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07 General Solution of the One-Dimensional Wave Equation

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7. General Solution of the One-Dimensional Wave Equation.

We will now find the "general solution" to the one-dimensional wave equation (5.11). What this means is that we will find a formula involving some "data" — some arbitrary functions — which provides every possible solution to the wave equation.* We can find the general solution of the (one-dimensional) wave equation as follows.

Make a change of variables $t = t(s, u)$, $x = x(s, u)$:

$$t = \frac{(u - s)}{2v},$$
$$x = \frac{(u + s)}{2},$$
(7.1)

with inverse $u = u(x, t)$, $s = s(x, t)$:

$$u = x + vt,$$
$$s = x - vt.$$
(7.2)

(*Note: this change of variables is reminiscent of the change of variables used for our problem involving two coupled oscillators. This is purely coincidence.*)

Exercise: Check that the formulas for u and s represent the inverse of the transformation.

The function $q(x, t)$ can now be viewed as a new function $\tilde{q}(u, s)$, which is obtained by substitution:

$$\tilde{q}(u, s) = q(x(s, u), t(s, u)).$$
(7.3)

Likewise, given $\tilde{q}(u, s)$ we can reconstruct $q(x, t)$ via

$$q(x, t) = \tilde{q}(u(x, t), s(x, t)).$$
(7.4)

For example, if $q(x, t) = xt$, then

$$\tilde{q}(u, s) = \left(\frac{(u + s)}{2} \right) \left(\frac{(u - s)}{2v} \right) = \frac{1}{4v}(u^2 - s^2)$$
(7.5)

As an exercise you can take this $\tilde{q}(u, s)$ and reconstruct $q(x, t)$ via

$$q(x, t) = \tilde{q}(u(x, t), s(x, t)) = xt.$$
(7.6)

A note on notation:

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* We will tacitly assume that all solutions are continuous and have continuous derivatives to sufficient order.

Using different symbols q and $\tilde{q}$ for the displacement function in different coordinates is not that common in physics texts and could be viewed as a somewhat fanatical devotion to notational consistency. The reason for using different symbols is that q and $\tilde{q}$ are in general different functions, mathematically speaking, as the example above demonstrates.* On the other hand, one can reasonably take the point of view that we have a single physical quantity — the displacement q — which assigns numbers to each location in space and instant of time, and we can express this same quantity in different coordinate systems. From this last more physical point of view it is reasonable to use the same symbol (q) for the displacement whether expressed in (t,x) or (u,v) coordinates. Because this last point of view is quite common, we will eventually adopt it, *i.e.,* we will eventually use the symbol q to denote wave displacement in any coordinate system. But for the purposes of explaining the chain rule calculation below, it is more instructive and less confusing (if more cumbersome) to keep clear which function is which.

The derivatives of $q(x,t)$ are related to derivatives of $\tilde{q}(u,s)$ by the chain rule of calculus. So, for example, we have (*exercise*)

$$
\begin{aligned}
\frac{\partial q}{\partial t} &= \frac{\partial \tilde{q}}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial \tilde{q}}{\partial s}\frac{\partial s}{\partial t} \\
&= v\left(\frac{\partial \tilde{q}}{\partial u} - \frac{\partial \tilde{q}}{\partial s}\right),
\end{aligned}
\tag{7.7}
$$

and

$$
\frac{\partial^2 q}{\partial t^2} = v^2\left(\frac{\partial^2 \tilde{q}}{\partial u^2} - 2\frac{\partial^2 \tilde{q}}{\partial u \partial s} + \frac{\partial^2 \tilde{q}}{\partial s^2}\right),
\tag{7.8}
$$

where we used (7.2) and its consequence:

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= v \\
\frac{\partial s}{\partial t} &= -v,
\end{aligned}
$$

Similarly, we have

$$
\frac{\partial^2 q}{\partial x^2} = \left(\frac{\partial^2 \tilde{q}}{\partial u^2} + 2\frac{\partial^2 \tilde{q}}{\partial u \partial s} + \frac{\partial^2 \tilde{q}}{\partial s^2}\right).
\tag{7.9}
$$

It is important to keep in mind when manipulating these chain rule formulas that equality between functions, *e.g.,* as in (7.7), holds provided we express the functions in terms of the *same* coordinates using the relations (7.1)–(7.2).

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* For a more blatant example, consider the familiar exponential function $\exp(u)$. Under the change of variables $u = \ln(\sin(v))$ we get $\exp(u(v)) = \sin(v)$. We certainly would not want to retain the original notation (exp) in this case!

Take the results (7.7)–(7.9) and substitute them into the wave equation (5.11) to get the corresponding equation for $\tilde{q}(s, u)$. You will find that the wave equation (5.11) then takes the very simple form (*exercise*)

$$\frac{\partial^2 \tilde{q}}{\partial u \partial s} = 0. \tag{7.10}$$

This equation for $\tilde{q}(u, s)$ is completely equivalent to the original wave equation (5.11) for $q(x, t)$. What this means is that if you have a solution $q(x, t)$ to (5.11) then it defines, via the change of independent variables $(x, t) \to (u, s)$, a solution to (7.10). Conversely, if you have a solution to (7.10) then it defines a solution (via the inverse coordinate transformation) to (5.11).

We can easily solve equation (7.10). It says that $\frac{\partial \tilde{q}}{\partial u}$ is independent of s, *i.e.*,

$$\frac{\partial \tilde{q}}{\partial u} = h(u), \tag{7.11}$$

where $h(u)$ is any function of u. You can view this result as coming from integrating an equation that says the derivative with respect to s of $\frac{\partial \tilde{q}}{\partial u}$ is zero, in which case $h(u)$ is the "integration constant". Equation (7.11) can be viewed as a first-order differential equation for $\tilde{q}$ with a given function $h(u)$ and is easily integrated. Hold s fixed and integrate both sides of the equation with respect to u to find

$$\tilde{q}(s, u) = \int du\, h(u) + g(s).$$

Here $g(s)$ is an arbitrary function. Since the function $h(u)$ was arbitrary, the integral of $h(u)$ is just some other arbitrary function of u; call it $f(u)$. The solution to the wave equation in terms of $\tilde{q}(s, u)$ is thus of the form:

$$\tilde{q}(s, u) = f(u) + g(s), \tag{7.12}$$

where f and g are any functions of one variable. You can easily check that this form for $\tilde{q}(s, u)$ does indeed solve the wave equation expressed in the form (7.10) (*exercise*). Thus $\tilde{q}(s, u)$ solves (7.10) if and only if it takes the form (7.12).

We can now go back to our original time and space coordinates to see that (*exercise*)

$$q(x, t) = f(x + vt) + g(x - vt) \tag{7.13}$$

is the solution to the wave equation expressed in the original coordinates. Because of the equivalence of (5.11) and (7.10) (via the coordinate transformation), and since we have found a formula for all solutions to (7.10), it follows that (7.13) is the general solution to (5.11). We have thus completely solved (or "integrated") the wave equation in one spatial dimension.

So, to solve the wave equation we only need pick a couple of functions of one variable, call them $f(z)$ and $g(z)$. Aside from requiring them to be suitably differentiable (so we can plug them into the wave equation!) they can be chosen to be any functions you like. We then set $z = x + vt$ in $f(z)$ and $z = x - vt$ in $g(z)$ and add the results together to get a solution to the wave equation. Note that we can always add a constant to f and subtract that same constant from g without changing the form of the solution. Thus f and g, while convenient for specifying a solution, are slightly redundant.

Let us go back and see how our elementary solutions in §6 fit in with our general form for the solution (7.13). The traveling wave solution (6.7) is easy to check; it is a solution in which (*exercise*)

$$f(z) = 0, \qquad g(z) = A\cos(\frac{2\pi}{\lambda}z).$$ (7.14)

We see that g leads to "right-moving" solutions, *i.e.*, the cosine wave profile $g(z)$ moves toward increasing x values with speed v (*exercise*). Likewise, f leads to "left-moving" solutions. The standing wave solution (6.4) is a superposition of a left-moving and right-moving sinusoidal traveling wave solution to (5.11). To obtain this solution we set (*exercise*)

$$f(z) = g(z) = \frac{A}{2}\sin(\frac{2\pi}{\lambda}z).$$ (7.15)

Evidently, given (7.13), every solution to the one-dimensional wave equation can be viewed as a superposition of a left moving and right moving wave profile. These profiles are determined by the choice of the functions f and g. Each of these component wave profiles move to the left and right, respectively, without changing their shape. The actual wave of interest is, of course, obtained by superposing the displacements defined by the left-moving and right-moving wave profiles.

We have obtained all solutions to the wave equation in one dimension, but not all of these solutions need be appropriate for a given physical situation. This is because one will typically have to impose boundary conditions. Normally, these boundary conditions will involve fixing the value of $q(x, t)$ at the endpoints of the allowed range of x, and this implies restrictions on the functions f and g appearing in (7.13). For fixed endpoints,

$$q(0, t) = 0 = q(L, t),$$ (7.16)

we have for all z (*exercise*)

$$f(z) + g(-z) = 0, \quad g(z) = -f(2L - z).$$ (7.17)

If we impose periodic boundary conditions on the solution and its first derivatives,

$$q(0, t) = q(L, t), \quad dq(0, t) = dq(L, t)$$ (7.18)

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we have (*exercise*)

$$f(z + L) = f(z) + c, \quad g(z + L) = g(z) - c, \tag{7.19}$$

where c is any constant.

7.1 The Initial Value Formulation

There are evidently quite a few solutions to the wave equation in one dimension. Roughly speaking, there are as many solutions as there are pairs of functions of one variable! Why do we get so many solutions? One way to answer this question is to recall the oscillator model from which the wave equation was derived. Physically, we expect that the motion for N particles is determined by the initial positions and velocities of the particles. Consequently, a viable mathematical description of the particles' motion must be such that the solutions are uniquely determined once we have specified these initial conditions. This means that there must be $2N$ free parameters in the general solution which can be adjusted to meet the $2N$ initial conditions. The number N is the number of "degrees of freedom" in the one-dimensional chain of oscillators. In the continuum limit of our chain of oscillators we are taking a limit as $N \to \infty$. One sometimes says that the continuum description has an infinite number of degrees of freedom – essentially one degree of freedom for each point on the (one-dimensional) medium of vibration.

In the continuum limit we naturally expect that the initial conditions will be the initial displacement at each x, denoted by $q(x, t = 0)$, and the initial rate of displacement at each x, denoted by $\frac{\partial q}{\partial t}(x, t = 0)$. Further, we expect to be able to choose these *initial data* any way we like*, and that the solution to the wave equation will be uniquely determined in terms of these data. This is indeed the case, which we shall demonstrate below. The gist of the demonstration is that specification of the initial conditions corresponds mathematically to picking two functions of one variable. The general solution depends upon two functions of one variable, and these functions can be determined by the initial conditions.

The above discussion concerning initial data for the wave equation can be made more explicit and precise as follows. Let us call the initial displacement profile $a(x)$, *i.e.*,

$$q(x, t = 0) = a(x), \tag{7.20}$$

and call the initial velocity profile $b(x)$†:

$$\frac{\partial q}{\partial t}(x, t = 0) = b(x). \tag{7.21}$$

---

* Up to suitable requirements on the continuity of the initial data functions and their derivatives. And boundary conditions, if any, should be imposed.

† Here

$$\frac{\partial q}{\partial t}(x, t = 0) \equiv \left(\frac{\partial q}{\partial t} \right)_{t=0}.$$

We imagine that the functions $a(x)$ and $b(x)$ have been specified, and we demand that our solution (7.13) to the wave equation matches these initial conditions at $t = 0$. This means we must choose f and g such that (*exercise*)

$$f(x) + g(x) = a(x) \tag{7.22}$$

and

$$v[f'(x) - g'(x)] = b(x), \tag{7.23}$$

where the prime indicates differentiation with respect to the argument (x) of the function, that is, $f'(w) = \frac{df}{dw}$.

We can solve (7.23) by integrating:

$$f(x) - g(x) = \frac{1}{v} \int_{x_0}^{x} dy\, b(y), \tag{7.24}$$

where x_0 is any constant. It is now a simple matter to solve for f and g from (7.24) and (7.22) (*exercise*):

$$\begin{aligned}
f(x) &= \frac{1}{2}\left[a(x) + \frac{1}{v}\int_{x_0}^{x} dy\, b(y)\right] \\
g(x) &= \frac{1}{2}\left[a(x) - \frac{1}{v}\int_{x_0}^{x} dy\, b(y)\right].
\end{aligned} \tag{7.25}$$

The constant x_0 is not determined by the initial data, but x_0 does not actually contribute to $q(x,t)$. As we pointed out above, we can always add a constant to f and subtract that constant from g without altering the solution to the wave equation; this redundancy is equivalent to the arbitrariness of x_0 (*exercise*). We have now obtained the general solution to the wave equation expressed in terms of any initial conditions $a(x)$ and $b(x)$. It takes the form:

$$q(x,t) = \frac{1}{2}\left[a(x + vt) + a(x - vt) + \frac{1}{v}\int_{x-vt}^{x+vt} dy\, b(y)\right]. \tag{7.26}$$

Notice how the two integrals featuring in the solution for f and g in (7.25) have been combined. You should definitely make sure that you can obtain (7.26) from our previous formulas. You can check that this $q(x,t)$ given in (7.26) does indeed produce the desired initial data at $t = 0$. That this function solves the wave equation is easily seen since this function is of the form (7.13) (*exercise*). This formula is known as *d'Alembert's formula* for the solution to the wave equation in one spatial dimension. Notice that the parameter x_0 used to define f and g does indeed drop out of the formula for q.

It is perhaps worth noting that d'Alembert's formula (7.26) proves *existence and uniqueness* of solutions to the wave equation with prescribed initial conditions $(q(x,0), \frac{\partial q(x,0)}{\partial t})$. Existence is clear: we exhibited a formula for the general solution, namely, (7.26). Uniqueness is also clear: there is only one solution defined by the initial data. Note in particular

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that the arbitrary constant x_0 dropped out of the formula for q. If x_0 had survived in (7.26), then the solution would not have been uniquely determined by the initial data.

To summarize, we have shown that the solutions to the one-dimensional wave equation are uniquely determined by the choice of initial data — displacement and displacement velocity at each x — and that we can choose the initial data any way we like. These qualitative features will generalize to higher dimensions, but the explicit form of the general solution we have found is applicable only in one spatial dimension.

7.2 Gaussian Wave Packet

Let us look at a simple illustration of the initial value problem. Suppose we want to find a wave which has for its initial displacement

$$q(x,0) \equiv a(x) = A \exp\left(-\frac{x^2}{a_0^2}\right). \tag{7.27}$$

This function is called a *Gaussian*.* As an *exercise* you can show that it is peaked at $x = 0$, where it has its maximum value $q = A$, and falls to $1/e \approx 0.37$ of its maximum at $x = \pm a_0$. We thus say that the *height* of the Gaussian is A, and the *width* of the Gaussian is $2a_0$. We have specified the initial wave displacement profile; to completely specify the wave we should give its initial velocity as well. Let us suppose that the entire medium is initially at rest:

$$\frac{\partial q(x,0)}{\partial t} \equiv b(x) = 0. \tag{7.28}$$

This mathematical situation could be a model for a guitar string under tension which has been "plucked" into a Gaussian of height A and width $2a_0$ at $t = 0$, after which the string is released from rest. (Of course, when you pluck a guitar string the initial shape of the string is not really ever going to be a Gaussian. But you get the idea.) Of course, we are neglecting the boundary condition that the string displacement is zero at the bridges of the guitar. Indeed, the Gaussian function – like any exponential function – never vanishes (*exercise*). Our simple model is, however, reasonably viable provided the width of the Gaussian wave is much smaller than the length of the string, and provided we only consider wave propagation for times short enough so that reflective effects are not relevant. If that has too many caveats for you, think in terms of the surface of a body of water which has a Gaussian pulse in its height stimulated by some means.

In any case, the solution to the wave equation with the specified boundary conditions is easily obtained from our general formula (7.26); we have (*exercise*)

$$q(x,t) = \frac{A}{2}\left[\exp\{-\frac{(x+vt)^2}{a_0^2}\} + \exp\{-\frac{(x-vt)^2}{a_0^2}\}\right]. \tag{7.29}$$

---

* Newbs call the graph of this function the "bell shaped curve".

At each t, that is, if we take a photograph of the resulting wave, we find a superposition of Gaussian pulses, each of width $2a_0$ and height $A/2$ centered at $x = \pm vt$. In effect, the initial Gaussian pulse "splits" into two similar Gaussians with half the original height, which move off in opposite directions. For a guitar string the Gaussian waves reflect repeatedly from the ends and set up a standing wave.
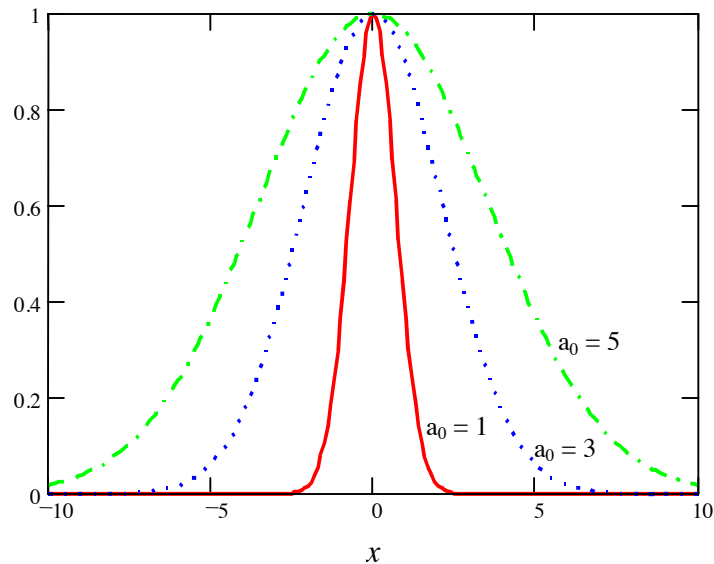

Figure 10. Gaussian function $\exp\left(-(x/a_0)^2\right)$ for three different values of the width parameter a_0.
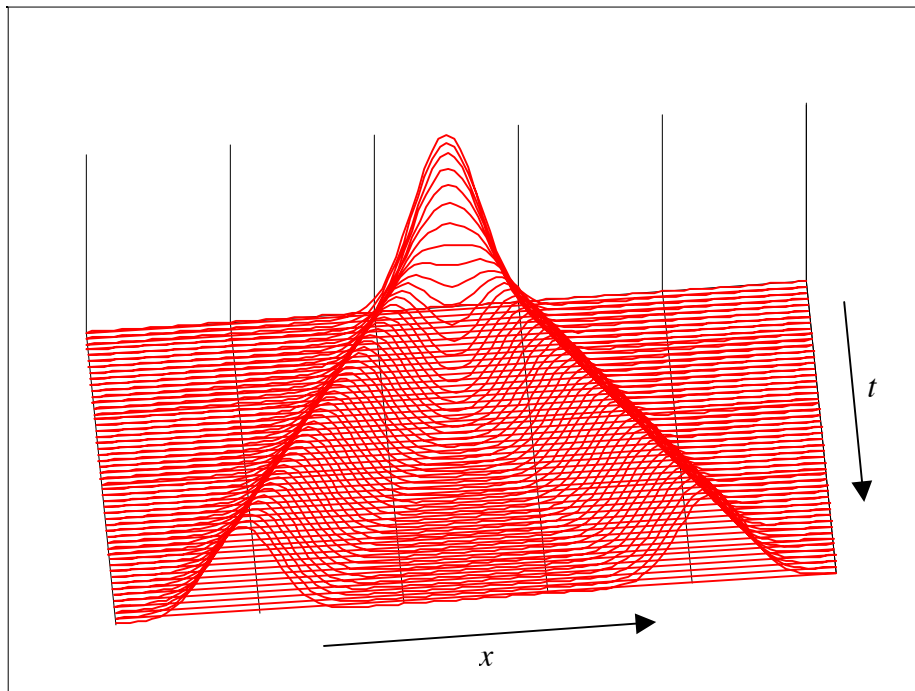
Figure 11. Solution to wave equation for initial Gaussian pulse displacement [$a(x)$] and zero initial velocity [$b(x) = 0$].

7.3 Linearity and superposition

Just as the oscillator equation(s) were linear and homogeneous, so is the wave equation. What this means is the following. If $q_1(x,t)$ and $q_2(x,t)$ are solutions of the wave equation, then

$$q_3(x,t) = c_1 q_1(x,t) + c_2 q_2(x,t)$$

is also a solution, where c_1 and c_2 are any two constants. We say that q_3 is obtained by a *linear superposition* of q_1 and q_2. Because solutions to the wave equation are completely

determined by their initial data it is natural to ask how the initial data for the superposition are related to the data for q_1 and q_2. As an exercise you can check that the initial data for q_3 are obtained by taking the same superposition of initial data for q_1 and q_2.

It is worth pointing out that this "superposition property" is a signal that the objects under consideration (solutions to the one-dimensional wave equation) form a *vector space* (see Appendix B for a review of vector spaces). Recall that a vector space is (i) a set of objects (the vectors), (ii) a set of "scalars" (real or complex numbers) which can be used via a commuting "scalar multiplication" to make new vectors from old vectors, (iii) a rule for addition of the vectors that is closed, commutative, associative, and distributive with respect to the scalar multiplication, and (iv) a zero vector (additive identity). In the present example, let us take the set to be all functions $q(x,t)$ which solve the wave equation. If $q(x,t)$ is a solution, then so is any constant multiple of $q(x,t)$, consequently scalar multiplication can be defined by simply multiplying $q(x,t)$ by numbers. As we have noted, the sum of any two solutions is another solution, so the rule for addition is the usual point-wise addition of functions. Finally, the function $q(x,t) = 0$ is a solution, *i.e.,* a member of the set, and plays the role of the zero vector. All of this mathematical finery may seem like overkill but, as you know, vector spaces have many nice properties (existence of a basis, *etc.*) and in more sophisticated applications it is good to know when the objects under consideration form a vector space.

Exercise: Show that the fixed endpoint and periodic boundary conditions can be imposed without destroying the superposition (vector space) property.