) ds dv + \frac{3}{4}a^{2} du dr - \frac{1}{4}a^{2} du ds + \frac{1}{4}a^{2} du du - \frac{1}{4}a^{2} (2e^{s/2+u/2+v/2+r/2} - 1) du dv + \frac{1}{4}a^{2} (2e^{s/2+u/2+v/2+r/2} + 3) dv dr - \frac{1}{4}a^{2} (1 + 2e^{s/2+u/2+v/2+r/2}) dv ds - \frac{1}{4}a^{2} (2e^{s/2+u/2+v/2+r/2} - \frac{1}{4}) dv du - \frac{1}{4}a^{2} (4e^{s/2+u/2+v/2+r/2} - 1 + 2e^{s+u+v+r}) dv dv$$

• Using 
$$t = \frac{-r+s+u+v}{2}, x = \frac{r+s+u+v}{2}, y = \frac{r-s+u+v}{2}, z = v$$

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• Here Gödel is again:

 $g = -a^2 dt dt - a^2 e^x dt dz + a^2 dx dx + a^2 dy dy - a^2 e^x dz dt - 1/2 a^2 e^{2x} dz dz$ 

• It's been shown that (33.17) with  $\epsilon = -1$  is the ONLY metric in Petrov with equivalent Killing vectors:

 $\tilde{g} = 2 dx^1 dx^4 + k_{22} e^{-2x3} dx^2 dx^2 - e^{-x3} dx^2 dx^4 + k_{22} dx^3 dx^3 + 2 dx^4 dx^1 - e^{-x3} dx^4 dx^2$ 

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Compute their respective Riemann invariants:

• Gödel, we have R = -1 and



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Petrov's (33.17), 
$$\epsilon = -1$$
, we have  $R = \frac{-2}{k_{22}}$  and

$$r_{1} = \frac{1}{4k_{22}^{2}} \qquad m_{1} = -\frac{1}{12k_{22}^{3}}$$

$$r_{2} = 0 \qquad m_{2} = \frac{1}{36k_{22}^{4}}$$

$$r_{3} = \frac{1}{64k_{22}^{4}} \qquad m_{3} = \frac{1}{36k_{22}^{4}}$$

$$m_{1} = \frac{1}{6k_{22}^{2}} \qquad m_{4} = 0$$

$$m_{2} = \frac{1}{36k_{22}^{3}} \qquad m_{5} = -\frac{1}{108k_{22}^{5}}$$

• These are different metrics. Petrov incorrectly *normalized*.

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Introduction	A Little Background	I he Idea	The Example	Conclusion

- A corrected and complete classification is in a database.
- A classifier has been coded in Maple that makes comparisons against database.
- Many metrics from Stephani's compilation have been identified in Petrov.
- Software has been written to help find explicit equivalences.
- Using that software, all homogeneous spaces of dimension 3-5 have explicit equivalences to metrics in Petrov.

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- Many thanks to my advisor Dr. Ian Anderson and the Differential Geometry Group at USU!
- Thank You!

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