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## 16 The Curl

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## 16. The Curl.

In §17–§20 we will study the underpinnings of the propagation of light waves, radio waves, microwaves, *etc.* All of these are, of course, examples of electromagnetic waves, just differing in their wavelength. The (non-quantum) description of all known electromagnetic phenomena is provided by the *Maxwell equations*. These equations are normally presented as differential equations for the electric field  $\vec{E}(\vec{r}, t)$  and the magnetic field  $\vec{B}(\vec{r}, t)$ . You may have been first introduced to them in an equivalent integral form. In differential form, the Maxwell equations involve the divergence operation, which we mentioned before, and another vector differential operator, known as the *curl*. In preparation for our discussion of electromagnetic waves, we explore this vector differential operator in a little detail.

### 16.1 Vector Fields

Like the divergence, the curl operates on a *vector field*. To begin, recall that a vector field is different from what one usually thinks of as simply a “vector”. A vector is an arrow. A vector has magnitude and direction. A vector is an ordered set of 3 numbers, *etc.* More abstractly, a vector is an element of a vector space (see Appendix B). A *vector field* is an assignment of a vector to each point of space (and instant time). You pick the location, and the vector field gives you an arrow at that location. So a vector field is really an infinite collection of vectors. The electric and magnetic fields are examples of vector fields, although one sometimes gets lazy and simply calls them vectors.

Let  $\vec{V}$  be a vector field in 3-dimensional space. Thus, at each point  $\vec{r}$  we have assigned a vector, denoted by  $\vec{V}(\vec{r})$ . We can break the vector field into its Cartesian components

$$\vec{V}(\vec{r}) = V^x(\vec{r})\hat{x} + V^y(\vec{r})\hat{y} + V^z(\vec{r})\hat{z} \quad (16.1)$$

at  $\vec{r}$ . Because these components will, in general, vary with the choice of  $\vec{r}$  the *components* of  $\vec{V}$ , *i.e.*,  $V^x(\vec{r})$ ,  $V^y(\vec{r})$ ,  $V^z(\vec{r})$ , are functions. This is a signal that we are dealing with a vector field rather than a vector.\* Perhaps you should keep in mind the analogous case with a function (also called a “scalar field”). A function  $f$  assigns a number  $f(\vec{r})$  (or scalar) to each point  $\vec{r}$ , just as a vector field  $\vec{V}$  assigns a vector  $\vec{V}(\vec{r})$  to each point  $\vec{r}$ .

Since the components of a vector field are allowed to be functions, the possibility arises for using operations of differentiation and integration when dealing with vector fields. We

\* It is possible to have a *constant vector field*, in which the Cartesian components happen to be constants. A constant vector field is completely determined by a single vector, namely, the value of the constant vector field at any one point. This allows the (somewhat confusing) custom in which one treats a constant vector field as an ordinary vector. The familiar unit vectors  $\hat{x}$ ,  $\hat{y}$ ,  $\hat{z}$  are examples of vectors which are also used as (constant) vector fields.

have already seen some examples: the divergence and the divergence theorem. Let us now look at another important differential operation that can be applied to vector fields.

### 16.2 The Curl

The curl is a linear differential operator that takes a vector field and produces a new vector field. It is defined as follows. Let  $\vec{V}$  be a vector field, as given in (16.1). The curl of  $\vec{V}$ , denoted by  $\nabla \times \vec{V}$ , is defined as that vector field whose Cartesian components are given by

$$\nabla \times \vec{V} = \left( \frac{\partial V^z}{\partial y} - \frac{\partial V^y}{\partial z} \right) \hat{x} + \left( \frac{\partial V^x}{\partial z} - \frac{\partial V^z}{\partial x} \right) \hat{y} + \left( \frac{\partial V^y}{\partial x} - \frac{\partial V^x}{\partial y} \right) \hat{z}. \quad (16.2)$$

Just as with divergence and gradient (see (9.3) and (9.4)), some texts denote the curl using “curl”:

$$\nabla \times \vec{V} \equiv \text{curl } \vec{V}. \quad (16.3)$$

### 16.3 Why is the Curl Formula So Complicated?

At this point, there are (at least) two questions you may have. First, why is this rather complicated formula (16.2) useful? And second, how can you remember a formula like this? The second question is easiest, so we start with it. You will note that the formula is very much like the cross product, where

$$\vec{A} \times \vec{B} = (A^y B^z - A^z B^y) \hat{x} + (A^z B^x - A^x B^z) \hat{y} + (A^x B^y - A^y B^x) \hat{z}. \quad (16.4)$$

The pattern of terms is the same in the cross product and the curl: we have *cyclic permutations* of  $xyz$ , that is

$$xyz \longrightarrow yzx \longrightarrow zxy.$$

Have a look; you really only have to remember one component of the formula, say, the  $x$ -component and then use the cyclic permutation rule (*exercise*). I think you can see why people use the notation  $\nabla \times$  for the curl.

Of all the possible things one could write down, why the curl? Well, you could also have asked this about the gradient or the divergence. The answer comes in several layers. First of all, consider the options: we can of course define any differential operator we want, *e.g.*, we could make up an operator called  $\$$  which is defined via

$$\$ \vec{A} = \frac{\partial A^y}{\partial z} \hat{x}.$$

(We won’t dignify this silly formula with an equation number.) This formula certainly defines a vector field by differentiating the vector field  $\vec{A}$ , but the result carries no particularly interesting information; indeed, it carries a lot of useless information. For example,

this vector  $\vec{A}$  always points along the  $x$ -axis, but the orientation of the  $x$ -axis in space is arbitrary. Thus the output of our fictitious operator  $\$$  is rather arbitrary and hence not particularly useful. Differential operators like gradient, divergence, and curl are singled out because they capture intrinsic, coordinate independent information about the objects they differentiate. Thus, the gradient of a function is a vector field that can be used to compute the rate of change of a function in any direction (via the directional derivative). In addition, the gradient has the nice geometrical interpretation as being orthogonal to the level surfaces of the function. The gradient is essentially the only linear differential operator one can apply to a function that yields a vector field and does not depend on extraneous information — just the function. Similarly, the divergence and curl capture useful information about the vector field they act on — and nothing else. Although we won't prove it here, they are essentially unique as linear differential operators that act on a vector field and give a coordinate independent result. But what is the meaning of that result?

#### 16.4 Geometric Meaning of the Curl

It is far from obvious from (16.2) what the curl actually tells you about the vector field. We can give (without proof) an interpretation of the curl which is analogous to that which we gave for the divergence. To do this we need to revisit something you should have seen before: the line integral. Let us pause to define it.

Given a vector field  $\vec{V}$ , choose a closed curve  $C$ . We can now define the line integral or “circulation” of  $\vec{V}$  around  $C$ . This integral is obtained by taking the dot product of  $\vec{V}$  with the tangent to  $C$  at each point on the curve and then adding up (well, integrating) the results around the curve. If you prefer formulas, you can compute the circulation by writing the curve parametrically as

$$x = f(t), \quad y = g(t), \quad z = h(t), \quad t \in [a, b]. \quad (16.5)$$

The unit tangent vector  $\hat{T}$  has components

$$T^x = \frac{1}{N} \frac{df}{dt}, \quad T^y = \frac{1}{N} \frac{dg}{dt}, \quad T^z = \frac{1}{N} \frac{dh}{dt}, \quad (16.6)$$

where

$$N = \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2}. \quad (16.7)$$

The line element for the curve is

$$d\vec{l} = \hat{T} N dt. \quad (16.8)$$

The circulation is then the integral

$$\oint_C \vec{V} \cdot d\vec{l} = \int_a^b dt \left( V^x \frac{df}{dt} + V^y \frac{dg}{dt} + V^z \frac{dh}{dt} \right). \quad (16.9)$$

Evidently, the circulation measures how much the vector field is, well, circulating — or “curling” — in the direction of the curve. Now consider filling in the closed curve with a surface  $S$ , *i.e.*, consider a surface  $S$  which has  $C$  as its boundary.† (You have certainly used this idea when studying Faraday’s and Ampere’s law.) Each point of the surface will have a normal vector. Consider shrinking  $C$  and  $S$  to infinitesimal size around some point  $p$ . The component of the curl along the normal at  $p$  is the circulation per unit area in the limit as the area is made to vanish. It turns out that the circulation doesn’t depend upon how you fill in the surface. So, if you like, you can interpret any desired component of the curl of  $\vec{V}$  as the circulation per unit area of  $\vec{V}$  around the boundary of a very small circular disk whose normal is in the direction of the desired component.

There is an important theorem which captures this interpretation of the curl even when the curve is not becoming small — it is known as *Stokes’ Theorem*. Once again consider a closed curve  $C$  bounding a surface  $S$  with unit normal  $\hat{n}$  and area element  $dS$ . Define

$$d\vec{S} = \hat{n}dS. \quad (16.10)$$

Given a vector field  $\vec{V}$ , its curl is another vector field  $\nabla \times \vec{V}$ . Stokes’ Theorem equates the flux of  $\nabla \times \vec{V}$  through  $S$  to the circulation of  $\vec{V}$  around  $C$ :

$$\int_S (\nabla \times \vec{V}) \cdot d\vec{S} = \oint_C \vec{V} \cdot d\vec{l}. \quad (16.11)$$

If you consider the limit in which the surface shrinks to zero area you can more or less see from this formula how to recover the interpretation of the curl we gave in the previous paragraph.

### 16.5 Some Important Identities for Div, Grad, Curl

There are some very important identities relating the gradient, divergence and curl that we shall need, some of which you will verify in a homework problem. In the following  $\vec{V}$  is an arbitrary vector field and  $f$  is an arbitrary function. Our principal identities are:

$$\nabla \cdot (\nabla \times \vec{V}) = 0, \quad (16.12)$$

$$\nabla \times \nabla f = 0, \quad (16.13)$$

We see from (16.12) that if a vector field arises as the curl of another vector field, then its divergence is zero. Note, though, that not every vector field can be written as a curl,

† There is more than one way to do this — think of how you might fill in a circle. For example, you could fill it with a flat disk, or with a round hemisphere. As it happens, the circulation and the resulting interpretation of the curl are independent of how you fill in the surface.

otherwise every vector field would have vanishing divergence — a simple counterexample being the vector field (*exercise*):

$$\vec{V} = x\hat{x}. \quad (16.14)$$

We see from (16.13) that if a vector field arises as the gradient of a function, then its curl is zero. Note though, not every vector field can be expressed as a gradient of a function, otherwise every vector field would have vanishing curl — a simple counterexample being the vector field (*exercise*):

$$\vec{V} = y\hat{x} - x\hat{y}. \quad (16.15)$$

Normally, one can assume the converse to the results above – at least locally; see the *Note* below. So, if a vector has vanishing divergence, then it can be written as a curl (at least locally). What this means is that if a vector field  $\vec{v}$  satisfies in some region

$$\nabla \cdot \vec{v} = 0, \quad (16.16)$$

then there exists a vector field  $\vec{w}$  such that

$$\vec{v} = \nabla \times \vec{w}, \quad (16.17)$$

possibly in some sub-region. Because of (16.13) the vector field  $\vec{w}$  cannot be unique. Indeed we can take any function  $f$  and redefine  $\vec{w}$  via

$$\vec{w}' = \vec{w} + \nabla f \quad (16.18)$$

and still get (*exercise*)

$$\vec{v} = \nabla \times \vec{w}'. \quad (16.19)$$

Likewise, if a vector field  $\vec{w}$  has vanishing curl,

$$\nabla \times \vec{w} = 0, \quad (16.20)$$

then it must be a gradient, *i.e.*, (at least locally) there is a function  $f$  such that

$$\vec{w} = \nabla f. \quad (16.21)$$

The function  $f$  is unique up to an additive constant (*exercise*).

*Note:* Equations (16.12) and (16.13) always hold — they're identities. The converse results just described are only guaranteed to hold in regions of space free of any “holes”. Moreover, these latter converse results are not guaranteed to be compatible with any boundary conditions which may be imposed on the vector fields.

There is one more identity that we shall need. It involves the double-curl of a vector field:

$$\nabla \times (\nabla \times \vec{V}) = \nabla(\nabla \cdot \vec{V}) - \nabla^2 \vec{V}. \quad (16.22)$$

To use this formula you should use Cartesian components ( $V^x, V^y, V^z$ ) with the Laplacian of a vector field being computed component-wise, that is,

$$(\nabla^2 \vec{V})^x = \nabla^2(V^x),$$

and so forth. If a vector field has vanishing divergence, then any given component (*e.g.*, the  $x$  component) of its double-curl is just the Laplacian on that component of  $\vec{V}$  (*e.g.*,  $V^x$ ).

## 17. Maxwell Equations.

With our brief review of vector analysis out of the way, we can now discuss the Maxwell equations. We will use the Gaussian system of electromagnetic units and we will let  $c$  denote the speed of light in vacuum. The Maxwell equations are differential equations for the electric field  $\vec{E}(\vec{r}, t)$ , and the magnetic field  $\vec{B}(\vec{r}, t)$ , which are defined by the force they exert on a test charge  $q$  at the point  $\vec{r}$  at time  $t$  via the *Lorentz force law*:

$$\vec{F}(\vec{r}, t) = q \left( \vec{E}(\vec{r}, t) + \frac{1}{c} \vec{v}(t) \times \vec{B}(\vec{r}, t) \right), \quad (17.1)$$

where  $\vec{v}(t)$  is the particle's velocity at time  $t$ . Equation (17.1) is used to determine the motion of a charged particle in a given electromagnetic field assuming the effect of the particle on the field can be neglected. Equation (17.1) can also be used to measure the electromagnetic field by observing the motion of charged “test” particles.

The Lorentz force law tells us how the electromagnetic field affects electrically charged matter. The Maxwell equations tell us how the electrically charged matter affects the electromagnetic field. In macroscopic applications it is usually convenient to model the electric charges – the “sources” – of the electromagnetic field as a continuous electric charge density  $\rho(\vec{r}, t)$  and electric current density  $\vec{j}(\vec{r}, t)$ . (You may now note that we are anticipating with our notation that  $\rho$  and  $\vec{j}$  will satisfy a continuity equation corresponding to conservation of electric charge.) The Maxwell equations are

$$\nabla \cdot \vec{E} = 4\pi\rho, \quad (17.2)$$

$$\nabla \cdot \vec{B} = 0, \quad (17.3)$$

$$\nabla \times \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c} \vec{j}, \quad (17.4)$$