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09 The Wave Equation in 3 Dimensions

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9. The Wave equation in 3 dimensions.

We now turn to the 3-dimensional version of the wave equation, which can be used to describe a variety of wavelike phenomena, *e.g.,* sound waves, atmospheric waves, electromagnetic waves, and gravitational waves. One could derive this version of the wave equation much as we did the one-dimensional version by generalizing our line of coupled oscillators to a 3-dimensional array of oscillators. For many purposes, *e.g.,* modeling propagation of sound in a solid, this provides a useful discrete model of a three dimensional solid. We won't be able to go into that here. The point is, though, that if we take the continuum limit as before we end up with the *3-dimensional wave equation* for the displacement $q(\vec{r}, t)$ of the oscillator-medium at the point labeled $\vec{r} = (x, y, z)$ at time t:

$$\frac{1}{v^2}\frac{\partial^2 q}{\partial t^2} = \frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2} + \frac{\partial^2 q}{\partial z^2}. \tag{9.1}$$

The 3-dimensional wave equation is a linear, homogeneous partial differential equation with constant coefficients. It has one dependent variable (q) and four independent variables (t, x, y, z). Note that if we assume the wave displacement does not depend upon two of the four independent variables, *e.g.,* $q = q(x, t)$ we end up with the one-dimensional wave equation (*exercise*).

9.1 Gradient, Divergence and Laplacian

The right-hand side of (9.1) represents a very important differential operator, known as the *Laplacian*, so let us take a moment to discuss it. The Laplacian itself can be viewed as the composition of two other operators known as the gradient and the divergence, which we will also briefly discuss.

To begin, let $f(\vec{r}) = f(x, y, z)$ be a function of Cartesian coordinates for Euclidean space. The *gradient* of f is a vector *field*, *i.e.,* a vector at each position $\vec{r}$, denoted by ∇f, defined by

$$\nabla f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}, \tag{9.2}$$

where $\hat{x}$, $\hat{y}$, $\hat{z}$ are the orthogonal unit vector fields in the x, y, z directions, respectively.* What is the meaning of this vector field? Clearly at any given point, the x, y, z components of ∇f give the rate of change of f at that point in the x, y, z directions, respectively; the other variables being held fixed. More generally, at any given point $\vec{r}$, the rate of change of

* Some texts denote the gradient operation by "grad":

$$\nabla f \equiv \mathrm{grad} f. \tag{9.3}$$

f in the direction specified by a unit vector $\hat{n}$ is given via the *directional derivative* along $\hat{n}$, defined by

$$\nabla_{\hat{n}} f = \hat{n} \cdot \nabla f = n^x \frac{\partial f}{\partial x} + n^y \frac{\partial f}{\partial y} + n^z \frac{\partial f}{\partial z}.$$

A second important interpretation of the gradient is as follows. The direction of the gradient of f at any given point is the direction in which the function f has the greatest rate of change at that point, while the magnitude of the gradient (at the given point) is that rate of change. (Prove these last two facts as a nice *exercise.*)

A third important interpretation of the gradient is as follows. The gradient ∇f is always perpendicular to the surfaces $f =$ constant. For example, the function $f(x, y, z) = x^2 + y^2 + z^2$ is constant on spheres centered at the origin. The gradient of this function is orthogonal to those spheres. This you will study in a homework problem.

Finally, I mention that the gradient is an example of a linear differential operator.

The Laplacian is a combination of the gradient with another linear differential operator, called the *divergence*, which makes a function out of a vector field. Let

$$\vec{V}(\vec{r}) = V^x(\vec{r})\hat{x} + V^y(\vec{r})\hat{y} + V^z(\vec{r})\hat{z}$$

be a vector field (each of the three components (V^x, V^y, V^z) is a function of of position). The divergence, denoted $\nabla \cdot \vec{V}$, is the function*

$$\nabla \cdot \vec{V} = \frac{\partial V^x}{\partial x} + \frac{\partial V^y}{\partial y} + \frac{\partial V^z}{\partial z}. \tag{9.5}$$

You may have encountered the divergence in a discussion of the Gauss law of electromagnetism. Often, Gauss' law is formulated in terms of the electric flux through a closed surface. But if you take a spherical surface and shrink the surface to infinitesimal size at a point, the flux is given by the divergence of the vector field at that point. More precisely, consider a small closed ("Gaussian") surface enclosing a point. One can compute the flux of the vector field $\vec{V}$ through this surface (just as one computes electric flux). Now consider shrinking this surface to the chosen point. The limit as one shrinks the surface to the point of the flux divided by the enclosed volume is precisely the divergence of the vector field at that point. So, you can think of the divergence as a sort of flux per unit volume.

It is now easy to see that the right-hand side of the wave equation (9.1) is the divergence of the gradient of the function $q(\vec{r}, t)$ with t held fixed. The resulting differential operator,

* In some texts one denotes $\nabla\cdot$ by "div":

$$\nabla \cdot \vec{V} \equiv \mathrm{div}\vec{V}. \tag{9.4}$$

shown on the right-hand side of (9.1), is denoted by $\nabla^2 f$ and is the *Laplacian*:†

$$\nabla^2 f := \nabla \cdot (\nabla f) = \frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2} + \frac{\partial^2 q}{\partial z^2}. \tag{9.6}$$

The Laplacian is a linear differential operator that takes a (twice differentiable) function and produces another function.

The wave equation can thus be compactly written in terms of the Laplacian as

$$\frac{1}{v^2}\frac{\partial^2 q}{\partial t^2} = \nabla^2 q. \tag{9.7}$$

9.2 Solutions to the Three-Dimensional Wave Equation

Solutions of the 3-dimensional wave equation (9.7) are not any harder to come by than those of the 1-dimensional wave equation. Indeed, if we look for solutions that are independent of y and z, we recover the solutions obtained for the 1-dimensional equation. So, for example, the (complex) wave solution

$$q(\vec{r}, t) = Ae^{ik(x-vt)}, \tag{9.8}$$

satisfies the 3-dimensional wave equation (*exercise*) for any real number k. This is a (complex) wave traveling in the x direction with wavelength $\lambda = 2\pi/|k|$ and speed v. Of course, we can equally well write down solutions corresponding to traveling waves in the y and z directions (*exercise*). More generally, a (complex) wave with wavelength $\lambda = 2\pi/|k|$ propagating in the direction of the vector $\vec{k}$ with speed v is given by

$$q(\vec{r}, t) = Ae^{i(\vec{k}\cdot\vec{r}-\omega t)}, \tag{9.9}$$

where

$$\omega = |\vec{k}|v. \tag{9.10}$$

As usual, real solutions can be obtained by taking the real or imaginary parts. To check that (9.9) satisfies the wave equation, you can just plug (9.9) into (9.7) and check that the wave equation is satisfied. The dispersion relation (9.10) makes that happen.

Alternatively, you can check that (9.9) (with (9.10)) satisfies the wave equation by using the following (slightly tricky) argument, which you should ponder as an *exercise*. The dot product $\vec{k} \cdot \vec{r}$ is a scalar that is defined geometrically, that is, independently of the orientation of the (x, y, z) axes. Likewise, the Laplacian is the same no matter the orientation of these axes. So we can choose our axes anyway you like – the wave equation

† Sometimes one uses the notation $\Delta \equiv -\nabla^2$.

does not depend upon such choices. For a given vector $\vec{k}$, suppose you choose your x axis along the direction of $\vec{k}$. Then $\vec{k}\cdot\vec{r} = kx$, and $q(x,t)$ takes the form of the (complex) solution to the 1-d wave equation. We have already pointed out that if $q = q(x,t)$ the 3-d wave equation reduces back to the 1-d wave equation. Thus (9.9) solves (9.7) because we have reduced the situation back to the one dimensional case.

Solutions to the wave equation of the form (9.9) are called *plane waves* because at any given time they look the same as one moves along any plane $\vec{k}\cdot\vec{r} =$ constant. We'll have a little more to say on this in §10.

Note the relation between frequency and wavelength associated with the plane wave (9.9):

$$\omega = |\vec{k}|v = \frac{2\pi}{\lambda}v. \tag{9.10}$$

This is the dispersion relation for plane waves in three dimensions. We can view this restriction as fixing the frequency ω in terms of the magnitude $|\vec{k}|$ but leaving the *wave vector* $\vec{k}$ itself free. In effect, the dispersion relation (8.71), $\omega = |k|v$, still holds in the three-dimensional setting, provided we interpret $|k|$ as the magnitude of the wave vector. Indeed, as we shall see, each plane wave appearing in the Fourier decomposition of the general solution to (9.7) is, mathematically, just a complex sinusoidal solution of a one-dimensional wave equation.

Unfortunately, it is not quite as easy to write a simple formula for the *general* solution to the 3-dimensional wave equation as it was in the 1-dimensional case.* In particular, our trick of changing variables to $x \pm vt$ will not help here. However, Fourier analysis is easily generalized to any number of dimensions. The idea is that, given a function $f(x,y,z)$, we can take its Fourier transform one variable at a time. Let us briefly see how this works. For simplicity, we will only consider the case of waves on all of 3-dimensional space, *i.e.,* we will use the continuous version of the Fourier transform.

The Fourier transform $h(\vec{k}) = h(k_x, k_y, k_z)$ of $f(\vec{r}) = f(x,y,z)$ is defined by

$$h(\vec{k}) = (2\pi)^{-3/2}\int_{\text{all space}} d^3x\, e^{-i\vec{k}\cdot\vec{r}} f(\vec{r}), \tag{9.11}$$

where $\vec{k}\cdot\vec{r} = k_x x + k_y y + k_z z$, and the 2π factor is a conventional—and convenient—normalization. Every (square-integrable) function $f(\vec{r})$ can be expressed as

$$f(\vec{r}) = (2\pi)^{-3/2}\int_{\text{all space}} d^3k\, e^{i\vec{k}\cdot\vec{r}} h(\vec{k}) \tag{9.12}$$

for some (square-integrable) function $h(\vec{k})$. The integration region in each of these formulas is denoted by "all space", which means that each of (x,y,z) in (9.11) and each of (k^x, k^y, k^z)

* There *is* a formula analogous to (7.26), but it is a little too complicated to be worth going into here.

in (9.12) run from $-\infty$ to ∞. Once again, if $f(\vec{r})$ is real, then its Fourier transform satisfies $h^*(\vec{k}) = h(-\vec{k})$.

Note that $e^{i\vec{k}\cdot\vec{r}}$ can be viewed as a (complex) traveling plane wave profile at a fixed time. In this interpretation, the plane of symmetry is orthogonal to $\vec{k}$ and the wavelength $\lambda = 2\pi/k$, where

$$k = |\vec{k}| = \sqrt{(k^x)^2 + (k^y)^2 + (k^z)^2} \tag{9.13}$$

is the magnitude (length) of the vector $\vec{k}$. Because

$$e^{i\vec{k}\cdot\vec{r}} = e^{ik_x x} e^{ik_y y} e^{ik_z z},$$

3 you can think of the three-dimensional Fourier transform as simply taking 3 one-dimensional Fourier transforms, one for each spatial variable.

The essence of Fourier analysis is that every function can be expressed as a superposition of plane waves with (i) varying amplitudes (specified by $h(\vec{k})$), (ii) varying directions (specified by $\vec{k}/k$), and (iii) varying wavelengths (specified by k). The precise contributions from the ingredients (i)–(iii) depend upon the particular function being Fourier analyzed. We now use this basic fact from Fourier analysis to get a handle on the general solution of the wave equation; this will generalize our one-dimensional result.

Suppose $q(\vec{r}, t)$ is a solution to the wave equation. Let us define:

$$p(\vec{k}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\text{all space}} d^3x\, e^{-i\vec{k}\cdot\vec{r}} q(\vec{r}, t), \tag{9.14}$$

This is just the (inverse) Fourier transform of q at each time t. We therefore also have (*exercise*)

$$q(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\text{all space}} d^3k\, e^{i\vec{k}\cdot\vec{r}} p(\vec{k}, t).$$

It is easy to see that the wave equation for q implies (*exercise*)

$$\frac{\partial^2 p}{\partial t^2} + k^2 v^2 p = 0. \tag{9.15}$$

Just as in the one-dimensional case, for each value of k this is just the harmonic oscillator equation. The wave vector, in effect, labels degrees of freedom that oscillate with a frequency $\omega = |k|v$. This is not an accident, of course, given the relation between the wave equation and a collection of harmonic oscillators which we discussed earlier. The solution to (9.15) for any given $\vec{k}$ is then of the form

$$p(\vec{k}, t) = A(\vec{k}) \cos(\omega t) + B(\vec{k}) \sin(\omega t). \tag{9.16}$$

As usual, to fix the coefficients A and B we need to consider initial conditions. Suppose that at, say, $t = 0$ the wave has displacement and velocity profiles given by

$$q(\vec{r}, 0) = \alpha(\vec{r}), \quad \frac{\partial q(\vec{r}, 0)}{\partial t} = \beta(\vec{r}), \tag{9.17}$$

where $\alpha(\vec{r})$ and $\beta(\vec{r})$ have Fourier expansions given by:

$$\alpha(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \int_{\text{all space}} d^3k \, e^{i\vec{k}\cdot\vec{r}} a(\vec{k}) \tag{9.18}$$

and

$$\beta(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \int_{\text{all space}} d^3k \, e^{i\vec{k}\cdot\vec{r}} b(\vec{k}). \tag{9.19}$$

Then it is easy to see that*

$$A(\vec{k}) = a(\vec{k}) \quad B(\vec{k}) = \frac{1}{\omega} b(\vec{k}). \tag{9.20}$$

Evidently,

$$q(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\text{all space}} d^3k \, e^{i\vec{k}\cdot\vec{r}} \left\{ a(\vec{k}) \cos(\omega t) + b(\vec{k}) \frac{\sin(\omega t)}{\omega} \right\}, \tag{9.21}$$

where

$$a(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int_{\text{all space}} d^3x \, e^{-i\vec{k}\cdot\vec{r}} q(\vec{r}, 0), \tag{9.22}$$

and

$$b(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int_{\text{all space}} d^3x \, e^{-i\vec{k}\cdot\vec{r}} \frac{\partial q(\vec{r}, 0)}{\partial t} \tag{9.23}$$

satisfies the wave equation with the general choice of initial conditions (9.17). This justifies calling (9.21) the Fourier representation of the general solution to the 3-dimensional wave equation.

The general solution (9.21) to the 3-d wave equation can be viewed as a superposition of the elementary plane wave solutions that we studied earlier. To see this, just note that

$$\cos(\omega t) = \frac{1}{2}(e^{i\omega t} + e^{-i\omega t}), \quad \sin(\omega t) = \frac{1}{2i}(e^{i\omega t} - e^{-i\omega t}), \tag{9.24}$$

and then rearrange the terms in (9.21) to get the solution into the form

$$q(\vec{r}, t) = (2\pi)^{-3/2} \int_{\text{all k-space}} d^3k \, \left[c(\vec{k}) e^{i(\vec{k}\cdot\vec{r} - kvt)} + c^*(\vec{k}) e^{-i(\vec{k}\cdot\vec{r} - kvt)} \right], \tag{9.25}$$

* Note that the coefficient $B(\vec{k})$ is not defined as $k \to 0$ (exercise), but the product $B(\vec{k}) \sin(\omega t)$ is well-defined at $k = 0$.

where $c(\vec{k})$ are determined by the initial data (*exercise*). Physically, you can think of this integral formula as representing a (continuous) superposition of plane waves over their possible physical attributes. To see this, consider a plane wave of the form

$$q(\vec{r}, t) = Re\left[ce^{i(\vec{k}\cdot\vec{r}-\omega t)}\right], \tag{9.26}$$

where c is a complex number. You can check that this is a wave traveling in the direction of $\vec{k}$, with wavelength $\frac{2\pi}{k}$, and with amplitude $|c|$. The phase $\frac{c}{|c|}$ of the complex number c adds a constant to the phase of the wave *(exercise)*. The integral in (9.21) is then a superposition of waves in which one varies the amplitudes ($|c|$), relative phases ($c/|c|$), wavelengths ($2\pi/k$), and directions of propagation ($\vec{k}/k$) from one wave to the next.

Equivalently, every solution to the wave equation can be obtained by superimposing real plane wave solutions of the form

$$q(\vec{r}, t) = A\cos(\vec{k}\cdot\vec{r} - \omega t + \phi). \tag{9.27}$$

Here the (continuous) superposition takes place by varying the amplitude A, the wave vector $\vec{k}$ and the phase ϕ (*exercise*).

Exactly as we did for the space of solutions to the one-dimensional wave equation, we can view the space of solutions of the three-dimensional wave equation as a vector space (*exercise*). From this point of view, the plane waves form a basis for the vector space of solutions.

10. Why "plane" waves?

Let us now pause to explain in more detail why we called the elementary solutions (9.9) and (9.27) *plane waves*. The reason is that the displacement $q(\vec{r}, t)$ has the symmetry of a plane. To see this, fix a time t (take a "snapshot" of the wave) and pick a location $\vec{r}$. Examine the wave displacement q (at the fixed time) at all points in a plane that is (i) perpendicular to $\vec{k}$, and (2) passes through $\vec{r}$. The wave displacement will be the same at each point of this plane. To see this most easily, simply choose, say, the x-axis to be along the vector $\vec{k}$. The planes perpendicular to $\vec{k}$ are then parallel to the y-z plane. In these new coordinates the wave (9.27) takes the simple form (*exercise*)

$$q(\vec{r}, t) = A\cos(kx - \omega t + \phi). \tag{10.1}$$

Clearly, at a fixed t and x, $q(\vec{r}, t)$ is the same anywhere on the plane obtained by varying y and z.

A more formal — and perhaps more instructive — way to see the plane wave symmetry of (9.27) is to fix a time t and ask for the locus of points upon which the wave displacement