Problem Set 1

Charles G. Torre
charles.torre@usu.edu

PROBLEM SET 1

Problem 1.1

Verify that each of the three forms of harmonic motion

$$\begin{aligned} q(t) &= B\sin(\omega t + \phi) \\ &= C\cos(\omega t + \psi) \\ &= D\cos\omega t + E\sin\omega t. \end{aligned}$$

satisfies the harmonic oscillator equation. Give formulas relating the constants B, C, D, E, ϕ, ψ in each case, *i.e.,* given B and ϕ in the first form of the motion, how to compute C, ψ and D, E?

Problem 1.2

Consider a potential energy function $V(x) = ax^2 + bx^4$. Discuss the possible equilibrium positions and their stability in terms of a and b. For $a > 0, b > 0$ show that the frequency for oscillations near equilibrium is independent of b.

Problem 1.3

An oscillator starts from rest at the initial position q_0. Give formulas for the motion in each of the 3 trigonometric forms (1.1)–(1.3). Do the same for the case where the initial velocity v_0 is non-zero but the initial position is zero.

Problem 1.4

Suppose we changed the sign in the harmonic oscillator equation so that the equation becomes

$$\frac{d^2q}{dt^2} = \omega^2 q. \tag{1.40}$$

(a) What shape must the potential energy graph have in the neighborhood of an equilibrium point to lead to this (approximate) equation?

(b) Find the general solution to this equation. In particular, show that the solutions can grow exponentially with time (rather than oscillate). (This is what it means for the equilibrium to be *unstable*: if the system starts off ever so slightly from equilibrium, no matter how close the initial conditions are to equilibrium, the solutions can depart from the initial values by arbitrarily large amounts.)

(c) For what initial conditions will the exponential growth found in (b) *not* occur.

(d) Show that (1.40) and its solutions can be obtained from the general complex solution to the harmonic oscillator equation by letting the oscillator frequency become imaginary.

Problem 1.5

Find the absolute value and phase of the following complex numbers:

(a) $3 + 5i$

(b) 10

(c) $10i$.

Problem 1.6

Calculate z^2 and $|z|^2$ using the polar ($re^{i\theta}$) and Cartesian ($x + iy$) parameterizations of a complex number. Using the usual relationship between polar and Cartesian coordinates show that these results agree.

Problem 1.7

Prove that two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal.

Problem 1.8

Let ω be a given real number. Show that $q(t) = Ae^{i\omega t} + Be^{-i\omega t}$ is a real number for all values of t if *and only if* $A^* = B$. (*Hint:* A necessary condition is that $q(0)$ is real and that $\frac{dq(0)}{dt}$ is real.)

Problem 1.9

In the text it is shown that one can find the general *real* solution to the harmonic oscillator equation by taking the real part of a complex solution: $q = \mathrm{Re}(\alpha e^{i\omega t})$. It is pointed out that one can just as well take the imaginary part, $\tilde{q} = \mathrm{Im}(\alpha e^{i\omega t})$. Evidently we have two versions, q and $\tilde{q}$, of the general (real) solution to the harmonic oscillator equation. How are these two versions of the general real solution related?

Problem 1.10

With $q(t) = 2Re(\alpha e^{i\omega t})$, show that the initial position and velocities are given by $q(0) = 2Re(\alpha)$ and $v(0) = -2\omega Im(\alpha)$.

Problem 1.11

Using Euler's formula (1.28), prove the following trigonometric identities.

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$

and

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

(*Hint:* $e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}$.)

Problem 1.12

Generalize equation (1.9) to allow for an arbitrary equilibrium point q_0 and an arbitrary value for the potential there, $V(q_0) \equiv V_0$. (Just consider the Taylor expansion around an arbitrary point q_0.) Show that the corresponding generalization of (1.12) is again the harmonic oscillator equation, but now the dependent variable is the displacement from equilibrium $(q - q_0)$.

Problem 1.13

Show that the set of complex numbers forms a vector space.

Problem 1.14

Using the complex form of a real solution to the harmonic oscillator equation,

$$q(t) = \alpha e^{i\omega t} + \alpha^* e^{-i\omega t},$$

show that the solution can be expressed in the forms (1.1) and (1.3). (*Hint:* Make use of the polar representation of the complex number α and Euler's formula.)

Problem 1.15

The *energy* of a harmonic oscillator can be defined by

$$E = \frac{\gamma}{2}\left\{\left(\frac{dq}{dt}\right)^2 + \omega^2 q^2\right\},$$

where γ is a constant (needed to get the units right). The energy E is *conserved*, that is, it doesn't depend upon time. Prove this in the following two distinct ways.

(i) Substitute one of the general forms (1.1) – (1.3) of harmonic motion into E and show that the time dependence drops out.

(ii) Take the time derivative of E and show that it vanishes provided (1.6) holds (without using the explicit form of the solutions).

Problem 1.16

Consider the scalar product (1.19).

(i) Show that the solutions to the oscillator equation given by $\cos(t)$ and $\sin(t)$ are orthogonal and have unit length with respect to this scalar product.

(ii) More generally, show that the scalar product between two solutions is of the form

$$(q_1, q_2) = A_1 A_2 \cos(\alpha),$$

where A_1 and A_2 are the amplitudes of the two solutions and α is the difference in phase between the solutions. *Hint: Take the solutions to be of the form* $A\cos(\omega t + \phi)$.

(*Hint: This one is really easy!*)

Problem 1.17

It was pointed out in §1.2 that the functions $\cos(\omega t)$ and $\sin(\omega t)$ are orthogonal with respect to the scalar product (1.19). This implies they are linearly independent when viewed as elements of the vector space of solutions to the harmonic oscillator equation. Prove directly that these functions are linearly independent, *i.e.*, prove that if a and b are constants such that, for all values of t,

$$a\cos(\omega t) + b\sin(\omega t) = 0,$$

then $a = b = 0$.

Problem 1.18

Derive Euler's formula,

$$e^{i\theta} = \cos(\theta) + i\sin(\theta),$$

in two distinct ways.

(i) By comparing the Taylor series for the left and right hand sides, using:

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \quad \cos(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} (-1)^n x^{2n}, \quad \sin(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} (-1)^n x^{2n+1}$$

(ii) By using the fact that $f(\theta) \equiv e^{i\theta}$ is the *unique* solution of the differential equation:

$$f'' = -f,$$

with initial conditions

$$f(0) = 1, \quad f'(0) = i.$$

(*Hint:* Since f satisfies the harmonic oscillator equation it can be written in the form (1.3).)