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10 Why "Plane" Waves?

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where $c(\vec{k})$ are determined by the initial data (*exercise*). Physically, you can think of this integral formula as representing a (continuous) superposition of plane waves over their possible physical attributes. To see this, consider a plane wave of the form

$$q(\vec{r}, t) = Re\left[ce^{i(\vec{k}\cdot\vec{r}-\omega t)}\right], \tag{9.26}$$

where c is a complex number. You can check that this is a wave traveling in the direction of $\vec{k}$, with wavelength $\frac{2\pi}{k}$, and with amplitude $|c|$. The phase $\frac{c}{|c|}$ of the complex number c adds a constant to the phase of the wave *(exercise)*. The integral in (9.21) is then a superposition of waves in which one varies the amplitudes ($|c|$), relative phases ($c/|c|$), wavelengths ($2\pi/k$), and directions of propagation ($\vec{k}/k$) from one wave to the next.

Equivalently, every solution to the wave equation can be obtained by superimposing real plane wave solutions of the form

$$q(\vec{r}, t) = A\cos(\vec{k}\cdot\vec{r} - \omega t + \phi). \tag{9.27}$$

Here the (continuous) superposition takes place by varying the amplitude A, the wave vector $\vec{k}$ and the phase ϕ (*exercise*).

Exactly as we did for the space of solutions to the one-dimensional wave equation, we can view the space of solutions of the three-dimensional wave equation as a vector space (*exercise*). From this point of view, the plane waves form a basis for the vector space of solutions.

10. Why "plane" waves?

Let us now pause to explain in more detail why we called the elementary solutions (9.9) and (9.27) *plane waves.* The reason is that the displacement $q(\vec{r}, t)$ has the symmetry of a plane. To see this, fix a time t (take a "snapshot" of the wave) and pick a location $\vec{r}$. Examine the wave displacement q (at the fixed time) at all points in a plane that is (i) perpendicular to $\vec{k}$, and (2) passes through $\vec{r}$. The wave displacement will be the same at each point of this plane. To see this most easily, simply choose, say, the x-axis to be along the vector $\vec{k}$. The planes perpendicular to $\vec{k}$ are then parallel to the y-z plane. In these new coordinates the wave (9.27) takes the simple form (*exercise*)

$$q(\vec{r}, t) = A\cos(kx - \omega t + \phi). \tag{10.1}$$

Clearly, at a fixed t and x, $q(\vec{r}, t)$ is the same anywhere on the plane obtained by varying y and z.

A more formal — and perhaps more instructive — way to see the plane wave symmetry of (9.27) is to fix a time t and ask for the locus of points upon which the wave displacement is constant. At a fixed time, the wave displacement q is a function of 3 variables. As you know, the locus of points where a function takes the same values generically defines a surface. Since the spatial dependence of the plane wave is via the combination $\vec{k}\cdot\vec{r}$, when $t=$ constant the surfaces of fixed q are given by $\vec{k}\cdot\vec{r}=$ constant *(exercise)*. But the equation $k_x x + k_y y + k_z z =$ constant (with each of k_x, k_y, k_z a constant) is the equation for a plane *(exercise)*! This plane is everywhere orthogonal to the wave vector $\vec{k}$, which can be viewed as a constant vector field (*i.e.,* a vector field whose Cartesian components are the same everywhere). To see this, we recall from our discussion in §9 that the gradient of a function is always perpendicular to the surfaces upon which the function is constant. We just saw that the plane of symmetry (where q doesn't change its value) arises when the function $\vec{k}\cdot\vec{r}$ is constant. Thus, the plane of symmetry for the plane wave is orthogonal to the (constant) vector field

$$\nabla(\vec{k}\cdot\vec{r}) = \vec{k}. \tag{10.2}$$

The wave vector is thus normal to the planes of symmetry of a plane wave. As time evolves, the displacement profile on a given plane of symmetry moves along $\vec{k}$. In this way $\vec{k}$ determines the propagation direction of the plane wave. The wave vector thus determines the wavelength, the direction of motion, and the plane of symmetry of a plane wave.