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20 Polarization

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of infinite radius. If we consider an *isolated system*, so that the electric and magnetic fields vanish sufficiently rapidly at large distances (*i.e.,* "at infinity"), then the flux of the Poynting vector will vanish as the radius of A is taken to infinity. Thus the total electromagnetic energy of an isolated (and source-free) electromagnetic field is constant in time.

20. Polarization.

Our final topic in this brief study of electromagnetic waves concerns the phenomenon of *polarization*, which occurs thanks to the vector nature of the waves. More precisely, the polarization of an electromagnetic plane wave concerns the direction of the electric (and magnetic) vector fields. Let us first give a rough, qualitative motivation for the phenomenon. An electromagnetic plane wave is a traveling sinusoidal disturbance in the electric and magnetic fields. Let us focus on the behavior of the electric field since (i) typically the electric force on a charge is the most important influence of an electromagnetic wave, and (ii) we can in any case always reconstruct the behavior of the magnetic field from the electric field. Because the electric force on a charged particle is along the direction of the electric field, the response of charges to electromagnetic waves is sensitive to the direction of the electric field in a plane wave. Such effects are what we refer to when we discuss polarization phenomena involving light. Now comes the important part. It may appear to you that plane electromagnetic waves will always have a *linear polarization*, that is, a constant electric (and hence magnetic) field direction. However, consider superimposing two plane waves with the same propagation direction and wavelength but with different phases and directions for the electric and magnetic fields. Thanks to the linear-homogeneous nature of the source-free Maxwell equations, we know that this superposition will also be a solution of those equations. And, as we shall see, this superposition will be another plane wave of the type we have studied. Even though the direction of the electric field in each of the constituent waves is constant, the superposition of the two can have a time varying electric (and magnetic) field direction because the two constituent electric fields need not be in phase with each other. The net effect is a time varying electric (and magnetic) field direction and the resulting phenomena of circular and elliptical polarization. We now want to see how to describe this mathematically.

Let us choose our z-axis along the direction of propagation of the wave so that the Cartesian components of $\vec{k}$ are $(0, 0, k)$. Let us construct an electromagnetic plane wave by superimposing 2 plane waves with the same wave vector: $(\vec{E}_1, \vec{B}_1)$, with $\vec{E}_1$ directed along the x-axis, and $(\vec{E}_2, \vec{B}_2)$, with $\vec{E}_2$ directed along the y-axis. Further, let us work with the representation of the waves as complex-valued exponentials. This keeps the trigonometry from getting in our way; in particular, the phase information is contained in the complex amplitudes. Keep in mind that we should take the real part of the electric field at the end of the day.

With all the preceding as justification, we write

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = (E_1\hat{x} + E_2\hat{y})e^{i(kz-\omega t)}. \tag{20.1}$$

Here E_1 and E_2 are just two complex numbers, that is, for real numbers R_1,R_2, α, and β we have

$$E_1 = R_1 e^{i\alpha} \tag{20.2}$$

$$E_2 = R_2 e^{i\beta}. \tag{20.3}$$

We see that all we have given is the complex representation of the superposition of two real waves (*exercise*):

$$\vec{E}_1 = R_1 \cos(kz - \omega t + \alpha)\,\hat{x} \tag{20.4}$$

$$\vec{E}_2 = R_2 \cos(kz - \omega t + \beta)\,\hat{y}. \tag{20.5}$$

Different polarizations occur for different choices of the phase difference $\alpha - \beta$ and the amplitudes R_1 and R_2.

Note that even though we began by assuming the two electric fields were orthogonal, even if they weren't orthogonal we would have ended up with a similar result upon superposition. Specifically, in (20.1) E_1 would have been the x component of the superposition and E_2 would have been the y component of the superposition. So the formulas we have constructed represent the superposition of *any* two plane waves which have the same wave vector $\vec{k}$.

Let us now consider the behavior of the electric field for various choices of the relative phases and amplitudes.

20.1 Linear Polarization

The linear polarization case, which we studied in previous sections, occurs when $\beta = \alpha \pm n\pi$, where $n = 0, 1, 2, 3, \ldots$. We place no restrictions upon R_1 and R_2. In this case you can check (*exercise*) that the complex form of the total electric field is

$$\vec{E} = (R_1\hat{x} \pm R_2\hat{y})e^{i(kz-\omega t+\alpha)}. \tag{20.6}$$

Taking the real part gives (*exercise*)

$$\vec{E} = (R_1\hat{x} \pm R_2\hat{y})\cos(kz - \omega t + \alpha). \tag{20.7}$$

This is a wave in which the magnitude of the electric field oscillates in time and space, but with its direction held fixed.

20.2 Circular Polarization

Here we set $R_1 = R_2 \equiv R$ and $\beta = \alpha \pm \frac{\pi}{2}$. We find for the complex field (*exercise*)

$$\vec{E} = R(\hat{x} \pm i\hat{y})e^{i(kz-\omega t+\alpha)}. \tag{20.8}$$

Take the real part to find (*exercise*)

$$\vec{E} = \hat{x}R\cos(kz - \omega t + \alpha) \mp \hat{y}R\sin(kz - \omega t + \alpha). \tag{20.9}$$

Let us consider the direction of $\vec{E}$ as measured at a fixed point (x_0, y_0, z_0) in space. Physically, we move to this point and hold out our electric field probe. As time passes, what do we find? Well, of course we find that $\vec{E}$ always lies in the x-y plane, as you can see from (20.9). More interestingly, we find that the magnitude of $\vec{E}$ is constant while its direction moves in a circle with constant angular velocity ω. It is easy to verify that the magnitude of $\vec{E}$ is constant in time:

$$E^2 = R^2(\cos(kz - \omega t + \alpha))^2 + R^2(\sin(kz - \omega t + \alpha))^2 = R^2. \tag{20.10}$$

To see that the electric field direction moves with uniform circular motion at angular velocity ω, simply recall that such motion can always be mathematically characterized in the general form

$$u(t) = r\cos(\omega t + \xi) \tag{20.11}$$

$$v(t) = r\sin(\omega t + \xi), \tag{20.12}$$

where (u, v) are Cartesian coordinates in a two dimensional space (*exercise*). At a fixed location, the x and y components $\vec{E}$, given above in (20.9), are precisely of this form.

20.3 Elliptical Polarization

Finally, we consider the most general case in which $R_1 \neq R_2$ and $\alpha \neq \beta$. From our previous special case, you can see that it might be profitable to make the definition

$$\gamma = \beta - \frac{\pi}{2}, \tag{20.13}$$

in which case you can show as a nice *exercise* that the real electric field is given by:

$$\vec{E} = \hat{x}R_1\cos(kz - \omega t + \alpha) + \hat{y}R_2\sin(kz - \omega t + \gamma). \tag{20.14}$$

From this result it follows that the electric field direction—as measured at a fixed z—traces out an *ellipse*. There are a number of ways to see this. First of all, for fixed z, the x and y components of $\vec{E}$ are mathematically identical to a pair of general solutions to a one-dimensional harmonic oscillator (*exercise*). Put differently, the motion of the x-y components of $\vec{E}$ is mathematically identical to the x and y motions of a two-dimensional (isotropic) harmonic oscillator. It is a familiar result from classical mechanics that the superposition of these x-y motions is an ellipse in the x-y plane. In case this result is unfamiliar to you, let us recall the definition of an ellipse. One definition is the locus of points in the x-y plane such that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{20.15}$$

where a and b are some constants. You will be asked to show in the Problems that such a relationship is satisfied by the x and y components of $\vec{E}$.