17 Maxwell Equations

Charles G. Torre
Department of Physics, Utah State University, charles.torre@usu.edu

There is one more identity that we shall need. It involves the double-curl of a vector field:

$$\nabla \times (\nabla \times \vec{V}) = \nabla(\nabla \cdot \vec{V}) - \nabla^2 \vec{V}. \tag{16.22}$$

To use this formula you should use Cartesian components (V^x, V^y, V^z) with the Laplacian of a vector field being computed component-wise, that is,

$$(\nabla^2 \vec{V})^x = \nabla^2(V^x),$$

and so forth. If a vector field has vanishing divergence, then any given component (*e.g.,* the x component) of its double-curl is just the Laplacian on that component of $\vec{V}$ (*e.g.,* V^x).

17. Maxwell Equations.

With our brief review of vector analysis out of the way, we can now discuss the Maxwell equations. We will use the Gaussian system of electromagnetic units and we will let c denote the speed of light in vacuum. The Maxwell equations are differential equations for the electric field $\vec{E}(\vec{r}, t)$, and the magnetic field $\vec{B}(\vec{r}, t)$, which are defined by the force they exert on a test charge q at the point $\vec{r}$ at time t via the *Lorentz force law*:

$$\vec{F}(\vec{r}, t) = q\left(\vec{E}(\vec{r}, t) + \frac{1}{c}\vec{v}(t) \times \vec{B}(\vec{r}, t)\right), \tag{17.1}$$

where $\vec{v}(t)$ is the particle's velocity at time t. Equation (17.1) is used to determine the motion of a charged particle in a given electromagnetic field assuming the effect of the particle on the field can be neglected. Equation (17.1) can also be used to measure the electromagnetic field by observing the motion of charged "test" particles.

The Lorentz force law tells us how the electromagnetic field affects electrically charged matter. The Maxwell equations tell us how the electrically charged matter affects the electromagnetic field. In macroscopic applications it is usually convenient to model the electric charges – the "sources" – of the electromagnetic field as a continuous electric charge density $\rho(\vec{r}, t)$ and electric current density $\vec{j}(\vec{r}, t)$. (You may now note that we are anticipating with our notation that ρ and $\vec{j}$ will satisfy a continuity equation corresponding to conservation of electric charge.) The Maxwell equations are

$$\nabla \cdot \vec{E} = 4\pi\rho, \tag{17.2}$$

$$\nabla \cdot \vec{B} = 0, \tag{17.3}$$

$$\nabla \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{j}, \tag{17.4}$$

© C. G. Torre

$$\nabla \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0. \tag{17.5}$$

Our goal is to see how on earth it is possible to find wavelike phenomena coming from these equations. But first it is worth pausing to get a feel for the basic features of these equations.

17.1 The Basic Structure of the Maxwell Equations

First of all, the equations (17.2)–(17.5) are 8, coupled, first-order, partial differential equations (with constant coefficients in Cartesian coordinates) for the 6 unknown functions contained in the components of $\vec{E}$ and $\vec{B}$. One usually views the Maxwell equations as equations which are used to determine $\vec{E}$ and $\vec{B}$ for a given ρ and $\vec{j}$. In this setting the equations for $\vec{E}$ and $\vec{B}$ are "linear-inhomogeneous" thanks to the "source terms" defined by ρ and $\vec{j}$. Because the equations are inhomogeneous, it is not possible to superimpose solutions $(\vec{E}_1, \vec{B}_1)$ and $(\vec{E}_2, \vec{B}_2)$ to get new solutions without altering the charge and current densities *(exercise)*. On the other hand, given any solution to the Maxwell equations (for a given ρ and $\vec{j}$) one can add any solution of the homogeneous Maxwell equations (where $\rho = 0 = \vec{j}$) to get a new solution of the inhomogeneous equations (*Exercise:* Prove this.) As a special case of this last property, if one is solving the Maxwell equations in a region of space where $\rho = 0$ and $\vec{j} = 0$, then the equations are homogeneous and one *can* superimpose solutions.

The equations (17.2) and (17.3) represent 2 "scalar" equations, while equations (17.4) and (17.5) are "vector equations". A vector equation equates the components of two vectors. Thus the equations (17.4) and (17.5) each represent 3 (coupled) equations in which the x component of the left-hand side is equated to the x component of the right hand side, and so on.

Usually, the Maxwell equations, as presented above, are meant to be solved for $\vec{E}$ and $\vec{B}$ once the charge density and its motion (the current density) are specified. For example, one can let the charge density be that of a uniform ball of positive charge held fixed in space so that the current density vanishes. As you might guess, the solution of these equations has vanishing magnetic field and a Coulomb-type electrostatic field outside the ball. (Do you remember what happens inside the ball?) Note that this way of using the Maxwell equations assumes that the motion of the sources is completely known (or else, how could we specify ρ and $\vec{j}$?). For many purposes this is a reasonable physical assumption. But, strictly speaking, this way of describing electrodynamics is at best an approximate description. As you can imagine, many applications (*e.g.,* the electrodynamics of the ionosphere) will require us to also figure out how the sources are moving. This is a complicated problem and quite non-linear: the sources generate the electromagnetic field according to the Maxwell equations (17.2)–(17.5), the electromagnetic field affects the

© C. G. Torre

sources according to the Lorentz force law (17.1), but the motion of the charges determines the fields, *etc.* Needless to say, we will be content to study the case where the motion of the sources is prescribed, that is, explicitly given.

Note that only 4 of the 8 Maxwell equations, (17.2) and (17.4), involve the sources. These are often called the *inhomogeneous equations* because they are linear inhomogeneous in the unknowns. The other 4 which do not involve the sources, (17.3) and (17.5), are likewise known as the *homogeneous equations*. The inhomogeneous equation involving ρ shows that the charge density gives a divergence to the electric field. This is reasonable: electric charges create electric fields; this Maxwell equation tells what part of the vector field $\vec{E}$ is affected by charges. The last Maxwell equation (17.3), on the other hand, shows that the magnetic field never has a divergence. By analogy with the electric field, this equation can be viewed as stating that there is no such thing as "magnetic charge". The magnetic field can have a curl however, (17.4), and this arises from either a time varying electric field or from a current density (moving charges). Thus a moving charge creates a magnetic field and, as Maxwell first postulated, so does a time varying electric field. Finally, from (17.5), the electric field can also have a curl, but only if there is a time varying magnetic field—a phenomenon characterized by Faraday and one which is necessary for existence of civilization as we know it.

Note also that only 6 of the 8 equations, (17.4) and (17.5) involve a time derivative, that is, only 6 equations concern themselves with how the fields change in time. For this reason these equations are called the *evolution equations*. The remaining two divergence equations are called *constraint equations* since they restrict the fields at any given time. It can be shown that the constraint equations only need to be satisfied once, *i.e.,* at a single instant of time; the evolution equations will guarantee they will be satisfied at later times. This is important because otherwise we would be in danger of having too many equations (8) and not enough unknowns (6).

17.2 Continuity Equation and Conservation of Charge

We now consider an important consistency condition that must be satisfied by the sources if the Maxwell equations are to admit any solution at all. Besides being an important feature of the equations, this condition follows from a nice manipulation of vector differentiation. This consistency condition says that a necessary condition for the existence of a solution $(\vec{E}, \vec{B})$ to the Maxwell equations is that the sources $(\rho, \vec{j})$ must satisfy a continuity equation.

Take the time derivative of (17.2) and interchange time and space derivatives to get (*exercise*)

$$\nabla \cdot \frac{\partial \vec{E}}{\partial t} = 4\pi \frac{\partial \rho}{\partial t}. \tag{17.6}$$

© C. G. Torre

Compare this result with the divergence of (17.4) (*exercise*):

$$-\frac{1}{c}\nabla \cdot \frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\nabla \cdot \vec{j},\tag{17.7}$$

to find (*exercise*)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0.\tag{17.8}$$

This is our old friend the continuity equation. What this computation means is that the Maxwell equations have solutions for $\vec{E}$ and $\vec{B}$ only if the 4 functions $\rho(\vec{r}, t)$ and $\vec{j}(\vec{r}, t)$ are chosen to satisfy the continuity equation (17.8) given above. Recall that this equation is a differential version of a conservation law; the conserved quantity in this case being the electric charge. More precisely, the total charge Q contained in a volume V at time t, defined by

$$Q = \int_V dV\, \rho(\vec{r}, t),\tag{17.9}$$

changes in time according to the net flux of the current density $\vec{j}$ through the boundary S of V:

$$\frac{dQ}{dt} = -\oint_S d\vec{S} \cdot \vec{j}.\tag{17.10}$$

If the net flux of charge through the boundary (which may be "at infinity") vanishes, then the charge contained in V is constant in time. When we use the Maxwell equations to solve for the electromagnetic field due to a given charge distribution, that distribution must be specified so that charge is conserved in the sense of (17.8) or else the equations cannot have a solution.*

Given the continuity equation, we can now consider the status of the constraint equations (17.2) and (17.3). It is straightforward to show that if they are satisfied at one time, say $t = 0$, by the initial values for $\vec{E}$ and $\vec{B}$, then they are automatically solved at later times provided (i) the electromagnetic field at later times satisfies the evolution equations, and (ii) (17.8) is satisfied by the sources. See the Problems for details.

18. The Electromagnetic Wave Equation.

Let us now see how the Maxwell equations (17.2)–(17.5) predict the existence of electromagnetic waves. For simplicity we will consider a region of space and time in which

---

* It is no accident that the Maxwell equations, in effect, force the conservation of electric charge. Indeed, our current field theoretic description of all fundamental interactions (electromagnetic, weak, strong, and gravitational) is geared to force such conservation laws through the use of variational principles and the "principle of local gauge invariance". Unfortunately, a discussion of such ideas would be beyond the scope of this course.