Foundations of Wave Phenomena

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11 Separation of Variables

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11. Separation of variables.

There is yet another way to find the general solution to the wave equation which is valid in 1, 2, or 3 (or more!) dimensions. This method is quite important and, as we shall see, can often be used for other linear homogeneous differential equations. This technique for solving the wave equation is called the method of *separation of variables.*

We begin by assuming that the solution of the wave equation is a superposition of elementary solutions $q(\mathbf{r}, t)$ each of which can be expressed as a product:

$$q(\mathbf{r}, t) = T(t)X(x)Y(y)Z(z). \tag{11.1}$$

We shall show that each of these 4 functions satisfies a relatively simple ordinary differential equation. To this end, we insert the trial solution (11.1) into the wave equation to obtain (*exercise*):

$$\frac{1}{v^2}T''XYZ - TX''YZ - TXY''Z - TXYZ'' = 0, \tag{11.2}$$

where the prime on a function denotes a derivative with respect to the argument of that function, *e.g.*, $X' = \frac{dX(x)}{dx}$. Away from points where $q = 0$, we can divide both sides of this equation by q to get

$$\frac{1}{v^2}\frac{T''}{T} - \frac{X''}{X} - \frac{Y''}{Y} - \frac{Z''}{Z} = 0. \tag{11.3}$$

Note that each term in (11.3) is a function, respectively, of the *different* variables (t, x, y, z). Because of this property, we say that the wave equation is "separable", or "separates" in Cartesian coordinates. As we shall see, the property of being separable really refers not just to the equation, but also to the coordinate system employed. Typically, one can only expect linear, homogeneous partial differential equations to separate.

Because the variables are separated in (11.3), this equation represents a very stringent requirement; it amounts to saying that each of the functions T''/T, X''/X, Y''/Y, Z''/Z is in fact a constant! To see this, let us isolate the T''/T term on one side of the equation. We we then get an equation of the form

$$f(t) = g(x, y, z), \tag{11.4}$$

where we are denoting

$$f(t) = \frac{1}{v^2}\frac{T''}{T} \tag{11.5}$$

and

$$g(x, y, z) = \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z}. \tag{11.6}$$

Because we can vary x, y, z independently of t, (11.4) holds if and only if $f(t) = a = g(x, y, z)$, where a is a constant. To see this directly, simply take the partial derivative of (11.4) with respect to t to find $f' = 0$ (*exercise*).

So, at this point we know that for some constant a

$$\frac{1}{v^2} \frac{T''}{T} = a, \tag{11.7}$$

and

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = a. \tag{11.8}$$

But we can play the same game again! We write

$$\frac{X''}{X} = a - \frac{Y''}{Y} - \frac{Z''}{Z}, \tag{11.9}$$

and we see that both sides must equal a constant, say, b:

$$\frac{X''}{X} = b. \tag{11.10}$$

Applying this logic repeatedly, we conclude that (*exercise*)

$$\begin{aligned}
\frac{1}{v^2} \frac{T''}{T} &= a, \\
\frac{X''}{X} &= b, \\
\frac{Y''}{Y} &= c, \\
\frac{Z''}{Z} &= d,
\end{aligned} \tag{11.11}$$

where a, b, c, d are constants ("separation constants") and

$$a - b - c - d = 0, \quad i.e., \quad a = (b + c + d). \tag{11.12}$$

The single *partial* differential equation (9.7) has thus been turned into 4 *ordinary* differential equations (11.11) involving 3 arbitrary separation constants.

It is now a simple manner to solve the quartet (11.11) of linear, homogeneous ordinary differential equations for T, X, Y, Z. They are all of the form

$$f'' = -\alpha^2 f, \tag{11.13}$$

which is the harmonic oscillator equation again! This equation has (complex) solutions (*exercise*)

$$f(u) = \beta e^{\pm i \alpha u},$$

where where α and β are constants. To recover our plane wave "basis" for the general solution to the wave equation we set

$$b = -k_x^2, \quad c = -k_y^2, \quad d = -k_z^2, \tag{11.14}$$

and

$$a = -k_x^2 - k_y^2 - k_z^2 = -k^2. \tag{11.15}$$

Thus our separation of variables (complex) solution is a plane wave of the form

$$q(\mathbf{r}, t) = Ae^{ik_x x}e^{ik_y y}e^{ik_z z}e^{i\omega t} = Ae^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \tag{11.16}$$

where

$$\omega = \pm kv \tag{11.17}$$

as before. Thus the "separation constants" make up the frequency of the wave and the components of the wave vector.

We can build up the general solution to the wave equation by superposition of these elementary separation of variables solutions and arrive again at the Fourier expansions of §9 (*exercise*).