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## Taylor's Theorem and Taylor Series (Appendix A)

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## Appendix A. Taylor's Theorem and Taylor Series

Taylor's theorem and the Taylor series constitute one of the more important tools used by mathematicians, physicists and engineers. They provides a means of approximating a function in the vicinity of any chosen point in terms of polynomials. To begin, we present *Taylor's theorem*, which is an identity satisfied by any function f(x) that has continuous derivatives of, say, order (n + 1) on some interval  $a \le x \le b$ . Taylor's theorem asserts that

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \ldots + \frac{1}{n!}f^{(n)}(a)(x-a)^n + R_{n+1}, \quad (A.1)$$

where  $R_{n+1}$  – the *remainder* – can be expressed as

$$R_{n+1} = \frac{1}{(n+1)!} (x-a)^{n+1} f^{(n+1)}(\xi), \qquad (A.2)$$

for some  $\xi$  with  $a \leq \xi \leq b$ . Here we are using the notation

$$f^{(k)}(c) = \frac{d^k f}{dx^k}\Big|_{x=c}.$$
 (A.3)

The number  $\xi$  is not arbitrary; it is determined (though not uniquely) via the mean value theorem of calculus. For our purposes we just need to know that it lies between a and b. The equation (A.1) is an identity; it involves no approximations.

The idea is that for many functions the value of n can be chosen so that the remainder is sufficiently small compared to the polynomial terms that we can omit it. In this case we get Taylor's approximation:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \ldots + \frac{1}{n!}f^{(n)}(a)(x-a)^n.$$
(A.4)

Typically, the approximation is reasonable provided x is close enough to a and none of the derivatives of f get too large in the region of interest. As you can see, if (x - a) is small, *i.e.*,  $x - a \ll 1$ , successive powers of (x - a) become smaller and smaller so that one need only keep a few terms in the polynomial expansion to get a good approximation.

If you can prove that

$$\lim_{n \to \infty} R_n = 0, \tag{A.5}$$

then it makes sense to consider expressing f(x) as a power series:

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!}f^{(n)}(a)(x-a)^n, \quad (A.6)$$

which is known in this context as the *Taylor series* for f. (Note that here we use the definitions 0! = 1 and  $f^{(0)}(x) = f(x)$ .) Normally the Taylor series of a function will

converge in some neighborhood of x = a and diverge outside of this neighborhood.<sup>\*</sup> In any case, for a sufficiently "well-behaved" function, one can usually get a good approximation to the function using (A.4) even with n being relatively small. How small n needs to be depends, in large part, on how big (x - a) is. Often times one can get away with just choosing n = 1 or perhaps n = 2 for x sufficiently close to a.

As a simple example, consider the sine function  $f(x) = \sin(x)$ . Let us approximate the sine function in the vicinity of x = 0, so that we are taking a = 0 in the above formulas. The *zeroth-order* approximation amounts to using n = 0 in (A.4). We get

$$\sin(x) \approx \sin(0) = 0. \tag{A.7}$$

This is obviously not a terribly good approximation. But you can check (using your calculator *in radian mode*) that if x is nearly zero, so is sin(x). A better approximation, the *first-order* approximation, arises when n = 1 in (A.4). We get (exercise)

$$\sin(x) \approx \sin(0) + \cos(0)x = x. \tag{A.8}$$

Again, you can check this approximation on your calculator. If x is kept sufficiently small (in radians), this approximation does a pretty good job. As x gets larger the approximation gets less accurate. For example, at x = 0.1 the error in the approximation is about 0.2%. At x = 0.75, the error is about 10%. The *second-order* approximation is identical to the first-order approximation, as you can check explicitly (exercise). The third-order approximation (exercise),

$$\sin(x) \approx x - \frac{1}{6}x^3 \tag{A.9}$$

is considerably better than the first-order approximation. It gives good results out to, say, x = 1.7, where the error is about 11%. Incidentally, the remainder term for the sine function satisfies (A.5) (exercise), and we can represent the analytic function  $\sin(x)$  by its (everywhere convergent) Taylor series:

$$\sin(x) = x - \frac{1}{6}x^3 + \ldots + \frac{1}{n!}x^n + \ldots, \quad n \text{ odd.}$$

<sup>\*</sup> As usual, "convergence" in this context means that the sequence of partial sums approaches the stated value in the limit. Functions that can be represented by a convergent Taylor series are called *analytic*.