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18 The Electromagnetic Wave Equation

Charles G. Torre

Department of Physics, Utah State University, Charles.Torre@usu.edu

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Compare this result with the divergence of (17.4) (*exercise*):

$$-\frac{1}{c}\nabla\cdot\frac{\partial\vec{E}}{\partial t}=\frac{4\pi}{c}\nabla\cdot\vec{j}, \quad (17.7)$$

to find (*exercise*)

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\vec{j}=0. \quad (17.8)$$

This is our old friend the continuity equation. What this computation means is that the Maxwell equations have solutions for \vec{E} and \vec{B} only if the 4 functions $\rho(\vec{r}, t)$ and $\vec{j}(\vec{r}, t)$ are chosen to satisfy the continuity equation (17.8) given above. Recall that this equation is a differential version of a conservation law; the conserved quantity in this case being the electric charge. More precisely, the total charge Q contained in a volume V at time t , defined by

$$Q=\int_V dV\rho(\vec{r}, t), \quad (17.9)$$

changes in time according to the net flux of the current density \vec{j} through the boundary S of V :

$$\frac{dQ}{dt}=-\oint_S d\vec{S}\cdot\vec{j}. \quad (17.10)$$

If the net flux of charge through the boundary (which may be “at infinity”) vanishes, then the charge contained in V is constant in time. When we use the Maxwell equations to solve for the electromagnetic field due to a given charge distribution, that distribution must be specified so that charge is conserved in the sense of (17.8) or else the equations cannot have a solution.*

Given the continuity equation, we can now consider the status of the constraint equations (17.2) and (17.3). It is straightforward to show that if they are satisfied at one time, say $t=0$, by the initial values for \vec{E} and \vec{B} , then they are automatically solved at later times provided (i) the electromagnetic field at later times satisfies the evolution equations, and (ii) (17.8) is satisfied by the sources. See the Problems for details.

18. The Electromagnetic Wave Equation.

Let us now see how the Maxwell equations (17.2)–(17.5) predict the existence of electromagnetic waves. For simplicity we will consider a region of space and time in which

* It is no accident that the Maxwell equations, in effect, force the conservation of electric charge. Indeed, our current field theoretic description of all fundamental interactions (electromagnetic, weak, strong, and gravitational) is geared to force such conservation laws through the use of variational principles and the “principle of local gauge invariance”. Unfortunately, a discussion of such ideas would be beyond the scope of this course.

there are no sources (*i.e.*, we consider the propagation of electromagnetic waves in vacuum). Thus we set $\rho = 0 = \vec{j}$ in our space-time region of interest. Now all the Maxwell equations are linear, homogeneous.

Begin by taking the curl of (17.4) to find (*exercise*)

$$\nabla \times (\nabla \times \vec{B}) - \frac{1}{c} \nabla \times \frac{\partial \vec{E}}{\partial t} = 0. \quad (18.1)$$

Of course we assume that the components of the electromagnetic field are at least twice continuously differentiable functions of space and time. We are therefore allowed to assume that partial derivatives commute and we have, using the identity (16.22), (*exercise*)

$$\nabla(\nabla \cdot \vec{B}) - \nabla^2 \vec{B} - \frac{1}{c} \frac{\partial}{\partial t} \nabla \times \vec{E} = 0. \quad (18.2)$$

Next we use the fact that (i) the magnetic field is divergence-free (see (17.3)) and (ii) the equation (17.5) to find (*exercise*)

$$-\nabla^2 \vec{B} + \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} = 0. \quad (18.3)$$

Thus (each component of) \vec{B} satisfies the wave equation with wave velocity c ! As a homework problem you will play an analogous game with the electric field to show

$$-\nabla^2 \vec{E} + \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0. \quad (18.4)$$

So, each component of \vec{E} and \vec{B} satisfies a wave equation, *i.e.*, we can have traveling wave disturbances in the electromagnetic field. As you know, such disturbances correspond to light waves, radio waves, microwaves, *etc.*

From our computations above it is tempting to think that electromagnetic fields satisfying the Maxwell equations are really just 6 copies of the waves we have been studying all along, one for each component of \vec{E} and \vec{B} . In particular it appears that each component of the electric and magnetic fields goes its own way, independently of the others. This is not true. What we *have* shown is that, given an \vec{E} and \vec{B} solving the (source-free) Maxwell equations, the electric and magnetic fields necessarily satisfy uncoupled wave equations. What we have *not* shown (and is not true) is that, given electric and magnetic fields solving uncoupled wave equations (18.3) and (18.4), we obtain solutions to the Maxwell equations. Put differently, that \vec{E} and \vec{B} solve the wave equation is *necessary but not sufficient* for them to solve the source-free Maxwell equations. We will now show how to use the solutions to (18.3) and (18.4) to build *bona fide* solutions to the Maxwell equations in the form of electromagnetic waves. For simplicity, we will restrict our attention to electromagnetic plane waves. (More general waves can be obtained by superposition in the usual way.)

The strategy is to take a plane wave solution to (18.4), restrict it to satisfy (17.2) (with $\rho = 0$) and then define the magnetic field so that the remaining Maxwell equations are satisfied. Here are the details.

Because each component of \vec{E} satisfies the wave equation, we can consider a plane wave electric field of the form:

$$\vec{E}(\vec{r}, t) = \vec{E}_0 \cos(\vec{k} \cdot \vec{r} - \omega t + \phi). \quad (18.5)$$

Here the constant vector field \vec{E}_0 determines the amplitude and (constant) direction of the electric field, and the phase ϕ is a constant. You can check that (18.5) is a solution to (18.4) if and only if the dispersion relation is (*exercise*)

$$\omega = kc. \quad (18.6)$$

Of course we could also use a sine function, or take the real part of an exponential, or take linear combinations (the source-free Maxwell equations are linear homogeneous), but this is good enough for our discussion. I emphasize that while each component of (18.5) solves the wave equation when (18.6) holds, (18.5) does not yet necessarily define a solution to the Maxwell equations — they still need to be imposed. To begin with, let us impose $\nabla \cdot \vec{E} = 0$. You can easily check that this equation means we have to choose the wave vector to be orthogonal to the amplitude vector (*good exercise*):

$$\vec{k} \cdot \vec{E}_0 = 0. \quad (18.7)$$

Thus the direction of the electric field in the electromagnetic plane wave is always perpendicular to the direction of propagation of the wave. As you probably know, electromagnetic waves are “transverse”.

The remaining Maxwell equations still need to be imposed, and they involve the magnetic field. We can build a divergence-free ($\nabla \cdot \vec{B} = 0$) magnetic field plane wave just as we did for \vec{E} , but we must keep in mind the other 2 Maxwell equations

$$\nabla \times \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} = 0 \quad (18.8)$$

$$\nabla \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0. \quad (18.9)$$

Let us first look at (18.8). From (18.5) the time derivative of \vec{E} yields a sine function, and this is to be equated to a curl of \vec{B} , which involves first spatial derivatives. This suggests we should also try a cosine for the magnetic field:

$$\vec{B} = \vec{B}_0 \cos(\vec{k}' \cdot \vec{r} - \omega' t + \psi). \quad (18.10)$$

To satisfy the wave equation we need the dispersion relation (*exercise*)

$$\omega' = k'c. \quad (18.11)$$

Because we want $\nabla \cdot \vec{B} = 0$, we have to choose the wave vector and amplitude such that

$$\vec{k}' \cdot \vec{B}_0 = 0. \quad (18.12)$$

It is easy to check that (*exercise*)

$$\begin{aligned} \nabla \times \vec{B} &= -\vec{B}_0 \times \nabla \left[\cos(\vec{k}' \cdot \vec{r} - \omega't + \psi) \right] \\ &= (\vec{B}_0 \times \vec{k}') \sin(\vec{k}' \cdot \vec{r} - \omega't + \psi). \end{aligned} \quad (18.13)$$

Next, we compute

$$\frac{1}{c} \frac{\partial \vec{E}}{\partial t} = \frac{\omega}{c} \vec{E}_0 \sin(\vec{k} \cdot \vec{r} - \omega t + \phi). \quad (18.14)$$

Thus (18.8) reduces to

$$\frac{\omega}{c} \vec{E}_0 \sin(\vec{k} \cdot \vec{r} - \omega t + \phi) = \vec{B}_0 \times \vec{k}' \sin(\vec{k}' \cdot \vec{r} - \omega't + \psi).$$

Since this must hold at each location and time, it follows (*exercise*) that

$$\phi = \psi, \quad (18.15)$$

$$\vec{k}' = \vec{k}, \quad (18.16)$$

and

$$\frac{\omega}{c} \vec{E}_0 = -\vec{k} \times \vec{B}_0. \quad (18.17)$$

Let \hat{n} be the unit vector in the direction of wave propagation:

$$\hat{n} = \frac{\vec{k}}{k}, \quad (18.18)$$

then we can write (*exercise*)

$$\vec{E}_0 = -\hat{n} \times \vec{B}_0. \quad (18.19)$$

Thus we see that the electric and magnetic fields must have the same wave vector, that is, they have the same wavelengths ($\lambda = 2\pi/k$) and propagation directions \hat{z} . Furthermore, from (18.19) and the dispersion relation (18.6) it follows that:

(1) The magnitudes of the electric and magnetic fields are equal* (*exercise*),

$$E_0 = B_0. \quad (18.20)$$

* Keep in mind we are using Gaussian units.

(2) \vec{E} and \vec{B} are orthogonal, $\vec{E} \cdot \vec{B} = 0$ (*exercise*).

(3) Each field is orthogonal to the propagation direction, $\vec{E} \cdot \hat{n} = 0 = \vec{B} \cdot \hat{n}$ (*exercise*).

Note that when \vec{E}_0 and \vec{B}_0 satisfy (18.19), they automatically satisfy (18.7) and (18.12).

Thus far we have only dealt with (18.8). As an exercise you can show the other evolution equation (18.9) gives identical information except we find that

$$\vec{B}_0 = \hat{n} \times \vec{E}_0. \quad (18.21)$$

However, this is equivalent to our earlier relation (18.19). To see this we first note that, given that \vec{E}_0 and \hat{n} are orthogonal (as follows from (18.7)), the relation (18.21) above says that: (1) (18.20) holds, and (2) \vec{B}_0 is perpendicular to both \vec{E}_0 and \hat{n} and with a direction given by the right-hand rule. The relation (18.19) also implies (18.20) and that the three vectors (\vec{E}_0 , \vec{B}_0 , \hat{n}) are mutually orthogonal; the only thing to check is that the right hand rule gives the same relation among the three vectors—which you can check. It is also possible to massage (18.21) directly into (18.19) using some vector identities; this will be explored in the homework.

At this point we have taken care of all the Maxwell equations. Let us gather all our results together. We can build a plane wave solution to the (source-free) Maxwell equations by the following procedure:

(i) Pick a wave propagation direction \hat{n} and a wavelength λ ; then define

$$\vec{k} = \frac{2\pi}{\lambda} \hat{n}. \quad (18.22)$$

(ii) Pick an electric field amplitude \vec{E}_0 , which is any (constant) vector (field) such that $\vec{E}_0 \cdot \hat{n} = 0$.

(iii) Set $\omega = kc$ and define

$$\vec{E}(\vec{r}, t) = \vec{E}_0 \cos(\vec{k} \cdot \vec{r} - \omega t + \phi), \quad (18.23)$$

$$\vec{B}(\vec{r}, t) = (\hat{n} \times \vec{E}_0) \cos(\vec{k} \cdot \vec{r} - \omega t + \phi). \quad (18.24)$$

Equations (18.23) and (18.24) give the electric and magnetic fields arising in a plane electromagnetic wave. This wave is *linearly polarized* (see §20). It can be shown that the general solution to the source-free Maxwell equations can be expressed as a superposition of these plane wave solutions over all possible values of the amplitude, phase, propagation direction, and wave-number.