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08 Fourier Analysis

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8. Fourier analysis.

We now would like to show that one can build any solution of the wave equation by superposing certain elementary solutions. The elementary solutions being referred to already appear in §6. These elementary solutions will form a very convenient “basis” for the vector space of solutions to the wave equation, just as the normal modes provided a basis for the space of solutions in the case of coupled oscillators. Indeed, as we shall see, the elementary solutions *are* the normal modes for wave propagation. The principal tools needed to understand this are provided by the methods of *Fourier analysis*, which is very useful in analyzing waves in any number of spatial dimensions. To begin, we will take a somewhat superficial tour of the key results of Fourier analysis. Then we’ll see how to use these results to better understand the solutions to the wave equation.

8.1 Fourier Series

Fourier series are a way of representing functions on an interval as a sum of sinusoidal functions, much in the same way as a vector in a vector space can be expanded in a basis. Although it is possible to proceed with more generality, let us restrict our attention to functions f on the interval $x \in [0, L]$ such that

$$f(0) = f(L) = 0. \quad (8.1)$$

(As you might guess, this discussion will be relevant when considering fixed endpoint boundary conditions of the form (5.12)). The main result we need is that, with appropriate technical assumptions about f ,[†] it is possible to express f as

$$f(x) = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} h_n \sin\left(\frac{n\pi x}{L}\right), \quad (8.2)$$

where h_n , $n = 1, 2, \dots$ are constants whose specification is equivalent to specification of f . This is usually called the *Fourier series* of f . (The factor in front of the sum is for later convenience.) Note that each term in the series involves the sine function $\sin\left(\frac{n\pi x}{L}\right)$, which vanishes at $x = 0, L$. This guarantees that the conditions (8.1) are satisfied.

Equation (8.2) is an infinite series and is to be interpreted as follows. There exists an (infinite) sequence of numbers h_n , $n = 1, 2, 3, \dots$, such that the sequence of partial sums,

$$F_N(x) = \sqrt{\frac{2}{L}} \sum_{n=1}^N h_n \sin\left(\frac{n\pi x}{L}\right) \quad (8.3)$$

[†] For example, it is enough to require f to be square-integrable:

$$\int_0^L dx (f(x))^2 < \infty.$$

converges to f ,

$$\lim_{N \rightarrow \infty} F_N \longrightarrow f. \quad (8.4)$$

Here “convergence” (the arrow) means that

$$\lim_{N \rightarrow \infty} \int_0^L dx (F_N - f)^2 = 0. \quad (8.5)$$

Notice that this is not quite the same as saying that at each x that values of $F_N(x)$ approach those of $f(x)$. For example, the limit of F_N can differ from f at isolated points.

The constants h_m , $m = 1, 2, 3, \dots$, are determined by the function f via the integrals

$$h_m = \sqrt{\frac{2}{L}} \int_0^L dx f(x) \sin\left(\frac{m\pi x}{L}\right). \quad (8.6)$$

As a nice *exercise* you can verify the identity*

$$\frac{2}{L} \int_0^L dx \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) = \delta_{nm}, \quad n, m \text{ integers.} \quad (8.7)$$

In case you haven’t encountered it before, δ_{nm} is the *Kronecker delta*, defined by

$$\delta_{nm} = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}. \quad (8.8)$$

The identity (8.7) formally produces (8.6) once you integrate both sides of the equation (8.2) against the sine function and interchange the order of integration and summation:

$$\begin{aligned} \sqrt{\frac{2}{L}} \int_0^L dx \sin\left(\frac{m\pi x}{L}\right) f(x) &= \frac{2}{L} \int_0^L dx \sin\left(\frac{m\pi x}{L}\right) \sum_{n=1}^{\infty} h_n \sin\left(\frac{n\pi x}{L}\right) \\ &= \sum_{n=1}^{\infty} h_n \frac{2}{L} \int_0^L dx \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \\ &= \sum_{n=1}^{\infty} \delta_{nm} h_n \\ &= h_m. \end{aligned} \quad (8.9)$$

(It is not always legitimate to interchange the integration and summation as we do here, but it can be shown that the formulas (8.6) are correct nonetheless.) Evidently, the sequence of numbers h_m , $m = 1, 2, 3, \dots$, contain the same information as the function f itself.† The

* If you are a mathematics fan you can try to prove this result directly. But it’s just as instructive — perhaps more so — for you to check this result using standard resources, *e.g.*, an integral table or using symbolic mathematics software.

† In this context it is worth noting that the square-integrability of f is equivalent to the square-summability of h_n :

$$\sum_{n=1}^{\infty} h_n^2 < \infty.$$

mapping (8.6) from a function f on an interval to the sequence $\{h_n\}$ is called the *Fourier transform of f* .

This is a good place to work a simple example. Consider a square pulse,

$$\eta(x) = \begin{cases} 1 & \text{if } a < x < b, \\ 0 & \text{otherwise,} \end{cases} \quad (8.10)$$

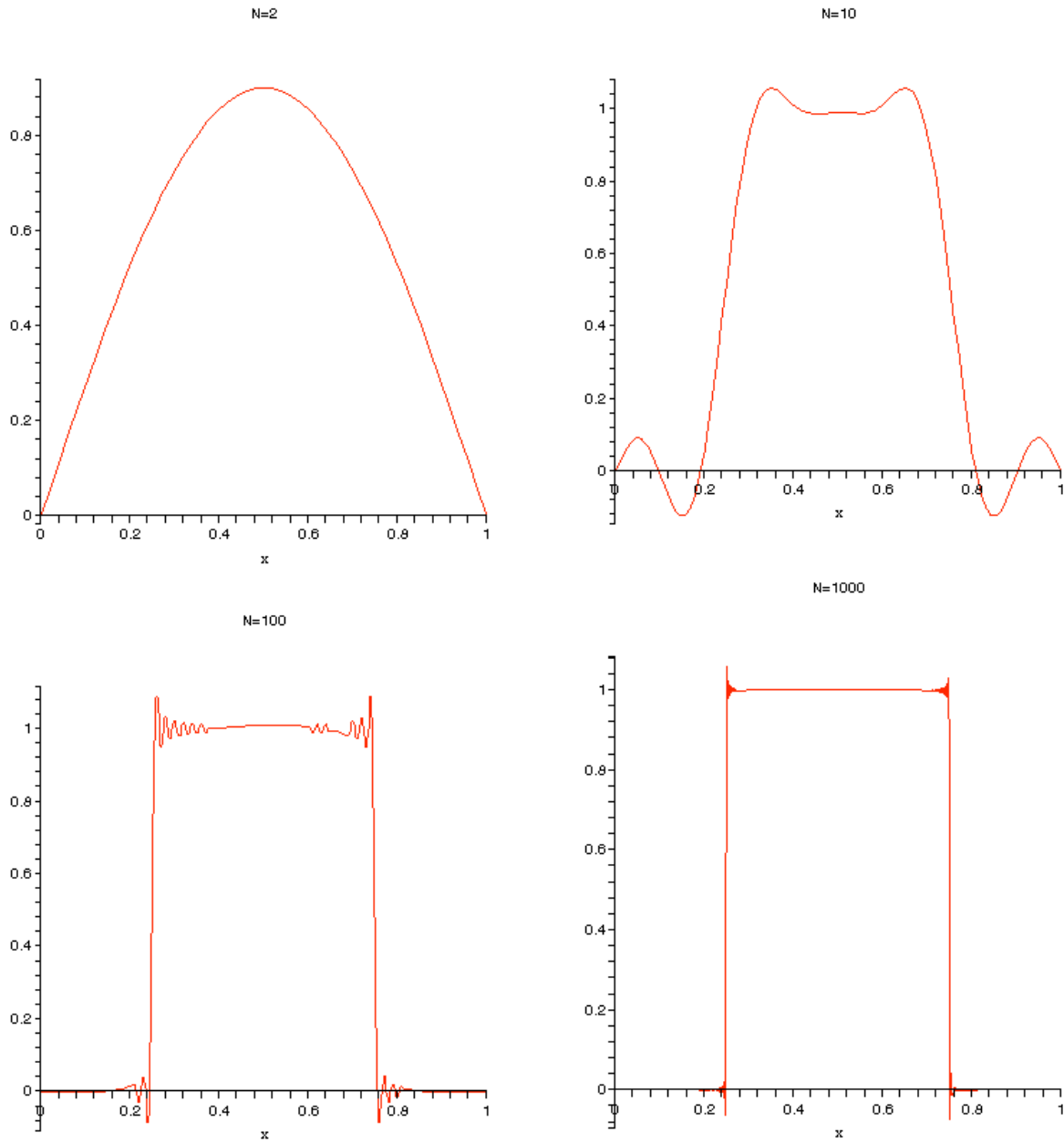
where $0 < a, b < L$. This is a great example because, while the graph of this function looks completely unlike a sinusoidal function, we will see that this function “is” a superposition of sinusoidal functions. To find the superposition, we compute the Fourier transform of η (*exercise*):

$$h_m = \frac{\sqrt{2L}}{m\pi} \left[\cos\left(\frac{m\pi a}{L}\right) - \cos\left(\frac{m\pi b}{L}\right) \right]. \quad (8.11)$$

The claim is, then, that the following series represents the square pulse:

$$\eta(x) = \sum_{m=1}^{\infty} \frac{2}{m\pi} \left[\cos\left(\frac{m\pi a}{L}\right) - \cos\left(\frac{m\pi b}{L}\right) \right] \sin\left(\frac{m\pi x}{L}\right). \quad (8.12)$$

Admittedly, it is hard to tell what is the shape of the function defined in (8.12)! Let us investigate (8.12) graphically. The following sequence of figures depict the partial sums (8.3) for various values of N with $L = 1$, $a = 1/4$, $b = 3/4$.



Fourier series approximation to a square pulse. Here $L=1$, $a=1/4$, $b=3/4$.

As you can see, for large enough values of N the superposition of sine functions gives a good approximation to the square pulse. The only place where things don't look quite so nice is at the discontinuities of $\eta(x)$, *i.e.*, at the corners. The relatively larger point-wise error in the Fourier representation of the function at the corners is due to the fact that we are trying to approximate a discontinuous function using a linear combination of smooth functions. While the point-wise error at the corners is getting better with larger N it is *not* true that the point-wise error vanishes as $N \rightarrow \infty$. What is guaranteed to vanish is the limit (8.5), which allows for the function and its Fourier representation to disagree at isolated points, *i.e.*, the Fourier series will agree with the function “almost everywhere”. It can be shown that for continuous functions the Fourier series will converge point-wise to $f(x)$, however. For discontinuous functions one can say precisely how the point-wise values of the series behave, but we won't go into that here.

The reason we used sine functions in the Fourier series rather than the obvious alternative — cosine functions — stems from the fact that we chose our interval of interest to begin at $x = 0$ and we demanded that $f(0) = 0$. It is possible to relax these assumptions; here we just state a generalization of our previous results. For any (square-integrable) function $f(x)$ on the finite interval $(-L, L)$ there exist (square-summable) numbers α_n and β_n , $n = 0, 1, 2, 3, \dots$, defined as

$$\alpha_0 = \frac{1}{2} \sqrt{\frac{1}{L}} \int_{-L}^L dx f(x) \quad \alpha_n = \sqrt{\frac{1}{L}} \int_{-L}^L dx f(x) \cos\left(\frac{n\pi x}{L}\right), \quad n \neq 0 \quad (8.13)$$

$$\beta_n = \sqrt{\frac{1}{L}} \int_{-L}^L dx f(x) \sin\left(\frac{n\pi x}{L}\right),$$

such that almost everywhere:*

$$f(x) = \sqrt{\frac{1}{L}} \sum_{n=0}^{\infty} \left[\alpha_n \cos\left(\frac{n\pi x}{L}\right) + \beta_n \sin\left(\frac{n\pi x}{L}\right) \right]. \quad (8.14)$$

Note that if we restrict attention to functions that vanish at the endpoints then we recover our previous results (*exercise*).

An elegant way to rewrite this last Fourier series is

$$f(x) = \sqrt{\frac{1}{2L}} \sum_{n=-\infty}^{\infty} \gamma_n e^{in\frac{\pi}{L}x}, \quad (8.15)$$

with

$$\gamma_n = \sqrt{\frac{1}{2L}} \int_{-L}^L dx e^{-in\frac{\pi}{L}x} f(x). \quad (8.16)$$

* Once again, the equality really means that the series converges to the function in the sense that the integral of the square of the difference between the series and the function vanishes in the limit.

Here we have combined the information contained in the two sequences of real numbers (α_n and β_n) into a single complex sequence (γ_n). You will work out the precise relationship between α_n , β_n and γ_n in the Problems. However, one point might confuse you. The real sequences are in some sense only half as big as the complex sequence because $n \geq 0$ in the real sequence case while n is any integer (positive and negative) in the complex sequence case. The resolution of this discrepancy lies in the fact that, since $f(x)$ is real, the constants γ_n must satisfy

$$\gamma_n^* = \gamma_{-n}, \quad (8.17)$$

which you can check (*exercise*) is necessary and sufficient for

$$f^*(x) - f(x) = 0. \quad (8.18)$$

Exercise: What is the Fourier transform of the function $f(x) = 0$?

Let us conclude this section by indicating that Fourier analysis and Fourier series have a very natural vector space interpretation. To keep this simple, consider the real vector space of functions on the interval $x \in [0, L]$ satisfying the boundary conditions (8.1). As a nice exercise you can check that the usual addition of functions and scalar multiplication turn this set into a vector space. We will equip this vector space with the scalar product*

$$(f, g) = \int_0^L dx fg. \quad (8.19)$$

You will prove in the Problems that (8.19) does indeed define a scalar product. The relation (8.2) can be interpreted as the statement that the functions (vectors)

$$e_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \quad (8.20)$$

form a basis for the vector space, *i.e.*, for any vector f :

$$f = \sum_{n=1}^{\infty} h_n e_n. \quad (8.21)$$

Note that this vector space is infinite-dimensional! The relation (8.7) can be interpreted as saying that the basis $\{e_n\}$ is orthonormal:

$$(e_n, e_m) = \delta_{nm}. \quad (8.22)$$

* For simplicity, we do not bother to attach vector notation arrows to the functions/vectors in this discussion.

And the relation (8.6) is just the usual way of extracting the components of a vector in an orthonormal basis:

$$h_n = (e_n, f). \quad (8.23)$$

So, in this context, Fourier analysis can be viewed as “just” vector algebra using an orthonormal basis! As an *exercise* you can contemplate the vector space interpretation of the more general form of the Fourier series shown in (8.13)–(8.16).

8.2 Fourier transforms and Fourier integrals

We have seen how to “Fourier analyze” functions on an interval. What happens if we want to Fourier analyze a function on the whole x -axis? We will now have a look at some of the salient features of this situation.

Consider a function $f(x)$ of one variable $x \in (-\infty, \infty)$. With appropriate technical assumptions about the behavior of the functions involved (we will at least require that f vanishes as $|x| \rightarrow \infty$ — see below for a little more on this) it can be shown that there is another *complex* function $h(k)$, again called the *Fourier transform* of $f(x)$, such that

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk h(k) e^{ikx}. \quad (8.24)$$

The Fourier transform, $h(k)$, of $f(x)$ is obtained via

$$h(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx f(x) e^{-ikx}. \quad (8.25)$$

The formulas (8.24) and (8.25) work by virtue of the *Fourier identity*

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dy f(y) e^{ik(x-y)}. \quad (8.26)$$

Note in particular that the function $f(x) = 0$ corresponds to the Fourier transform $h(k) = 0$ (*exercise*).

As before, the idea here is that the function $f(x)$ is completely represented by its Fourier transform $h(k)$, and *vice versa*. The Fourier transform expresses any function $f(x)$ as a (continuous) superposition of sinusoidal functions.

You can view (8.24) and (8.25) as a continuum limit of (8.15), (8.16) obtained by taking $L \rightarrow \infty$. Indeed, for large enough L one can view $\frac{n\pi}{L} \longleftrightarrow k$ as varying continuously with n to a better and better approximation since adding or subtracting one from n gives a very small change in $\frac{n\pi}{L}$. From this point of view, the Fourier sum over n in (8.15) becomes an integral over k in this limit. The restriction (8.17) becomes

$$h^*(k) = h(-k). \quad (8.27)$$

We will discuss the normalizing factor $1/\sqrt{2\pi}$ a little later.

Some technical details

As before, the restriction (8.27) is needed to keep $f(x)$ a real function. To see this, we demand

$$f(x) = f^*(x). \quad (8.28)$$

In terms of the Fourier representation (8.24) this gives (*exercise*)

$$\int_{-\infty}^{\infty} dk h(k) e^{ikx} = \int_{-\infty}^{\infty} dk h^*(k) e^{-ikx}. \quad (8.29)$$

In the integral on the right hand side, make the change of variables $k \rightarrow -k$. We then get (*exercise*)

$$\int_{-\infty}^{\infty} dk h(k) e^{ikx} = \int_{-\infty}^{\infty} dk h^*(-k) e^{-ikx}, \quad (8.30)$$

or

$$\int_{-\infty}^{\infty} dk [h(k) - h^*(-k)] e^{ikx} = 0. \quad (8.31)$$

Recalling that a function vanishes if and only if its Fourier transform vanishes, this means that f is a real function if and only if

$$h(k) = h^*(-k). \quad (8.32)$$

One nice set of technical assumptions one can use to justify the continuous form of the Fourier transform is as follows. Suppose that $f(x)$ satisfies

$$\int_{-\infty}^{\infty} dx |f(x)|^2 < \infty, \quad (8.33)$$

which says that f is “square integrable”. Note that this implies that f vanishes as $x \rightarrow \pm\infty$, but is otherwise a fairly weak requirement. For example, f need not be continuous. It can be shown that this guarantees the existence of the Fourier transform h , and that furthermore h is also square-integrable. Indeed, it can be shown that

$$\int_{-\infty}^{\infty} dx |f(x)|^2 = \int_{-\infty}^{\infty} dk |h(k)|^2, \quad (8.34)$$

which is known as *Plancherel’s identity*. With this set up, the Fourier transform relation is the statement that $f(x)$ and its Fourier integral representation (8.24) will agree “almost everywhere”, which essentially means that they will agree except at isolated points. More precisely, with $h(k)$ defined by (8.25), if we define

$$E_A(x) := f(x) - \frac{1}{\sqrt{2\pi}} \int_{-A}^A dk h(k) e^{-ikx}, \quad (8.35)$$

then it can be shown that

$$\lim_{A \rightarrow \infty} \int_{-\infty}^{\infty} dx |E_A(x)|^2 = 0. \quad (8.36)$$

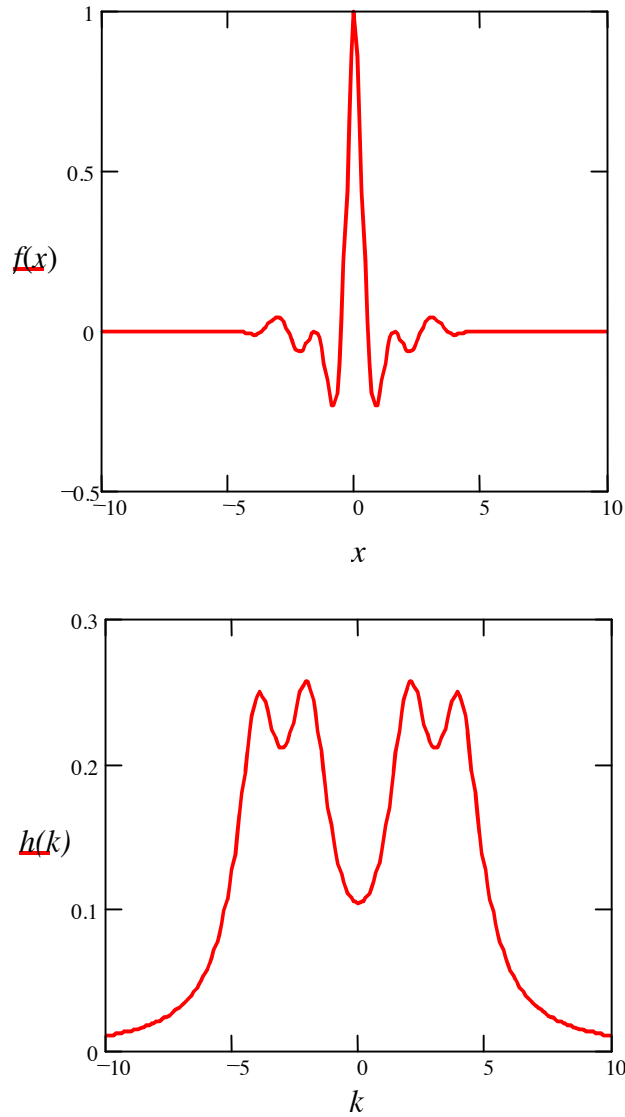


Figure 12. An example function $f(x) = \cos(x)\cos(3x)\exp(-|x|)$ and its Fourier transform $h(k)$.

It is not hard to see that all the preceding results will in fact extend to the case where $f(x)$ is a complex-valued function. Indeed, we could have phrased the discussion in terms of complex-valued functions from the start. One way to see this from the results shown above is to split f into its real and imaginary parts — which are given by real-valued functions — and Fourier analyze each of these parts separately. All of the preceding formulas still hold, except for (8.28)–(8.32), of course.

Once again, we can interpret the formal structure of the preceding discussion in terms of vector spaces. For novelty, let us briefly describe this in the case where we consider complex-valued functions. So, think of the set of all (square-integrable) complex functions of $x \in (-\infty, \infty)$ as a complex vector space. Indeed, one can multiply complex numbers functions by complex numbers (scalars), add functions to make new functions, *etc.* Thus, in particular, each function is to be viewed as a vector. Using this analogy, we can think of the functions $e_k(x) = \frac{1}{\sqrt{2\pi}}e^{ikx}$ as a family of vectors labeled by k . Then, the Fourier expansion (8.24) can be viewed as an expansion of the vector f in the basis provided by e_k . The sum becomes an integral because of the continuous nature of k . From this heuristic point of view (which can be made quite rigorous), the Fourier transform $h(k)$ can be viewed as providing the components of f in the basis e_k .

8.3 The Delta Function

The Fourier identity (8.26) can be conveniently expressed in terms of another useful mathematical object known as the *delta function*, which was devised by the great physicist Paul Dirac. We introduce the delta function by rewriting equation (8.26) as

$$f(x) = \int_{-\infty}^{\infty} dy \delta(x - y) f(y). \quad (8.37)$$

Evidently,

$$\delta(x - y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ik(x-y)}. \quad (8.38)$$

The “integral” appearing in (8.38) is just a formality since it does not really exist in the conventional sense of a Riemann integral (*exercise*). One should think instead of (8.38) as only being a shorthand expression, to be used inside an integral, such as appears in (8.37) or (8.26). Virtually everything one needs to know about the delta function can be deduced from the defining formula (8.37). For example, you will show in the problems that

$$\delta(ax) = \frac{1}{|a|} \delta(x).$$

Despite its name, the delta function is not really a function. Indeed, a simply *exercise* shows

$$f(0) = \int_{-\infty}^{\infty} dx \delta(x) f(x). \quad (8.39)$$

No ordinary function $\delta(x)$ could satisfy (8.39) for any $f(x)$. The delta “function” is an example of a mathematical object called a *distribution* or *generalized function*. A distribution is a linear operation which associates a number to a (nice enough type of) function. In a systematic development of the idea of a distribution one usually *defines* the delta

function as any linear operation that satisfies (8.37) for appropriate functions $f(x)$. One then uses the Fourier identity to give an explicit realization (8.38) of such an operation for an appropriate class of functions.

Some physics/engineering texts will assert that $\delta(x - y)$ is a function which vanishes when $x \neq y$ and is infinite when $x = y$. Of course, strictly speaking, this kind of definition is nonsense. But the idea being promoted here is that one can view the delta function in terms of a limit of well-defined functions. This limit is then used (inside an integral) to define formulas such as (8.37). For example, one can define

$$D_\epsilon(x) = \frac{1}{2\pi} \int_{-\frac{1}{\epsilon}}^{\frac{1}{\epsilon}} dk e^{ik(x)} = \frac{1}{\pi} \frac{\sin(\frac{x}{\epsilon})}{x}. \quad (8.40)$$

As ϵ becomes small, D_ϵ is narrowly peaked around zero with an increasing maximum. In the limit as $\epsilon \rightarrow 0$, D_ϵ the peak approaches infinity and the width of the peak approaches zero. So, D_0 is no longer a function*, but the fundamental property (8.37) still holds in the limit:

$$\lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} dy D_\epsilon(x - y) f(y) = f(x), \quad (8.41)$$

so that we write

$$\delta(x) = \lim_{\epsilon \rightarrow 0} D_\epsilon(x). \quad (8.42)$$

Another great example of this type involves the Gaussian function. As you will explore in the Problems, we have

$$\delta(x) = \lim_{\epsilon \rightarrow 0^+} \left[\frac{1}{\sqrt{\pi\epsilon}} e^{-\frac{x^2}{\epsilon}} \right].$$

Here it is pretty obvious that for small ϵ the function is narrowly peaked about zero with the limit corresponding to taking the maximum value to infinity and the width to zero.

A convenient mental picture of the delta function arises by comparing it to the *Kronecker delta*. Recall that the Kronecker delta, denoted in this context by δ_j^i , where $i, j = 1, 2, \dots, d$ is defined by

$$\delta_j^i = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}. \quad (8.43)$$

Fourier's identity (8.26), or (8.37), is then a continuous version of the identity

$$v^i = \sum_{j=1}^d \delta_j^i v^j, \quad (8.44)$$

with integration replacing summation.

* If it were, it would vanish away from zero and diverge at zero!

Using the delta function we can interpret Fourier analysis formula (8.24) as an expansion in an orthonormal basis of vectors:

$$e_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx},$$

where k labels the basis vectors. Here “vectors” means the set of complex-valued functions equipped with the usual addition and scalar multiplication operations. The scalar product of two functions is defined as

$$(f, g) = \int_{-\infty}^{\infty} dx f^*(x)g(x),$$

so that (*exercise*)

$$(e_k, e_l) = \delta(k - l).$$

Now you can see why the normalization factor is $\frac{1}{\sqrt{2\pi}}$ for the continuous Fourier transform.

Be careful not to take the analogy between the delta function and the Kronecker delta too literally! For example, it is *not* the case that $\delta(k - l)$ vanishes when $k \neq l$ and is unity when $k = l$. Indeed, since δ is not really a function, it makes no sense to speak of its values.

8.4 Applications to the Wave Equation

Let us now see how to use Fourier analysis to solve the wave equation. To begin, we consider the application of Fourier series to solve the wave equation. Let us consider a string of length L that is free to vibrate except that its ends are held fixed. We model this situation using a displacement $q(x, t)$ satisfying the wave equation along with the *boundary conditions*

$$q(0, t) = q(L, t) = 0. \quad (8.45)$$

This is the continuum limit of the chain of oscillators considered previously. A Fourier series form of the solution is obtained as follows. We note that at each time t the wave profile $q(x, t)$ is a function of $x \in [0, L]$ satisfying the boundary conditions (8.45). Thus we can express $q(x, t)$ as a Fourier series at each time t :

$$q(x, t) = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} q_n(t) \sin\left(\frac{n\pi x}{L}\right).$$

As a nice *exercise* you can verify that the wave equation then implies that

$$\sum_{n=1}^{\infty} \left[q_n''(t) + \left(\frac{n\pi v}{L}\right)^2 q_n(t) \right] \sin\left(\frac{n\pi x}{L}\right) = 0.$$

Here we have a function of x (for each t) that vanishes; the Fourier transform must likewise vanish:

$$q_n''(t) + \left(\frac{vn\pi}{L}\right)^2 q_n(t) = 0, \quad n = 1, 2, \dots$$

This is the harmonic oscillator equation! More precisely, for each n we get a harmonic oscillator equation with angular frequency given by $vn\pi/L$. You can interpret this result as showing that the sinusoidal Fourier basis elements are a set of “normal modes”. We easily get

$$q_n(t) = \text{Re}(\gamma_n e^{i\omega_n t}),$$

where the coefficients γ_n are complex and

$$\omega_n = \frac{n\pi v}{L}. \quad (8.46)$$

Thus we obtain

$$q(x, t) = \text{Re} \left(\sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} \gamma_n \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n t} \right). \quad (8.47)$$

The coefficients γ_n are determined by the initial conditions. Let us suppose that the initial wave profile is of the form

$$q(x, 0) = a(x), \quad \frac{\partial q(x, 0)}{\partial t} = b(x), \quad (8.48)$$

where a and b are some given functions satisfying

$$a(0) = a(L) = 0 = b(0) = b(L). \quad (8.49)$$

We can write

$$a(x) = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi x}{L}\right), \quad (8.50)$$

$$b(x) = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} \beta_n \sin\left(\frac{n\pi x}{L}\right), \quad (8.51)$$

where

$$\alpha_n = \sqrt{\frac{2}{L}} \int_0^L dx a(x) \sin\left(\frac{n\pi x}{L}\right), \quad (8.52)$$

$$\beta_n = \sqrt{\frac{2}{L}} \int_0^L dx b(x) \sin\left(\frac{n\pi x}{L}\right). \quad (8.53)$$

It is straightforward to check that (i) $q(x, t)$ solves the wave equation and (ii) matches the initial conditions provided

$$\text{Re}(\gamma_n) = \alpha_n, \quad \text{Im}(\gamma_n) = -\frac{\beta_n}{\omega_n}. \quad (8.54)$$

As a nice exercise you should compare these Fourier series to their discrete counterparts which we found when studying the chain of oscillators.

As an example of the above results, let us return to the square pulse profile,

$$\begin{aligned} a(x) &= \begin{cases} 1, & \text{if } \frac{L}{4} < x < \frac{3L}{4}; \\ 0, & \text{otherwise,} \end{cases} \\ b(x) &= 0. \end{aligned} \quad (8.55)$$

This is a square pulse of width $L/2$, at rest, centered at $x = L/2$.^{*} We have (exercise)

$$\begin{aligned} \alpha_n &= \frac{\sqrt{2L}}{n\pi} \left(\cos\left(\frac{n\pi}{4}\right) - \cos\left(\frac{3n\pi}{4}\right) \right) \\ &= \begin{cases} -i^{n+1} \frac{\sqrt{8L}}{n\pi} \sin\left(\frac{n\pi}{4}\right), & n \text{ odd} \\ = 0 & n \text{ even} \end{cases} \end{aligned} \quad (8.56)$$

and

$$\beta_n = 0. \quad (8.57)$$

Since we only need to take n to be odd, we set $n = 2k + 1$, $k = 0, 1, 2, \dots$ to get (exercise)

$$q(x, t) = \sum_{k=0}^{\infty} (-1)^k \frac{4}{(2k+1)\pi} \sin\left(\frac{(2k+1)\pi}{4}\right) \sin\left(\frac{(2k+1)\pi x}{L}\right) \cos\left(\frac{(2k+1)\pi vt}{L}\right) \quad (8.58)$$

Let us now turn our attention to solutions to the wave equation for all $x \in (-\infty, \infty)$, assuming that $q(x, t)$ is square-integrable for all time t . As before, we can Fourier analyze $q(x, t)$ at each time with respect to the x variable and derive a harmonic oscillator type equation for the Fourier transform. This approach will be addressed in the Problems. Here we illustrate a different approach. We consider the function

$$f(u) = f(x + vt), \quad (8.59)$$

which appears in the general solution (7.13) to the one-dimensional wave equation (the other function, $g(s)$, is handled in an identical fashion). We now know that there is a function $h(k)$ such that we can write

$$f(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk h(k) e^{iku}, \quad (8.60)$$

^{*} Strictly speaking, such initial conditions are beyond the scope of our original view of the wave equation since they correspond to discontinuous solutions — such functions can't be differentiated! You can view the current example as an idealized model of a realistic solution whose initial data are ever so slightly rounded at the corners. Both the general solution to the wave equation (3.10) as well as the Fourier series version (8.47) extend without difficulty to such discontinuous “solutions”. Thus one can use Fourier analysis to, in effect, *define* the wave equation for such functions.

or, equivalently,

$$f(x + vt) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk h(k) e^{ik(x+vt)}. \quad (8.61)$$

This way of expressing f as an integral can be viewed as a way of writing f as a continuous superposition of the elementary solutions (6.8) with coefficients $h(k)$. Of course, you probably think of a superposition as a *sum* with constant coefficients. But you can think of an integral as a limit of a sum, so it is reasonable to view the Fourier representation of $f(u)$ as just a continuous generalization of a superposition of elementary solutions. If you like, you can check that this form of $f(u)$ does indeed satisfy the wave equation by direct differentiation, but you already can see that the integral must satisfy the wave equation because it is a function of $x + vt$. A similar line of reasoning expresses $g(s)$ as

$$g(x - vt) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk w(k) e^{ik(x-vt)}. \quad (8.62)$$

Thus the general solution to the 1-dimensional wave equation, (7.13), can be characterized by $h(k)$ and $w(k)$, the Fourier transforms of f and g .

In our discussion we have not yet taken into account the fact that $q(x, t)$ must be real. This is achieved by choosing $f(u)$ and $g(s)$ to be real. As discussed earlier (see (8.32)) to do this we must require

$$h^*(k) = h(-k) \quad \text{and} \quad w^*(k) = w(-k). \quad (8.63)$$

To summarize, we can express the general solution of the wave equation as a superposition of elementary traveling wave solutions via

$$q(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \left(h(k) e^{ik(x+vt)} + w(k) e^{ik(x-vt)} \right), \quad (8.64)$$

where $h^*(k) = h(-k)$ and $w^*(k) = w(-k)$. We see that to specify a solution we still must choose two functions of one variable ($h(k)$ and $w(k)$). Again, this is equivalent to choosing initial data.

Let us make this last point explicit. Let the initial time be $t = 0$ (so that t is the elapsed time). The “initial data” are given by the initial displacement profile, $q(x, 0)$, and the initial velocity profile, $\frac{\partial q(x, 0)}{\partial t}$. These two functions of x are given by

$$q(x, 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk (h(k) + w(k)) e^{ikx} \quad (8.65)$$

and

$$\frac{\partial q(x, 0)}{\partial t} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk (h(k) - w(k)) (ikv) e^{ikx} \quad (8.66)$$

Evidently, using the inverse Fourier transform (8.25),

$$h(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \frac{1}{2} \left(q(x, 0) + \frac{1}{ikv} \frac{\partial q(x, 0)}{\partial t} \right) e^{-ikx}, \quad (8.67)$$

$$w(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \frac{1}{2} \left(q(x, 0) - \frac{1}{ikv} \frac{\partial q(x, 0)}{\partial t} \right) e^{-ikx}. \quad (8.68)$$

8.5 The Dispersion Relation

In §4 we introduced the idea of a *dispersion relation*, which could be interpreted as a relation between wavelength and wave speed for sinusoidal waves. There is likewise a dispersion relation for solutions of the 1-d wave equation. The (complex form of the) sinusoidal solution of the wave equation (which we just saw in the Fourier expansion) was displayed in (6.8):

$$q(x, t) = Ae^{i(kx \pm \omega t)}, \quad (8.69)$$

where $k = \frac{2\pi}{\lambda}$ and $\omega = |k|v$. The speed v of this wave is simply $\frac{\omega}{k}$, which gives the given constant v ,

$$\frac{\omega}{|k|} = v, \quad (8.70)$$

independent of wavelength. The dispersion relation

$$\omega = |k|v \quad (8.71)$$

shows that in the superposition (8.64) all the component waves are moving together at the same speed. This implies that whatever shape the wave displacement has at one time it will retain for all time. If waves with different wavelengths moved at different speeds, the superposition would be, in effect, changing in time and the shape of the wave would change in time. Because this *doesn't* happen here, one says that there is no dispersion in the 1-d wave equation.

8.6 Example: Gaussian Wave Packet

As a final example, let us return to the Gaussian wave profile displayed in (7.29). Recall that this wave arose by choosing a Gaussian displacement profile (7.27) at time $t = 0$ and a vanishing initial velocity profile (7.28). We have

$$\frac{A}{2} e^{-\frac{(x+vt)^2}{a^2}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk h(k) e^{ik(x+vt)}, \quad (8.72)$$

and

$$\frac{A}{2} e^{-\frac{(x-vt)^2}{a^2}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk w(k) e^{ik(x-vt)}. \quad (8.73)$$

To solve for $h(k)$ and $w(k)$ we use the Fourier transform formula (8.25). Set $u = x + vt$. Then we have (*exercise*)

$$h(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} du \frac{A}{2} e^{-\frac{u^2}{a^2}} e^{-iku}. \quad (8.74)$$

Note that the t dependence has dropped out! This is not too surprising since the formula for $h(k)$ should hold for all t , and we might as well choose $t = 0$ where we again get (8.74) (*exercise*).

You may not have encountered an integral like (8.74) before. It is an example of a *Gaussian integral*. Note that it is really a combination of the integral of $\exp(-u^2/a^2) \cos(ku)$ and $\exp(-u^2/a^2) \sin(ku)$. The integrals exist because the exponentials die off fast enough at large u so that the “area under the curve” is finite. There are several ways of performing such integrals, which you will probably see in detail in a more advanced course. Here we shall simply state the result; see the problems for some related results. It will be worth your while to learn how to reliably get such results using a computer. In any case, it can be shown that

$$\int_{-\infty}^{\infty} dx e^{-\alpha^2 x^2} \cos(\beta x) = \frac{\sqrt{\pi}}{\alpha} \exp\left(-\frac{\beta^2}{4\alpha^2}\right), \quad (8.75)$$

and

$$\int_{-\infty}^{\infty} dx e^{-\alpha^2 x^2} \sin(\beta x) = 0. \quad (8.76)$$

Here we assume $\alpha > 0$. Notice that (8.75) contains as a special case the famous integral (*exercise*)

$$\int_{-\infty}^{\infty} dx e^{-\alpha^2 x^2} = \frac{\sqrt{\pi}}{\alpha},$$

To use these results we identify $\alpha = 1/a$ and $\beta = k$; we then find

$$h(k) = \frac{1}{2\sqrt{2}} Aa \exp\left(-\frac{k^2 a^2}{4}\right). \quad (8.77)$$

Notice that this also a Gaussian, but in “ k -space”. Thus *the Fourier transform of a Gaussian is also a Gaussian*. A similar computation shows that $w(k) = h(k)$ (*exercise*). Note that both $h(k)$ and $w(k)$ are real functions and satisfy

$$h(-k) = h(k) = w(-k) = w(k), \quad (8.78)$$

as needed for a real solution $q(x, t)$ (*exercise*).

It is worth noting that while both the wave profiles in x -space and k -space are Gaussians, their widths behave in an opposite manner. The width of the x -space Gaussian increases with a while the width of the corresponding k -space Gaussian decreases with a .

This is in fact a general “rule of thumb” concerning Fourier transforms. The more “localized” are the non-zero values of a function, the more “de-localized” (or “spread out”) are the values of its Fourier transform. This holds whether or not the original function is in x -space or k -space. You may be familiar with a result of this type from quantum mechanics where it leads to one manifestation of the uncertainty principle.

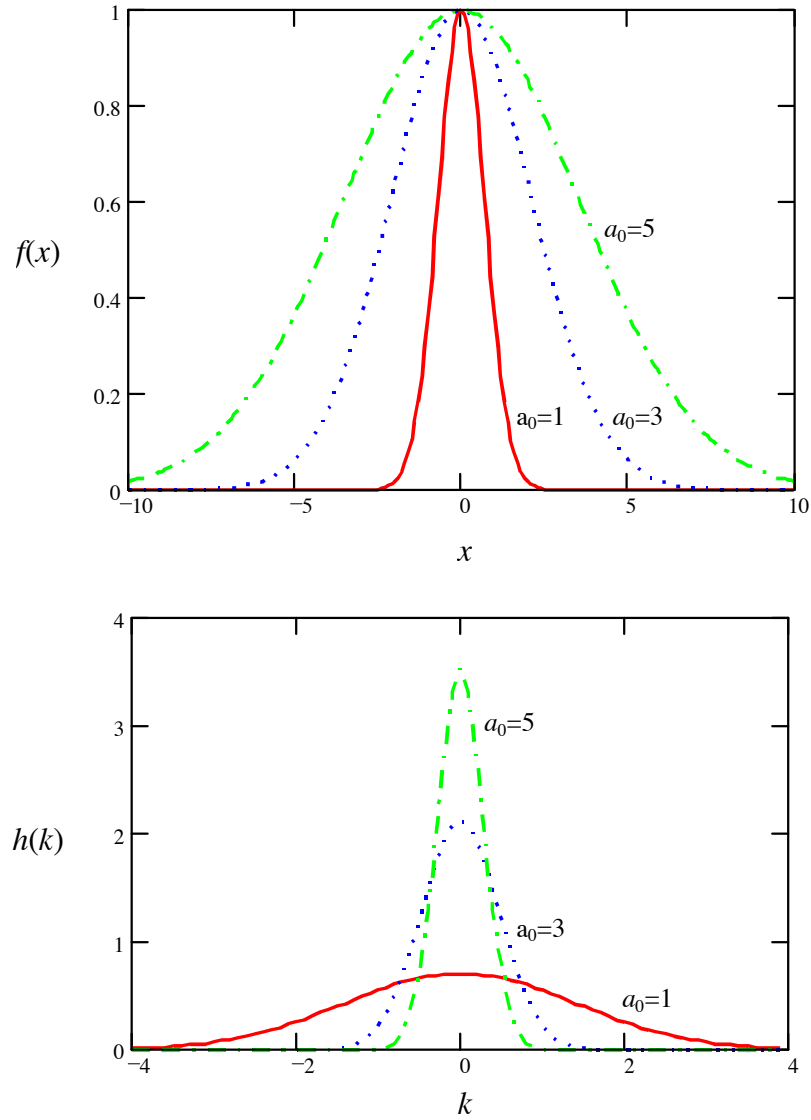


Figure 13. Three Gaussian functions $f(x)$ and their Fourier transforms $h(k)$, which are also Gaussian functions. Notice that as $f(x)$ becomes broader its Fourier transform becomes narrower.