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19 Electromagnetic Energy

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19. Electromagnetic Energy.

In a previous physics course you should have encountered the interesting notion that the electromagnetic field carries energy and momentum. If you have ever been sunburned you have experimental confirmation of this fact! We are now in a position to explore this idea quantitatively. In physics, the notions of energy and momentum are of interest mainly because they are conserved quantities. We can uncover the energy and momentum quantities associated with the electromagnetic field by searching for conservation laws. As before, such conservation laws will appear embodied in a continuity equation. Thus we begin by investigating a continuity equation for energy and momentum. As was mentioned earlier, there are systematic methods to search for continuity equations associated with a system of differential equations such as the Maxwell equations, but we will not be able to get into that here. Instead, we simply verify the continuity equation associated to energy-momentum conservation.

We will restrict our attention to the source-free case ($\rho = 0$, $\vec{j} = 0$). With sources, there can be an exchange of energy-momentum of the electromagnetic field with that of the sources and the story is a little longer than justified for this course*. Since we have used the symbols ρ and $\vec{j}$ to denote the electric charge density and electric current density, to avoid confusion we use the symbols U and $\vec{S}$ to denote the energy density and energy current density (also known as the "Poynting vector") for the electromagnetic field. These are defined by (still in Gaussian units)

$$U = \frac{1}{2}(E^2 + B^2), \tag{19.1}$$

$$\vec{S} = c\vec{E} \times \vec{B}. \tag{19.2}$$

The claim is that $(U, \vec{S})$ satisfy

$$\frac{\partial U}{\partial t} + \nabla \cdot \vec{S} = 0 \tag{19.3}$$

when $\vec{E}$ and $\vec{B}$ satisfy the (source-free) Maxwell equations.

To verify this, we need to use some vector identities. If $\vec{A}$ and $\vec{B}$ are any two vector fields that depend upon a variable t, we have that (*exercise*)

$$\begin{aligned} \frac{\partial}{\partial t}(A^2) &= \frac{\partial}{\partial t}(\vec{A} \cdot \vec{A}) \\ &= 2\vec{A} \cdot \frac{\partial \vec{A}}{\partial t}, \end{aligned} \tag{19.4}$$

and (a longer *exercise* – you needn't do this one unless you are ambitious)

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot \nabla \times \vec{A} - \vec{A} \cdot \nabla \times \vec{B}. \tag{19.5}$$

* As you might expect on physical grounds, in order to get a conservation law in this case, one must keep track of the energy and momentum of the sources.

Using these vector identities and the Maxwell equations, we have (*exercise*)

$$\begin{aligned}\frac{\partial U}{\partial t} &= \vec{E}\cdot\frac{\partial \vec{E}}{\partial t} + \vec{B}\cdot\frac{\partial \vec{B}}{\partial t}\\ &= c\vec{E}\cdot\nabla\times\vec{B} - c\vec{B}\cdot\nabla\times\vec{E},\end{aligned} \tag{19.6}$$

and

$$\nabla\cdot\vec{S} = c\left(\vec{B}\cdot\nabla\times\vec{E} - \vec{E}\cdot\nabla\times\vec{B}\right), \tag{19.7}$$

from which the result (19.3) follows immediately. Note we only needed the evolution equations to obtain (19.3).

We interpret $U(\vec{r},t)$ as the *electromagnetic energy density at the point $\vec{r}$ and time t* and the Poynting vector* $\vec{S}$ as the energy current density. Define the total electromagnetic energy $\mathcal{E}$ in the volume V to be the integral of the energy density U:

$$\mathcal{E} := \int_V dV\, U = \int_V dV\,\frac{1}{2}\left(E^2 + B^2\right). \tag{19.8}$$

According to the continuity equation (19.3), the time rate of change of $\mathcal{E}$ is controlled by the flux of the Poynting vector $\vec{S}$ through the boundary A of V:

$$\frac{d\mathcal{E}}{dt} = -\int_A d\vec{A}\cdot\vec{S}, \tag{19.9}$$

where

$$d\vec{A} = \hat{N}\, dA \tag{19.10}$$

with $\hat{N}$ being the unit normal to the boundary surface of V.

It is worth computing the energy density and Poynting vector for the plane wave constructed above in (18.23) and (18.24). You will find (*exercise*)

$$U = (E_0)^2\cos^2(\vec{k}\cdot\vec{r} - \omega t + \phi), \tag{19.11}$$

$$\vec{S} = c(E_0)^2\cos^2(\vec{k}\cdot\vec{r} - \omega t + \phi)\,\hat{n}. \tag{19.12}$$

Note that the Poynting vector lies along the direction of propagation $\hat{n} = \vec{k}/k$ of the plane wave. Thus the flow of energy is along $\hat{n}$. The continuity equation guarantees that if you integrate U over a volume you will find that the total energy is changing according to the net flux of $\vec{S}$ through the volume. This you will experiment with in a homework problem.

If, at the boundary of the region V, the fields are such that net the Poynting flux vanishes, then the energy contained in V will be constant in time. In particular, if the volume V in (19.8) is taken to be all space, we can view the boundary A as a sphere

* You should keep in mind that the Poynting vector is really a vector *field*.

of infinite radius. If we consider an *isolated system*, so that the electric and magnetic fields vanish sufficiently rapidly at large distances (*i.e.,* "at infinity"), then the flux of the Poynting vector will vanish as the radius of A is taken to infinity. Thus the total electromagnetic energy of an isolated (and source-free) electromagnetic field is constant in time.

20. Polarization.

Our final topic in this brief study of electromagnetic waves concerns the phenomenon of *polarization*, which occurs thanks to the vector nature of the waves. More precisely, the polarization of an electromagnetic plane wave concerns the direction of the electric (and magnetic) vector fields. Let us first give a rough, qualitative motivation for the phenomenon. An electromagnetic plane wave is a traveling sinusoidal disturbance in the electric and magnetic fields. Let us focus on the behavior of the electric field since (i) typically the electric force on a charge is the most important influence of an electromagnetic wave, and (ii) we can in any case always reconstruct the behavior of the magnetic field from the electric field. Because the electric force on a charged particle is along the direction of the electric field, the response of charges to electromagnetic waves is sensitive to the direction of the electric field in a plane wave. Such effects are what we refer to when we discuss polarization phenomena involving light. Now comes the important part. It may appear to you that plane electromagnetic waves will always have a *linear polarization*, that is, a constant electric (and hence magnetic) field direction. However, consider superimposing two plane waves with the same propagation direction and wavelength but with different phases and directions for the electric and magnetic fields. Thanks to the linear-homogeneous nature of the source-free Maxwell equations, we know that this superposition will also be a solution of those equations. And, as we shall see, this superposition will be another plane wave of the type we have studied. Even though the direction of the electric field in each of the constituent waves is constant, the superposition of the two can have a time varying electric (and magnetic) field direction because the two constituent electric fields need not be in phase with each other. The net effect is a time varying electric (and magnetic) field direction and the resulting phenomena of circular and elliptical polarization. We now want to see how to describe this mathematically.

Let us choose our z-axis along the direction of propagation of the wave so that the Cartesian components of $\vec{k}$ are $(0, 0, k)$. Let us construct an electromagnetic plane wave by superimposing 2 plane waves with the same wave vector: $(\vec{E}_1, \vec{B}_1)$, with $\vec{E}_1$ directed along the x-axis, and $(\vec{E}_2, \vec{B}_2)$, with $\vec{E}_2$ directed along the y-axis. Further, let us work with the representation of the waves as complex-valued exponentials. This keeps the trigonometry from getting in our way; in particular, the phase information is contained in the complex amplitudes. Keep in mind that we should take the real part of the electric field at the end