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Problem Set 10

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PROBLEM SET 10

Problem 10.1

Show that if ϕ is any solution of the sine-Gordon equation then $\tilde{\phi} = -\phi$ and $\tilde{\phi} = \phi + 2n\pi$, where n is an integer, are also solutions to the sine-Gordon equation. (These are *discrete symmetries* of the sine-Gordon equation.)

Problem 10.2

Verify the continuity equation (21.17) for the sine-Gordon equation.

Problem 10.3

Plot the soliton solution (21.12) and its energy density as a function of x, verifying that the soliton is a localized "lump" of energy. How do these graphs change as you vary λ and m? (Here is a good place to use a computer.)

Problem 10.4

Verify the identities (21.14) and (21.15).

Problem 10.5

The results of this problem will be used in the next problem. Define

$$u = \frac{m}{2}(t - x), \quad v = \frac{m}{2}(t + x), \quad \tilde{\phi}(u, v) = \frac{\sqrt{\lambda}}{m}\phi(t(u, v), x(u, v)).$$

Show that the sine-Gordon equation can be written as

$$\frac{\partial^2 \tilde{\phi}}{\partial u \partial v} + \sin(\tilde{\phi}) = 0. \tag{21.30}$$

Problem 10.6

Using the notation of the previous problem, show that the quantities

$$\rho = -\frac{1}{2}\left(\frac{\partial^2 \tilde{\phi}}{\partial u^2}\right)^2 + \frac{1}{8}\left(\frac{\partial \tilde{\phi}}{\partial u}\right)^4,$$

and

$$j = \frac{\partial^2 \tilde{\phi}}{\partial u^2}\frac{\partial^2 \tilde{\phi}}{\partial u \partial v}\sin(\tilde{\phi}) + \frac{\partial^2 \tilde{u}}{\partial u^2}\sin(\tilde{\phi}) - \frac{1}{2}\left(\frac{\partial \tilde{\phi}}{\partial u}\right)^2 \cos(\tilde{\phi}).$$

satisfy a continuity equation

$$\frac{\partial\rho}{\partial v} + \frac{\partial j}{\partial u} = 0,$$

when $\tilde{\phi}$ satisfies (21.30). The resulting conservation law is but one of an infinite hierarchy of conserved quantities that are intimately related to the stability of the sine-Gordon solitons.

The End

Congratulations. You started off trying to understand trig functions and the harmonic oscillator equation. A relatively short time later you were contemplating non-linear wave equations, solitons, and infinitely many conservation laws. And we're just getting started...