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19 Electromagnetic Energy

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19. Electromagnetic Energy.

In a previous physics course you should have encountered the interesting notion that the electromagnetic field carries energy and momentum. If you have ever been sunburned, you have experimental confirmation of this fact! We are now in a position to explore this idea quantitatively. In physics, the notions of energy and momentum are of interest mainly because they are conserved quantities. We can uncover the energy and momentum quantities associated with the electromagnetic field by searching for conservation laws. As before, such conservation laws will appear embodied in a continuity equation. Thus we begin by investigating a continuity equation for energy and momentum. As was mentioned earlier, there are systematic methods to search for continuity equations associated with a system of differential equations such as the Maxwell equations, but we will not be able to get into that here. Instead, we simply write down the continuity equation associated to energy-momentum conservation.

We will restrict our attention to the source-free case ($\rho = 0$, $\vec{j} = 0$). With sources, there can be an exchange of energy-momentum of the electromagnetic field with that of the sources and the story is a little longer than justified for this course*. Since we have used the symbols ρ and $\vec{j}$ to denote the electric charge density and electric current density, to avoid confusion we use the symbols U and $\vec{S}$ to denote the energy density and energy current density (also known as the "Poynting vector") for the electromagnetic field. These are defined by (still in Gaussian units)

$$U = \frac{1}{2}(E^2 + B^2), \tag{19.1}$$

$$\vec{S} = c\vec{E} \times \mathbf{B}. \tag{19.2}$$

The claim is that $(U, \vec{S})$ satisfy

$$\frac{\partial U}{\partial t} + \nabla \cdot \vec{S} = 0 \tag{19.3}$$

when $\vec{E}$ and $\mathbf{B}$ satisfy the (source-free) Maxwell equations.

To verify this, we need to use some vector identities. If $\vec{A}$ and $\vec{B}$ are any two vector fields that depend upon a variable t, we have that (*exercise*)

$$\begin{aligned} \frac{\partial}{\partial t}(A^2) &= \frac{\partial}{\partial t}(\vec{A} \cdot \vec{A}) \\ &= 2\vec{A} \cdot \frac{\partial \vec{A}}{\partial t}, \end{aligned} \tag{19.4}$$

and (a longer *exercise* – you needn't do this one unless you are ambitious)

$$\nabla \cdot (\vec{A} \times \vec{B}) = \mathbf{B} \cdot \nabla \times \vec{A} - \vec{A} \cdot \nabla \times \mathbf{B}. \tag{19.5}$$

* As you might expect on physical grounds, in order to get a conservation law in this case, one must keep track of the energy and momentum of the sources.

Using these vector identities and the Maxwell equations, we have (*exercise*)

$$\begin{aligned}\frac{\partial U}{\partial t} &= \vec{E}\cdot\frac{\partial \vec{E}}{\partial t} + \mathbf{B}\cdot\frac{\partial \mathbf{B}}{\partial t}\\ &= c\vec{E}\cdot\nabla\times\mathbf{B} - c\mathbf{B}\cdot\nabla\times\vec{E},\end{aligned} \tag{19.6}$$

and

$$\nabla\cdot\vec{S} = c\left(\mathbf{B}\cdot\nabla\times\vec{E} - \vec{E}\cdot\nabla\times\mathbf{B}\right). \tag{19.7}$$

from which the result (19.3) follows immediately. Note we only needed the evolution equations to obtain (19.3).

We interpret $U(\vec{r}, t)$ as the *electromagnetic energy density at the point* $\vec{r}$ *and time* t and the Poynting vector* $\vec{S}$ as the energy current density. Define the total electromagnetic energy $\mathcal{E}$ in the volume V to be the integral of the energy density U:

$$\mathcal{E} := \int_V dV\, U = \int_V dV\, \frac{1}{2}\left(E^2 + B^2\right). \tag{19.8}$$

According to the continuity equation (19.3), the time rate of change of $\mathcal{E}$ is controlled by the flux of the Poynting vector $\vec{S}$ through the boundary A of V:

$$\frac{d\mathcal{E}}{dt} = -\int_A d\vec{A}\cdot\vec{S}, \tag{19.9}$$

where

$$d\vec{A} = \hat{N}\, dA \tag{19.10}$$

with $\hat{N}$ being the unit normal to the boundary surface of V.

It is worth computing the energy density and Poynting vector for the plane wave constructed above in (18.23) and (18.24). You will find (*exercise*)

$$U = (E_0)^2\cos^2(\vec{k}\cdot\vec{r} - \omega t + \phi), \tag{19.11}$$

$$\vec{S} = c(E_0)^2\cos^2(\vec{k}\cdot\vec{r} - \omega t + \phi)\,\hat{n}. \tag{19.12}$$

Note that the Poynting vector lies along the direction of propagation $\hat{n} = \vec{k}/k$ of the plane wave. Thus the flow of energy is along $\hat{n}$. The continuity equation guarantees that if you integrate U over a volume you will find that the total energy is changing according to the net flux of $\vec{S}$ through the volume. This you will experiment with in a homework problem.

If, at the boundary of the region V, the fields are such that net the Poynting flux vanishes, then the energy contained in V will be constant in time. In particular, if the volume V in (19.8) is taken to be all space, we can view the boundary A as a sphere

* You should keep in mind that the Poynting vector is really a vector *field*.