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# 21 Non-Linear Wave Equations and Solitons

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## 21. Non-linear Wave Equations and Solitons.

In 1834 the Scottish engineer John Scott Russell observed at the Union Canal at Hermiston a well-localized* and unusually stable disturbance in the water that propagated for miles virtually unchanged. The disturbance was stimulated by the sudden stopping of a boat on the canal. He called it a "wave of translation"; we call it a *solitary wave.* As it happens, a number of relatively complicated – indeed, non-linear – wave equations can exhibit such a phenomenon. Moreover, these solitary wave disturbances will often be stable in the sense that if two or more solitary waves collide then after the collision they will separate and take their original shape. Solitary waves which have this stability property are called *solitons.* The terminology stems from a combination of the word solitary and the suffix "on" which is used to signify a particle (think of the proton, electron, neutron, *etc.* ). We shall discuss a little later the sense in which a soliton is like a particle. Solitary waves and solitons have become very important in a variety of physical settings, for example: hydrodynamics, non-linear optics, plasmas, meteorology, and elementary particle physics, to name a few. Our goal in this chapter is to give a very brief — and completely superficial — introduction to solitonic solutions of non-linear wave equations.

To begin, let me point out that the humble wave equation in one dimension already provides an illustration of some of the phenomena we want to explore, principally by virtue of its linearity.† We have already seen that the solutions to the wave equation

$$\left(\frac{\partial^2}{\partial t^2} - v^2\frac{\partial^2}{\partial x^2}\right) q(x,t) = 0 \tag{21.1}$$

take the general form

$$q(x,t) = f(x+vt) + g(x-vt), \tag{21.2}$$

where the functions $f$ and $g$ are determined by initial conditions. Let us suppose that we choose our initial conditions so that the solution has $f = 0$, so that the wave is simply the displacement profile $q = g(x-vt)$, that is, a traveling disturbance with the shape dictated by the curve $y = g(x)$ translating rigidly to the "right" (toward positive $x$) at speed $v$. Let us also suppose that $g$ is a function that is localized in some region, so that it has a finite width. We have a "pulse", which travels to the right, unchanged in shape. Thus the pulse is a solitary wave. To visualize this, imagine that you and your friend are holding a rope taut between you and you shake the end of the rope one time. The result is a pulse which travels toward your friend (with a speed depending upon the density and tension of the rope). This "pulse" has its shape described by the function $g$. Now suppose we also allow $f$ to be a localized non-trivial function, you then get a pulse traveling to the

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* About 10 meters long and half a meter in height.

† The truly remarkable thing about solitonic behavior is that there are highly non-linear equations which also can exhibit it.

left superposed with the pulse traveling to the right. Suppose the two pulses are initially separated, the one described by $f$ sitting off at large, positive $x$, and the one described by $g$ sitting off at large negative $x$, say. The pulses will approach each other at a relative speed of $2v$ and at some point they will overlap, giving a wave profile which is, evidently, the algebraic sum of the individual pulses. Eventually, the pulses will become separated with the pulse described by $g$ moving off toward large positive values of $x$ and the pulse described by $f$ moving to large negative values of $x$. The pulses "collide", but after the collision they retain their shape – their "identity", if you will. To visualize this, return to the rope experiment. You shake the rope once, and your friend also shakes the rope once. Each of you produce a pulse which travels toward the other person, overlap for a time, then separate again unscathed. Try it! (You may need a long rope to see this work.) This is a simple example of solitonic behavior. In this example the solitonic behavior results from the linearity of the wave equation (superposition!) and its dispersion relation. Indeed, it is difficult to imagine such behavior emerging from anything *but* a linear equation. Our model of coupled oscillators that led to the wave equation is about as simple as it can be. As such, it does not take into account many details of the behavior of the medium that the waves propagate in. More realistic or alternate wave equations do not necessarily exhibit this solitary wave behavior because they lack the superposition property and or they lack the requisite dispersion relation.

For example, let us consider the Schrödinger equation for a free, non-relativistic, quantum mechanical particle with mass $m$ moving in one dimension:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}. \tag{21.3}$$

This equation, being linear, respects the superposition property, but you will recall it does not have the simple dispersion relation possessed by the wave equation. As we saw, the general solution of (21.3) is a superposition of traveling waves

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} C(k)e^{i(kx-\omega(k)t)}\,dk, \tag{21.4}$$

where $C(k)$ is determined by a choice of initial wave function, $\psi(x,0)$, and where

$$\omega(k) = \frac{\hbar k^2}{2m}. \tag{21.5}$$

The traveling waves appearing in the superposition have different speeds. To see this, just note that the wave with wavenumber $k$ has a speed given by

$$v(k) = \frac{\omega(k)}{k} = \frac{\hbar k}{2m}. \tag{21.6}$$

The consequence of this dispersion relation is that at any time $t \neq 0$ a well-localized initial wave function does not retain its shape. Indeed, the wave pulse will spread as time passes because its different frequency (or wavelength) components do not travel at the same speed. (Physically, this is a manifestation of the uncertainty principle: localizing the particle at $t = 0$ within the support of the initial pulse leads to an uncertainty in momentum which, at later times, reduces the localization of the particle.) One says that the Schrödinger waves exhibit *dispersion* because the wave profiles "disperse" as time runs. So we see that solitary wave behavior is not a universal feature of wave phenomena.

Let us look at another linear wave equation that exhibits dispersion. The following equation is known as the *Klein-Gordon equation* (in one spatial dimension):

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) q(x,t) + m^2c^2 q(x,t) = 0. \tag{21.7}$$

This equation can be used to describe a relativistic quantum particle with rest mass determined by $m$ (moving in one dimension). In this context, $c$ is the speed of light – it is *not* the speed of the waves. You can think of it as a relativistic generalization of the Schrödinger equation.

Just like the Schrödinger equation, the general solution of the Klein-Gordon equation can be constructed by superimposing sinusoidal wave solutions over amplitudes, phases and wavelengths. To see that dispersion arises, we simply compute the dispersion relation that arises for a sinusoidal wave. Consider a (complex) solution of the form

$$q(x,t) = Ae^{i(kx-\omega t)}. \tag{21.8}$$

It is not hard to see that this wave solves the Klein-Gordon equation if and only if

$$\omega^2 = c^2k^2 + m^2c^4 \quad \Longleftrightarrow \quad \omega = \pm c\sqrt{k^2 + m^2c^2}. \tag{21.9}$$

As you can see, the wave speed $\omega/k$ again depends upon $k$, leading to dispersion.

Evidently, dispersion in linear wave equations does not allow for solitary wave phenomena. Remarkably, one can compensate for dispersion by carefully altering the superposition property using non-linearities in the wave equation. A detailed study of non-linear wave equations is way beyond the scope of this text. My plan is to just have a quick at one, relatively simple non-linear partial differential equation to get a glimpse of how solitons can arise.

A relatively simple non-linear equation is given by a modification of the Klein-Gordon equation (in one spatial dimension) for a scalar field $\phi(x,t)$:

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + \frac{m^3}{\sqrt{\lambda}}\sin(\frac{\sqrt{\lambda}}{m}\phi) = 0. \tag{21.10}$$

For simplicity in what follows we have chosen units in which $c = 1$. The presence of the non-linearity is controlled by $m$. If $m = 0$ we have the usual wave equation. The new parameter $\lambda$ is going to be a "self-coupling constant". The equation (21.10) is sometimes called the "sine-Gordon equation", showing that mathematicians have a sense of humor to some extent. We will use this terminology, also. The sine-Gordon equation is a wave equation that includes a "self-interaction" thanks to the sine term. Because of this term the equation is *non-linear*. Consequently, the superposition property (wherein one can linearly combine two solutions to make a third solution) is *not* present. To see the relationship between the sine-Gordon equation and the Klein-Gordon equation, suppose that we restrict attention to solutions which always have a bounded, small magnitude for $\phi$. In this case, for sufficiently small $\phi$ we can make the approximation:

$$\sin(\frac{\sqrt{\lambda}}{m}\phi) \approx \frac{\sqrt{\lambda}}{m}\left(\phi - \frac{1}{6}\frac{\lambda}{m^2}\phi^3 + \cdots\right). \tag{21.11}$$

Using this Taylor expansion in the sine-Gordon equation, you will see that the first term in the expansion gives the Klein-Gordon equation (with $c = 1$) while the next (and higher) terms provide non-linearities. Physically, these describe a "self-interaction" of $\phi$. The strength of the self-interaction is defined by $\lambda$. Indeed, if you consider the limit as $\lambda \to 0$ in (21.10) you will recover the Klein-Gordon equation (*exercise*).

There is an extensive body of literature that analyzes the sine-Gordon equation and methods for its solution. Here we merely point out that the sine-Gordon equation admits the solution

$$\phi(x, t) = 4\frac{m}{\sqrt{\lambda}} \arctan\left(e^{m(x-x_0)}\right), \tag{21.12}$$

where $x_0$ is an arbitrary constant. This is the *static soliton solution* to the sine-Gordon equation. To verify this you should first note that this putative solution does not depend upon time (it is a *static* solution), so the time derivatives of $\phi$ in (21.12) vanish and we only need to compare the $x$ derivatives to the sine term. The key thing to check is that

$$\frac{d^2}{dy^2}(4\arctan(e^y)) = \sin(4\arctan(e^y)). \tag{21.13}$$

To check this, you will need the math facts:

$$\frac{d}{dy}\arctan(y) = \frac{1}{1+y^2}, \tag{21.14}$$

$$\sin(4\arctan(y)) = -4\frac{y(y^2-1)}{(1+y^2)^2}. \tag{21.15}$$

(These calculations are a good place to try your skills with some algebraic computing software!) With these results in hand, it is a simple matter to see that (21.12) does solve the sine-Gordon equation (*exercise*).

The solution we have exhibited to the sine-Gordon equation is not, at first glance, a solitary wave such as we discussed for the wave equation. To see this, just plot the graph of soliton solution (see Problems) to see that the wave profile is rather spread out. However, it has the property that its *energy density* is localized about $x = x_0$. The energy density of $\phi$ is defined as follows:

$$\mathcal{E} = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{m^4}{\lambda}\left[1 - \cos(\frac{\sqrt{\lambda}}{m}\phi)\right]. \tag{21.16}$$

This definition is used because it leads to the continuity equation

$$\frac{\partial\mathcal{E}}{\partial t} + \frac{\partial j}{\partial x} = 0 \tag{21.17}$$

when $\phi$ satisfies the sine-Gordon equation. Here we define the energy current density just as we did for the wave equation:

$$j = -\frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial x}. \tag{21.18}$$

(You will be asked to verify this continuity equation in the Problems.) To see the localization of energy, set $x_0 = 0$ (just for simplicity) and compute the energy density for the static soliton solution (21.12). You will find:

$$\mathcal{E}_{x_0=0} = 16\frac{m^4}{\lambda}\frac{e^{2mx}}{(1+e^{2mx})^2}. \tag{21.19}$$

This function is peaked about $x = 0$ and decays rapidly as $|x|$ grows. Thus the static soliton solution can be viewed as defining a "lump" of energy at $x = x_0$.

It is not too hard to see that the sine-Gordon equation actually allows for a (time-dependent) solution in which the soliton we have exhibited moves at any constant speed $V$. To see this, we employ an elegant trick, which is based upon the observation that if $\phi(x)$ is a (static) solution to the equation then so is

$$\Phi(x,t) = \phi(\frac{x \pm Vt}{\sqrt{1-V^2}}). \tag{21.20}$$

To check this, we just need the chain rule. We have

$$\frac{\partial\Phi(x,t)}{\partial x} = \frac{1}{\sqrt{1-V^2}}\phi'(\frac{x \pm Vt}{\sqrt{1-V^2}}) \tag{21.21}$$

and so

$$\frac{\partial^2\Phi(x,t)}{\partial x^2} = \frac{1}{1-V^2}\phi''(\frac{x \pm Vt}{\sqrt{1-V^2}}). \tag{21.22}$$

We also have

$$\frac{\partial\Phi(x,t)}{\partial t} = \pm\frac{V}{\sqrt{1-V^2}}\phi'(\frac{x \pm Vt}{\sqrt{1-V^2}}), \tag{21.23}$$

and

$$\frac{\partial^2\Phi(x,t)}{\partial t^2} = \frac{V^2}{1-V^2}\phi''(\frac{x \pm Vt}{\sqrt{1-V^2}}), \tag{21.24}$$

Here we using the notation

$$\phi'(z) = \left(\frac{d\phi(y)}{dy}\right)_{y=z}, \quad \phi''(z) = \left(\frac{d^2\phi(y)}{dy^2}\right)_{y=z}. \tag{21.25}$$

So the formulas with $\phi'$, *etc.* are the derivatives of the function $\phi$ evaluated at the point $\frac{x\pm Vt}{\sqrt{1-V^2}}$. Now, using these formulas, plug the result into the sine-Gordon equation to see that $\Phi$ satisfies this equation because, as we showed earlier,

$$\phi'' = \frac{m^3}{\sqrt{\lambda}}\sin(\frac{\sqrt{\lambda}}{m}\phi). \tag{21.26}$$

The interpretation of this result is that one has solutions (21.20) of the sine-Gordon equation which are "lumps" of energy, propagating without change in shape at any constant speed $V$. It is this feature of the solution, particularly its energy density, that justifies the description of (21.12) as a "solitary wave". One can view these solutions – these solitary waves – as a continuum model of a free particle.

The change of variables

$$x \to \frac{x \pm Vt}{\sqrt{1-V^2}}, \tag{21.27}$$

when complemented with

$$t \to \frac{t \pm Vx}{\sqrt{1-V^2}}, \tag{21.28}$$

is an example of a *Lorentz transformation*. It defines the relation between time and space as determined in two inertial reference frames that are moving at constant relative velocity $\pm V$, according to Einstein's special theory of relativity. Using a computation similar to that performed above, it can be shown that *any* solution of the sine-Gordon equation, $\phi(x,t)$ is transformed to another solution $\Phi(x,t)$ of the sine-Gordon equation by the Lorentz transformation:

$$\Phi(x,t) = \phi(\frac{x \pm Vt}{\sqrt{1-V^2}}, \frac{t \pm Vx}{\sqrt{1-V^2}}). \tag{21.29}$$

This state of affairs is characterized by the statement that *Lorentz transformations are symmetries of the sine-Gordon equation.* You can think of two solutions related by (21.29) as a single solution being viewed in two different reference frames moving at constant relative velocity.

So far we have only shown how to get solitary wave solutions to the sine-Gordon equation, and this is all we shall do here. However, it can be shown that there exist solutions to the sine-Gordon equation that have properties such as we saw when we looked at the wave equation and which justify calling these solutions "solitons". At very early times (mathematically: $t \to -\infty$) the solution takes the form of two solitary waves of the type just described, very far apart and approaching each other at relative speed $2V$. This (more complicated) solution is usually called a "two soliton solution". For the sake of our further discussion, let us call the solitary wave moving toward the left as "soliton 1" and the one moving to the right as "soliton 2". As time runs the solution has a relatively complicated wave profile as the two solitary waves overlap and "interact". At late times (mathematically: $t \to \infty$), the solution again takes the form of two solitons, with soliton 1 now moving off toward $x = -\infty$ and soliton 2 moving off toward $x = +\infty$. Thus this solution can be viewed as a continuum model of two particles which approach each other, interact, then continue on their way unscathed. Moreover, solutions of this type also exist for any number of solitons. It is this stability of the solitons as they propagate and interact with each other which is the defining feature of the soliton solutions. The structural stability of the individual solitons is due to another remarkable property exhibited by the sine-Gordon equation: it admits infinitely many conservation laws! This will be explored in the Problems.