3-20-2015

The Riemann curvature tensor, its invariants, and their use in the classification of spacetimes

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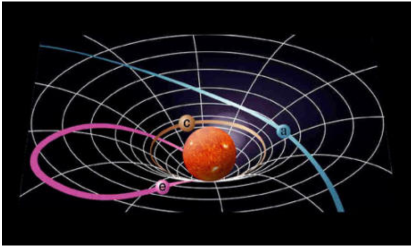
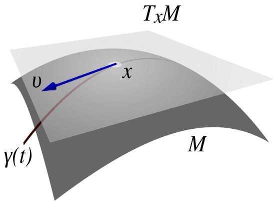


- "Spacetime tells matter how to move; matter tells spacetime how to curve" - John Wheeler
- The Einstein field equations:

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \Lambda g_{\alpha\beta} = \kappa T_{\alpha\beta}$$

- Solutions are components $g_{\alpha\beta}$ of metric tensor g
- Equivalence Problem: $g_1 \equiv g_2$?

- Let M be an n-dimensional differentiable manifold.



Definition

A **metric tensor field** g on M is a mapping $p \mapsto g_p$, where $p \in M$ and

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

is a symmetric, non-degenerate, bilinear form on the tangent space to p on M.

- Given a basis $\{X_\alpha(p)\}$ of T_pM, $g(p)$ has components

$$g(p)_{\alpha\beta} = g_p(X_\alpha(p), X_\beta(p))$$

- **In a neighborhood** U **of a point**,

$$g_{\alpha\beta}$$

are C^∞ functions of coordinates

Definition

A **spacetime** is a 4-dimensional manifold with metric tensor having signature $(3, 1)$, i.e. a Lorentzian signature.

- We refer to $g_{\alpha\beta}$ as the metric.

- The metric defines a unique **connection**

$$\Gamma_{\alpha\ \beta}^{\ \lambda} = \frac{g^{\lambda\mu}}{2}(\partial_\alpha g_{\beta\mu} + \partial_\beta g_{\alpha\mu} - \partial_\mu g_{\alpha\beta})$$

$$= \Gamma_{\beta\ \alpha}^{\ \lambda}$$

- $\Gamma_{\alpha\ \beta}^{\ \lambda}$ allows us to define a derivative, called the **covariant derivative**

...which returns a new tensor when applied to a given tensor.

- On a 4-dimensional manifold M

$$X : C^{\infty}(M, \mathbb{R}) \to C^{\infty}(M, \mathbb{R})$$

$$X = X^1(\mathbf{x})\partial_{x^1} + X^2(\mathbf{x})\partial_{x^2} + X^3(\mathbf{x})\partial_{x^3} + X^4(\mathbf{x})\partial_{x^4}$$

- The components of the covariant derivative of X are

$$X^{\nu}{}_{;\alpha} = \partial_{\alpha} X^{\nu} + \Gamma_{\alpha}{}^{\nu}{}_{\beta} X^{\beta}$$

- When performing repeated covariant differentiation and taking the difference, we get

$$X^{\nu}{}_{;\alpha;\beta} - X^{\nu}{}_{;\beta;\alpha} = (\partial_{\beta}\Gamma_{\mu}{}^{\nu}{}_{\alpha} - \partial_{\alpha}\Gamma_{\mu}{}^{\nu}{}_{\beta} + \Gamma_{\lambda}{}^{\nu}{}_{\beta}\Gamma_{\mu}{}^{\lambda}{}_{\alpha} - \Gamma_{\lambda}{}^{\nu}{}_{\alpha}\Gamma_{\mu}{}^{\lambda}{}_{\beta})X^{\mu}$$

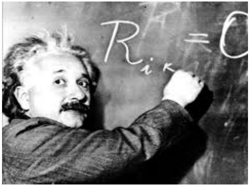
- Define

$$R_{\mu\ \alpha\beta}^{\ \nu} = \partial_\beta \Gamma_{\mu\ \alpha}^{\ \nu} - \partial_\alpha \Gamma_{\mu\ \beta}^{\ \nu} + \Gamma_{\lambda\ \beta}^{\ \nu} \Gamma_{\mu\ \alpha}^{\ \lambda} - \Gamma_{\lambda\ \alpha}^{\ \nu} \Gamma_{\mu\ \beta}^{\ \lambda}$$

- $R_{\mu\ \alpha\beta}^{\ \nu}$ are the components of the **Riemann curvature tensor**!

- $R_{\mu\ \alpha\beta}^{\ \nu} \neq 0$ indicates **curvature**.

- If all $R_{\mu\ \alpha\beta}^{\ \nu} = 0$, the spacetime is **flat**.



- The $R_{\mu}{}^{\nu}{}_{\alpha\beta}$ change when changing coordinates,
- BUT certain contractions and products and combinations of its contractions do NOT!
- Called **Riemann invariants**

Example

From $R_{\mu\alpha} := R_{\mu}{}^{\nu}{}_{\alpha\nu}$ *is created the* **Ricci scalar** $R := g^{\mu\alpha}R_{\mu\alpha}$

- From J. Carminati and R. Mclenaghan's paper *"Algebraic invariants of the Riemann tensor in a four-dimensional Lorentzian space"*

$$R := g^{ad} g^{bc} R_{abcd},$$

$$r_1 := \Phi_{AB\dot{A}\dot{B}} \Phi^{AB\dot{A}\dot{B}} = \tfrac{1}{4} S_a{}^b S_b{}^a,$$

$$r_2 := \Phi_{AB\dot{A}\dot{B}} \Phi^B{}_C{}^{\dot{B}}{}_{\dot{C}} \Phi^{CA\dot{C}\dot{A}} = -\tfrac{1}{8} S_a{}^b S_b{}^c S_c{}^a,$$

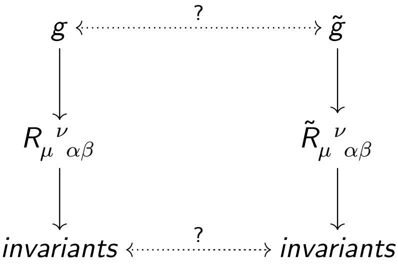
$$r_3 := \Phi_{AB\dot{A}\dot{B}} \Phi^B{}_C{}^{\dot{B}}{}_{\dot{C}} \Phi^C{}_D{}^{\dot{C}}{}_{\dot{D}} \Phi^{DA\dot{D}\dot{A}} = \tfrac{1}{16} S_a{}^b S_b{}^c S_c{}^d S_d{}^a,$$

$$w_1 := \Psi_{ABCD} \Psi^{ABCD} = \tfrac{1}{4} \overline{C}_{abcd} \overline{C}^{abcd},$$

$$w_2 := \Psi_{ABCD} \Psi^{CD}{}_{EF} \Psi^{EFAB} = -\tfrac{1}{8} \overline{C}_{abcd} \overline{C}^{cd}{}_{ef} \overline{C}^{efab},$$

- continued...

$$
\begin{aligned}
m_1 &:= \Psi_{ABCD}\Phi^{CD}{}_{\dot C\dot D}\Phi^{AB\dot C\dot D} = \tfrac{1}{4}\bar{C}_{acdb}S^{cd}S^{ab},\\
m_2 &:= \Psi_{ABCD}\Phi^{CD}{}_{\dot C\dot D}\Psi^{AB}{}_{EF}\Phi^{EF\dot C\dot D}\\
&= \tfrac{1}{4}\bar{C}_{acdb}S^{cd}\bar{C}^{a}{}_{ef}{}^{b}S^{ef},\\
m_3 &:= \Psi^{AB}{}_{CD}\Phi^{CD}{}_{\dot A\dot B}\bar{\Psi}^{\dot A\dot B}{}_{\dot C\dot D}\Phi_{AB}{}^{\dot C\dot D}\\
&= \tfrac{1}{4}\bar{C}_{acdb}S^{cd}\overset{+}{C}{}^{a}{}_{ef}{}^{b}S^{ef},\\
m_4 &:= \Psi_{A}{}^{B}{}_{DE}\Phi^{DE}{}_{\dot A}{}^{\dot B}\bar{\Psi}_{\dot B}{}^{\dot C}{}_{\dot D\dot E}\Phi_{B}{}^{C\dot D\dot E}\Phi_{C}{}^{A}{}_{\dot C}{}^{\dot A}\\
&= -\tfrac{1}{8}\bar{C}_{acdb}S^{cd}\overset{+}{C}{}^{b}{}_{efg}S^{ef}S^{ag},\\
m_5 &:= \Psi^{AB}{}_{CD}\Psi^{CD}{}_{EF}\Phi^{EF}{}_{\dot E\dot F}\bar{\Psi}^{\dot E\dot F}{}_{\dot C\dot D}\Phi_{AB}{}^{\dot C\dot D}\\
&= \tfrac{1}{4}\bar{C}_{aefb}\bar{C}^{a}{}_{cd}{}^{b}S^{cd}\overset{+}{C}{}^{g}{}_{ef}{}^{h}S^{ef},
\end{aligned}
$$



- THE KEY: Obvious when two metrics are NOT equivalent

- In **Einstein Spaces**, A. Z. Petrov gave a **"complete"** classification of spacetimes with symmetry according to group action.

- In **Exact Solutions of Einstein's Field Equations**, Stephani et al compiled many famous spacetimes.

- The Gödel metric SHOULD be in Petrov's classification, as it's a homogeneous space with isometry dimension 5.

The Gödel Metric

- $g = -a^2\, dt\, dt - a^2 \mathrm{e}^x\, dt\, dz + a^2\, dx\, dx + a^2\, dy\, dy - a^2 \mathrm{e}^x\, dz\, dt - 1/2\, a^2 \mathrm{e}^{2\,x}\, dz\, dz$

- $g = 1/4\, a^2\, dr\, dr + 1/4\, a^2\, dr\, ds + 3/4\, a^2\, dr\, du + 1/4\, a^2 (2\, \mathrm{e}^{s/2+u/2+v/2+r/2} + 3)\, dr\, dv + 1/4\, a^2\, ds\, dr + 1/4\, a^2\, ds\, ds - 1/4\, a^2\, ds\, du - 1/4\, a^2 (1 + 2\, \mathrm{e}^{s/2+u/2+v/2+r/2})\, ds\, dv + 3/4\, a^2\, du\, dr - 1/4\, a^2\, du\, ds + 1/4\, a^2\, du\, du - 1/4\, a^2 (2\, \mathrm{e}^{s/2+u/2+v/2+r/2} - 1)\, du\, dv + 1/4\, a^2 (2\, \mathrm{e}^{s/2+u/2+v/2+r/2} + 3)\, dv\, dr - 1/4\, a^2 (1 + 2\, \mathrm{e}^{s/2+u/2+v/2+r/2})\, dv\, ds - 1/4\, a^2 (2\, \mathrm{e}^{s/2+u/2+v/2+r/2} - 1)\, dv\, du - 1/4\, a^2 (4\, \mathrm{e}^{s/2+u/2+v/2+r/2} - 1 + 2\, \mathrm{e}^{s+u+v+r})\, dv\, dv$

- Using $t = \frac{-r+s+u+v}{2}, x = \frac{r+s+u+v}{2}, y = \frac{r-s+u+v}{2}, z = v$

- Here Gödel is again:

$$g = -a^2\,dt\,dt - a^2 e^x\,dt\,dz + a^2\,dx\,dx + a^2\,dy\,dy - a^2 e^x\,dz\,dt - 1/2\,a^2 e^{2\,x}\,dz\,dz$$

- It's been shown that (33.17) with $\epsilon = -1$ is the ONLY metric in Petrov with equivalent Killing vectors:

$$\tilde{g} = 2\,dx^1\,dx^4 + k_{22}\,e^{-2\,x3}\,dx^2\,dx^2 - e^{-x3}\,dx^2\,dx^4 + k_{22}\,dx^3\,dx^3 + 2\,dx^4\,dx^1 - e^{-x3}\,dx^4\,dx^2$$

Compute their respective Riemann invariants:

- Gödel, we have $R = -1$ and

$$r_1 = \frac{3}{16a^4}$$
$$r_2 = \frac{3}{64a^6}$$
$$r_3 = \frac{21}{1024a^8}$$
$$w_1 = \frac{1}{6a^6}$$
$$w_2 = \frac{1}{36a^6}$$

$$m_1 = 0$$
$$m_2 = \frac{1}{96a^8}$$
$$m_3 = \frac{1}{96a^8}$$
$$m_4 = -\frac{1}{768a^{10}}$$
$$m_5 = \frac{1}{576a^{10}}$$

- Petrov's (33.17), $\epsilon = -1$, we have $R = \frac{-2}{k_{22}}$ and

$$r_1 = \frac{1}{4k_{22}^2} \qquad m_1 = -\frac{1}{12k_{22}^3}$$
$$r_2 = 0 \qquad m_2 = \frac{1}{36k_{22}^4}$$
$$r_3 = \frac{1}{64k_{22}^4} \qquad m_3 = \frac{1}{36k_{22}^4}$$
$$w_1 = \frac{1}{6k_{22}^2} \qquad m_4 = 0$$
$$w_2 = \frac{1}{36k_{22}^3} \qquad m_5 = -\frac{1}{108k_{22}^5}$$

- These are different metrics. Petrov incorrectly *normalized*.

- A corrected and complete classification is in a database.
- A classifier has been coded in Maple that makes comparisons against database.
- Many metrics from Stephani's compilation have been identified in Petrov.
- Software has been written to help find explicit equivalences.
- Using that software, all homogeneous spaces of dimension 3-5 have explicit equivalences to metrics in Petrov.

- Many thanks to my advisor Dr. Ian Anderson and the Differential Geometry Group at USU!
- Thank You!