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13 Spherical Coordinates

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Recommended Citation

Torre, Charles G., "13 Spherical Coordinates" (2014). *Foundations of Wave Phenomena*. Book 10.
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13. Spherical Coordinates.

We can play similar games with another popular coordinate system: *spherical coordinates* (also called "spherical polar coordinates"). These coordinates are denoted (r, θ, ϕ) and are defined by

$$r = \sqrt{x^2 + y^2 + z^2} \tag{13.1}$$

$$\theta = \cos^{-1}(\frac{z}{r}) \tag{13.2}$$

$$\phi = \tan^{-1}(\frac{y}{x}). \tag{13.3}$$

Note that $r > 0$, $0 < \theta < \pi$, and $0 \leq \phi < 2\pi$. Careful! Spherical polar coordinates are not defined on the z-axis (*exercise*). A point labeled (r, θ, ϕ) has Cartesian coordinates (*exercise*)

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta. \end{aligned} \tag{13.4}$$

The spherical coordinates of a point p can be obtained by the following geometric construction. The value of r represents the distance from the point p to the origin (which you can put wherever you like). The value of θ is the angle between the positive z-axis and a line l drawn from the origin to p. The value of ϕ is the angle made with the x-axis by the projection of l into the x-y plane ($z = 0$). Note: for points in the x-y plane, where $\theta = \pi/2$, r and ϕ (not θ) are polar coordinates. The coordinates (r, θ, ϕ) are called the *radius, polar angle,* and *azimuthal angle* of the point p, respectively. It should be clear why these coordinates are called spherical. The points $r = a$, with $a =$ constant, lie on a sphere of radius a about the origin. Note that the angular coordinates can thus be viewed as coordinates on a sphere. Indeed, they label latitude and longitude (*exercise*).

It should be mentioned that many texts use a different labeling scheme for spherical coordinates in which the roles of θ and ϕ are interchanged. The convention being used here is found in most of the physics literature. The other convention is most common in mathematics texts. To be fair, the mathematicians' convention is a little more logical since the normal notation for polar coordinates in the x-y plane is (r, θ).

Spherical coordinates are, of course, particularly useful when studying wave phenomena exhibiting spherical symmetry. For example, the sound waves emitted by an exploding firework shell can be modeled as spherically symmetric with respect to an origin at the explosion's location. In other words, the compression/rarefaction of air at a point only depends on the distance from the point to the explosion, not on the angular location of the point relative to the explosion (ignoring obstacles, of course). Similarly, the light emitted from "point sources" (*i.e.,* sources which are sufficiently small compared to the distance

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to the points of interest) is best studied using spherical polar coordinates centered on the light source.

13.1 The Wave Equation in Spherical Coordinates

How do we find solutions to the wave equation in spherical coordinates? You might be able to guess how we are going to proceed: express the wave equation in spherical coordinates for a function $q(r, \theta, \phi, t)$ and solve by separation of variables. We will not go through

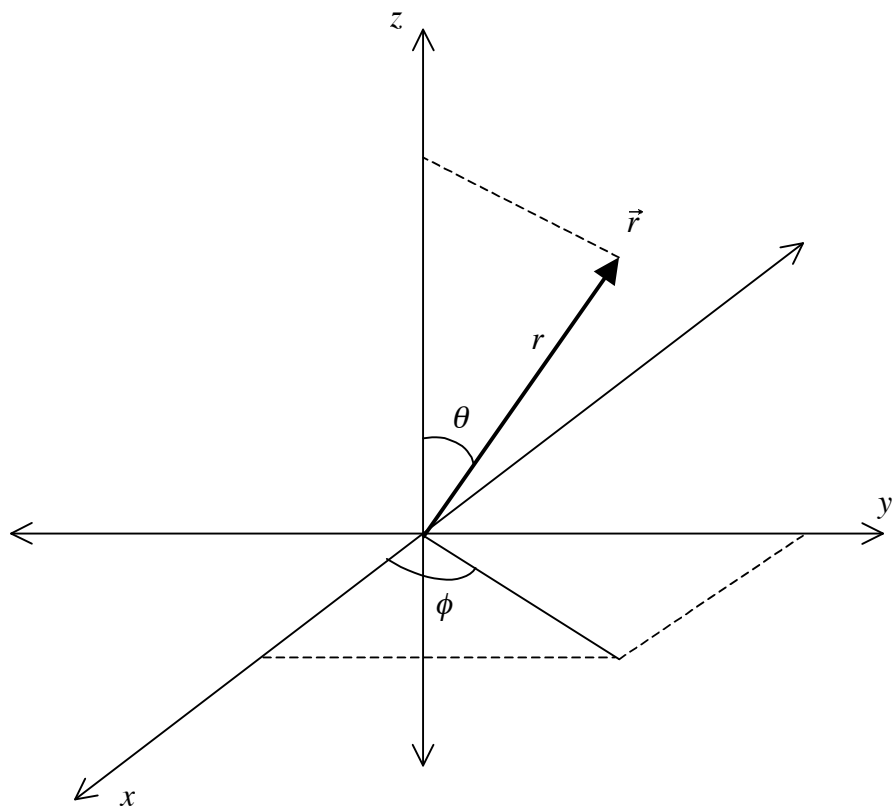

Figure 17. Illustration of spherical coordinates r , θ , and ϕ .

the chain rule derivation of the wave equation in spherical coordinates. It is the same kind of calculation we did for cylindrical coordinates. Here are the results. The Laplacian in

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spherical coordinates is

$$\nabla^2 f(r,\theta,\phi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}, \tag{13.5}$$

so the wave equation in spherical coordinates takes the form

$$\frac{1}{v^2}\frac{\partial^2}{\partial t^2}q(r,\theta,\phi,t) = \nabla^2 q(r,\theta,\phi,t), \tag{13.6}$$

where the Laplacian is given by (13.5) for each time t. Given the form of the Laplacian, this equation certainly looks formidable, but it can be converted into 4 tractable ordinary differential equations by separation of variables.

13.2 Separation of Variables in Spherical Coordinates

To solve (13.6) using the method of separation of variables we (i) assume the solution is of the form

$$q(r,\theta,\phi,t) = R(r)\Theta(\theta)\Phi(\phi)T(t), \tag{13.7}$$

(ii) substitute (13.7) into the wave equation, and (iii) divide the resulting equation by q. We obtain (*exercise*)

$$\frac{1}{v^2}\frac{T''}{T} = \frac{1}{r^2 R}\left(r^2 R'\right)' + \frac{1}{r^2\sin\theta}\frac{1}{\Theta}\left(\sin\theta\Theta'\right)' + \frac{1}{r^2\sin^2\theta}\frac{\Phi''}{\Phi}. \tag{13.8}$$

We now perform the familiar separation of variables analysis. Right away we see that there is a constant k such that (*exercise*)

$$T'' = -v^2k^2T \tag{13.9}$$

and hence the complex form of the solution for T is of the form (*exercise*)

$$T = Ae^{\pm ivkt}, \tag{13.10}$$

as before. This is not too surprising; wave phenomena are always characterized by harmonically varying displacement in time—this cannot be affected by a choice of spatial coordinates.

We now continue the analysis to see how to characterize the spatial dependence of the waves. If we use (13.9) in (13.8) and multiply both sides by $r^2\sin^2\theta$, the usual logic implies that there is a constant, denoted by a, such that (*exercise*)

$$\Phi'' = -(a^2+k^2)\Phi, \tag{13.11}$$

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and we solve this via

$$\Phi = Be^{\pm im\phi}, \quad m = 0, 1, 2, \ldots, \tag{13.12}$$

where

$$m^2 = a^2 + k^2. \tag{13.13}$$

Note that we have restricted m to integer values so that q is well-defined. Now we are left with

$$\frac{1}{r^2 \sin\theta} \frac{1}{\Theta} \left(\sin\theta\Theta'\right)' - \frac{m^2}{\sin^2\theta} = -\frac{1}{R}\left(r^2 R'\right)' - k^2 r^2. \tag{13.14}$$

Again, each side must equal a constant, which for later convenience is taken to be of the form $-l(l+1)$ for a constant l. We thus get an equation for Θ:

$$\frac{1}{\sin\theta}\left(\sin\theta\Theta'\right)' + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0, \tag{13.15}$$

and an equation for R:

$$\left(r^2 R'\right)' + \left[k^2 r^2 - l(l+1)\right] R = 0. \tag{13.16}$$

Notice that the equation (13.15) for Θ depends on l and m, but not k, while the equation (13.16) for R depends on l and k but not m.

Once again we have reduced the wave equation to 4 ordinary differential equations. We could easily solve two of the equations (for T and Φ), but the equations for Θ and R are a little more complicated.

The equation (13.15) for Θ defines, for each l and m, a "special function" somewhat analogous to the cosine, sine, or Bessel function. This new special function is called the *associated Legendre function*, denoted by $P_{lm}(\theta)$. While we won't go into the details here, an important result of analyzing the equation for Θ is that (i) the constant l must be a non-negative integer:

$$l = 0, 1, 2, ... \tag{13.17}$$

and (ii) for a given l, the allowed values of m are

$$m = -l, -l+1, ..., -1, l-1, 0, 1, ..., l. \tag{13.18}$$

The restrictions on l and m are needed for the solution to be well-defined at $\theta = 0, \pi$.* This result is the motivation for defining the angular coordinate separation constant in the

* As you can easily see, because spherical polar coordinates are not defined on the z axis the Laplacian in those coordinates is not defined there either. Strictly speaking, then, the equation can only be solved in the open interval $\theta \in (0, \pi)$ and the mathematics has no way of "knowing" that the solutions should exist at the endpoints of this interval. When we insist upon this we get the restriction on l and m. Requiring regularity at $\theta = 0, \pi$ also rejects another set of linearly independent solutions from consideration. (Generally speaking, there will be two linearly independent solutions to a second order ordinary differential equation.)

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form $l(l+1)$. We shall not try to prove the results (13.17) and (13.18); you will probably see a proof in a more advanced course.

Some examples of associated Legendre functions (with convenient normalizations) are

$$P_{00} = 1, \quad P_{10} = \cos\theta, \quad P_{11} = \sin\theta, \quad P_{20} = \frac{1}{2}(3\cos^2\theta - 1), \quad P_{2,1} = 3\sin\theta\cos\theta. \tag{13.19}$$

As a nice *exercise* you should verify that these functions do solve the equation for Θ with the indicated values of l and m. A general formula is obtained from

$$P_{lm}(x) = \frac{1}{2^l l!}(1-x^2)^{|m|/2}\left(\frac{d}{dx}\right)^{|m|+l}(x^2-1)^l \tag{13.20}$$

where $x = \cos\theta$.

For each l, the equation (13.16) for R — the "radial equation" — has solutions called *spherical Bessel functions* $R = j_l$ and *spherical Neumann functions* $R = n_l$. Note that these solutions depend upon the choice of k and l. All of these fancy special functions are well-studied and have well-understood properties. You can find a derivation of these solutions and discussion of their features in an upper level text. We will content ourselves with exhibiting a few of the spherical Bessel solutions and a general formula. The spherical Neumann functions are not defined at the origin (where they "become infinite") so we will not bother with them here.† The first 3 spherical Bessel functions are

$$j_0 = \frac{\sin kr}{kr}, \quad j_1 = \frac{\sin kr}{(kr)^2} - \frac{\cos kr}{kr}, \quad j_2 = \left(\frac{3}{(kr)^3} - \frac{1}{kr}\right)\sin kr - \frac{3}{(kr)^2}\cos kr. \tag{13.21}$$

You should check that these are solutions to the radial equation as a nice *exercise*. With $x = kr$ a general formula is

$$j_l(x) = (-x)^l\left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\sin x}{x}. \tag{13.22}$$

13.3 Some Simple Solutions to the Wave Equation in Spherical Coordinates

The simplest solution to the wave equation in spherical coordinates is obtained by setting $l = 0$ in the separation of variables solution. When $l = 0$ it follows that $m = 0$ (*exercise*) and it is easy to verify that in this case both Θ and

† As in the cylindrically symmetric case, the spherical Neumann functions are useful when considering solutions to the wave equation which do not include the origin.

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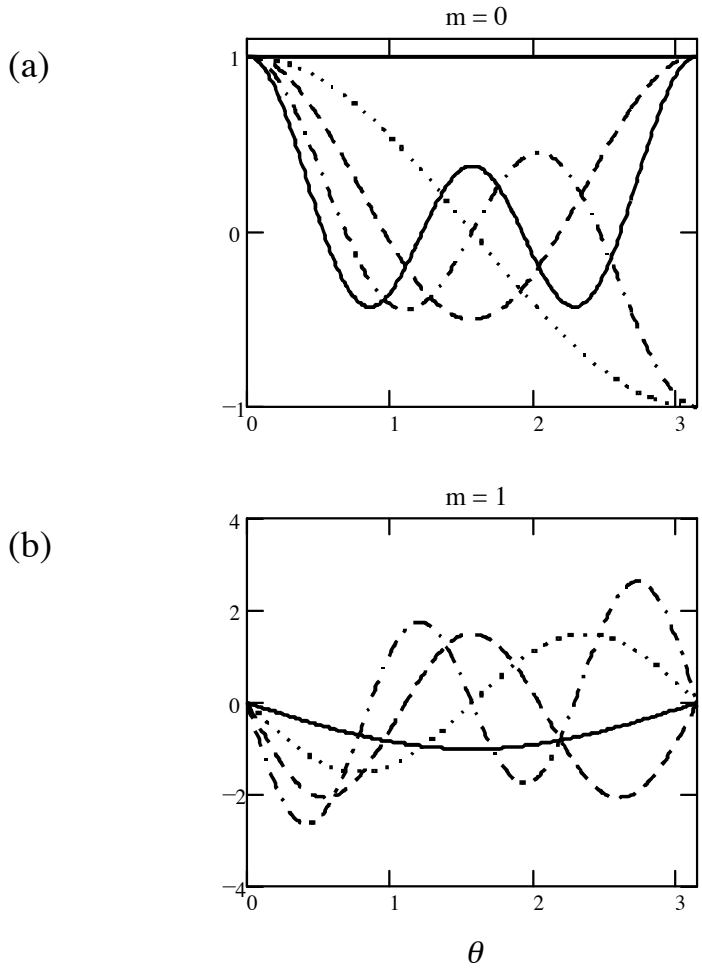

Figure 18. Some generalized Legendre functions. (a) five lowest order functions (in l) for $m = 0$. (b) four lowest order functions (in l) for $m = 1$.

Φ are constants (*exercise*). Evidently, by setting $l = m = 0$ we are selecting *spherically symmetric* solutions. The radial function in this case is given by the spherical Bessel function

$$R(r) = \frac{\sin kr}{kr}. \tag{13.23}$$

The other independent solution to the radial equation is the $l = 0$ spherical Neumann function, which is given by

$$R(r) = \frac{\cos kr}{kr}. \tag{13.24}$$

You can check that both of these radial functions solve the radial equation when $l = 0$. You can easily check as an *exercise* that the spherical Bessel function is well-behaved as $r \to 0$ (indeed, $\lim_{r\to 0} R(r) = 1$) while the spherical Neumann function is not defined as

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$r \to 0$, as we mentioned earlier. Since we are considering solutions valid everywhere in space, we shall not consider the spherical Neumann function solution any further.*

The function (13.23) is sinusoidal in r with decreasing amplitude as r grows (much like the ordinary Bessel function). Note that $R(0) = 1$ by l'Hospital's rule (*exercise*). The spherically symmetric solution to the wave equation thus takes the form of a sinusoidal profile in r with a harmonic time dependence. We have (*exercise*)

$$q(r, \theta, \phi, t) = A \cos(\omega t + \alpha) \frac{\sin kr}{kr}, \tag{13.25}$$

where

$$\omega = kv. \tag{13.26}$$

The solution (13.25) is *spherically symmetric*, which means that at each time t the solution is the same everywhere on any sphere $r = const.$, as you can easily see from the fact that the solution is independent of θ and ϕ. Physically, the solution (13.25) can be viewed as representing the displacement (compression) of air due to a source at $r = 0$ emitting sound at a fixed frequency $\omega/2\pi$. The decrease of the amplitude at large distances represents the decreasing intensity of the emitted sound, which is required by conservation of energy (see §14).

As another example, let us suppose that $l = 1$, $m = 0$ so that (*exercise*)

$$q(r, \theta, \phi, t) = A \cos(\omega t + \alpha) \cos\theta \left[\frac{\sin kr}{(kr)^2} - \frac{\cos kr}{kr} \right]. \tag{13.27}$$

In this case the solution depends upon r and θ but not ϕ. Such solutions are called *azimuthally symmetric* (or *axially symmetric*). As an *exercise* you should verify that this solution is well-defined at $r = 0$.

Exercise: Devise an example of a source of sound which would yield (at least ideally) azimuthally symmetric waves.

* If we consider solutions in a region not including the origin, then the solution to the wave equation with $l = 0$ is a superposition of the spherical Bessel and Neumann functions.

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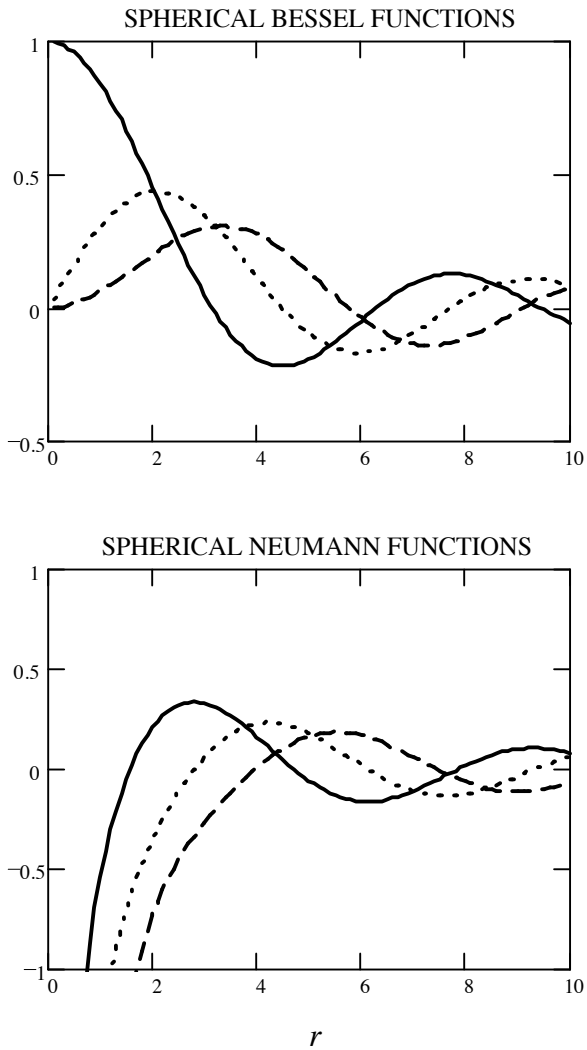

Figure 19. Spherical Bessel and spherical Neumann functions. In each graph the functions for $l = 0, 1,$ and 2 are shown. Note that the spherical Neumann functions are all singular at the origin while the spherical Bessel functions remain finite.

We will not elaborate this point, but once again it can be shown that the general solution of the wave equation can be obtained by superpositions of the separation of variables solutions over all values of k, l, m. Note that this involves (i) an integral over k from 0 to ∞, (ii) a sum over l from 0 to ∞ , and (iii) for each l, a sum over m from $-l$ to l. Thus the separation of variables solutions form a basis for the vector space of solutions to the wave equation.

Finally, we mention that various kinds of boundary conditions can be handled using

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separation of variables. For boundary conditions which are imposed on surfaces where one of the coordinates is constant, *e.g.,* on a sphere centered at the origin, one simply imposes the boundary conditions on the solutions to the relevant ordinary differential equation (see the Problems for an example). This will, in general, limit the range of the separation constants and/or the allowed superpositions.

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