Foundation for an Energy-Vorticity Turbulence Model with Application to Flow near Rough Surfaces[*]

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Based on a more direct analogy between turbulent and molecular transport, a foundation is presented for an *energy-vorticity* turbulence model. Whereas traditional *k-ε*, *k-ω*, and *k-ζ* models relate the eddy viscosity to a dissipation length scale associated with the smaller eddies having the highest strain rates, the proposed model relates the eddy viscosity to a *mean vortex wavelength* associated with the larger eddies primarily responsible for turbulent transport. A rigorous development of the turbulent-energy-transport equation from the Navier-Stokes equations includes exact relations for the viscous dissipation and molecular transport of turbulent kinetic energy. Application of Boussinesq's analogy between turbulent and molecular transport leads to a transport equation, which shows neither molecular nor turbulent transport of turbulent energy to be simple gradient diffusion. The new turbulent-energy-transport equation contains two closure coefficients; a viscous-dissipation coefficient and a turbulent-transport coefficient. To help evaluate closure coefficients and provide insight into the energy-vorticity turbulence variables, fully rough pipe flow is considered. For this fully developed flow, excellent agreement with experimental data for velocity profiles and friction factors is attained over a wide range of closure coefficients, provided that a given relation between the coefficients is maintained.

Nomenclature

$A_{\lambda 0}$	=	empirical coefficient, Eqs. (98) and (101)
$A_{\lambda 1}$	=	empirical coefficient, Eqs. (98) and (114)
$A_{\lambda 10}$	=	empirical coefficient, Eqs. (102) and (114)
$A_{\lambda 11}$	=	empirical coefficient, Eqs. (103) and (114)
$A_{\lambda 12}$	=	empirical coefficient, Eqs. (104) and (114)
$A_{\lambda 13}$	=	empirical coefficient, Eqs. (105) and (114)
$B_{\lambda 0}$	=	empirical coefficient, Eqs. (98) and (115)
$B_{\lambda 00}$	=	empirical coefficient, Eqs. (106) and (115)
$B_{\lambda 01}$	=	empirical coefficient, Eqs. (107) and (115)
$B_{\lambda 02}$	=	empirical coefficient, Eqs. (108) and (115)
$B_{\lambda 03}$	=	empirical coefficient, Eqs. (109) and (115)
$B_{\lambda 1}$	=	empirical coefficient, Eqs. (98) and (116)
$B_{\lambda 10}$	=	empirical coefficient, Eqs. (110) and (116)
$B_{\lambda 11}$	=	empirical coefficient, Eqs. (111) and (116)
$B_{\lambda 2}$	=	empirical coefficient, Eqs. (98) and (117)

$B_{\lambda 20}$	=	empirical coefficient, Eqs. (112) and (117)
$B_{\lambda 21}$	=	empirical coefficient, Eqs. (113) and (117)
C_f	=	Fanning friction factor also called the skin-friction coefficient
C_{0-7}	=	empirical coefficients, Eqs. (94) and (95)
C_{0-4}	=	empirical coefficients, Eqs. (104), (108), and (112)
$C_{\varepsilon 1}, C_{\varepsilon 2}$	=	turbulence model closure coefficients, Eqs. (7) and (8)
$C_{\zeta 1} - C_{\zeta 5}$	=	turbulence model closure coefficients, Eqs. (27) and (28)
$C_{\omega 1}, C_{\omega 2}$	=	turbulence model closure coefficients, Eqs. (17) and (18)
C_λ	=	turbulence model closure coefficient, Eq. (85)
C_μ	=	turbulence model closure coefficient, Eqs. (3) and (8)
C_ν	=	turbulence model closure coefficient, Eqs. (78) and (81)
D	=	pipe diameter
D_{0-3}	=	empirical coefficients, Eq. (95)
E	=	wall damping function, Eq. (10)
f_1, f_2	=	wall damping functions, Eq. (10) or (21)
f_k	=	wall damping function, Eq. (20)
f_μ	=	wall damping function, Eq. (9) or (19)
g_o	=	standard acceleration of gravity at sea level
$\bar{\bar{\mathbf{J}}}$	=	Jacobian tensor for a vector field
k	=	turbulent kinetic energy per unit mass, Eq. (1)
k_s	=	equivalent sand-grain roughness
k_s^+	=	wall-scaled dimensionless roughness, called the roughness Reynolds number, $k_s^+ = u_\tau k_s / \nu$
k_{wall}^+	=	dimensionless proportionality coefficient, Eq. (97)
$\hat{k}_s$	=	pipe-scaled dimensionless roughness, $\hat{k}_s = k_s / R$
ℓ_k	=	mean turbulent radius of gyration, Eq. (77)
p	=	instantaneous local pressure
$\bar{p}$	=	mean local pressure
$\widetilde{p}$	=	fluctuating local pressure, $\widetilde{p} \equiv p - \bar{p}$
$\hat{p}$	=	total hydrostatic pressure, $\hat{p} \equiv p + \rho g_o Z + \frac{2}{3}\mu \nabla \cdot \mathbf{V}$
$\bar{\hat{p}}$	=	mean total hydrostatic pressure, $\bar{\hat{p}} \equiv \bar{p} + \rho g_o Z + \frac{2}{3}\mu \nabla \cdot \overline{\mathbf{V}}$
$\widetilde{\hat{p}}$	=	fluctuating total hydrostatic pressure, $\widetilde{\hat{p}} \equiv \widetilde{p} + \frac{2}{3}\mu \nabla \cdot \widetilde{\mathbf{V}}$
$\check{\hat{p}}$	=	pseudo mean pressure, $\check{\hat{p}} \equiv \bar{p} + \rho g_o Z + \frac{2}{3}\rho[k + (\nu + \nu_t)\nabla \cdot \overline{\mathbf{V}}]$
R	=	pipe radius
$\hat{R}_s$	=	change of variables, $\hat{R}_s = R/(\gamma k_s)$
r	=	radial coordinate measured outward from the pipe centerline
$\hat{r}$	=	pipe-scaled dimensionless coordinate, $\hat{r} = r / R$
$\bar{\bar{\mathbf{S}}}$	=	strain-rate tensor for a vector field
$\overline{S}$	=	magnitude of the mean strain-rate tensor, Eq. (56)
$\widetilde{S}$	=	magnitude of the fluctuating strain-rate tensor, Eq. (57)
t	=	time
u_τ	=	shear velocity, $u_\tau = (\tau_w / \rho)^{1/2}$
$\mathbf{V}$	=	instantaneous local velocity vector
V_m	=	bulk velocity

$\overline{\mathbf{V}}$	=	mean velocity vector
$\overline{V}_z$	=	axial component of the mean velocity vector
$\widetilde{\mathbf{V}}$	=	fluctuating velocity vector, $\widetilde{\mathbf{V}} \equiv \mathbf{V} - \overline{\mathbf{V}}$
V	=	magnitude of the instantaneous local velocity vector
$\overline{V}$	=	magnitude of the mean velocity vector
$\widetilde{V}$	=	magnitude of the fluctuating velocity vector
$\widetilde{V}_x, \widetilde{V}_y, \widetilde{V}_z$	=	Cartesian components of the fluctuating velocity vector
y	=	normal coordinate measured into the fluid from a wall
y^+	=	wall-scaled dimensionless coordinate, $y^+ = u_\tau y / \nu$
$\hat{y}$	=	pipe-scaled dimensionless coordinate, $\hat{y} = y / R$
Z	=	geopotential altitude, $Z = R_E H / (R_E + H)$, H is geometric altitude and R_E is the radius of the Earth
γ	=	Nikuradse constant
$\overline{\overline{\boldsymbol{\delta}}}$	=	Kronecker delta
δ	=	deviation function, Eqs. (94) and (95)
ε	=	turbulent-energy-dissipation parameter, Eq. (2)
ε_o	=	wall damping function, Eq. (11)
ε_v	=	turbulent-energy-dissipation parameter, $\varepsilon_v \equiv \varepsilon - \varepsilon_o$
$\widetilde{\varepsilon}$	=	exact turbulent-energy dissipation per unit mass, Eq. (59)
ζ	=	turbulent-energy-dissipation parameter in the Robinson-Hassan k-ζ turbulence model, Eq. (22)
ζ	=	mean fluctuating enstrophy, Eq. (80)
κ	=	von Kármán constant
λ	=	mean vortex wavelength, Eq. (83)
μ	=	dynamic molecular viscosity
μ_t	=	dynamic eddy viscosity
ν	=	kinematic molecular viscosity
ν_t	=	kinematic eddy viscosity
ρ	=	fluid density
σ_k	=	turbulence model closure coefficient, Eqs. (6), (8), (18), (28), and (52)
σ_ε	=	turbulence model closure coefficient, Eqs. (7) and (8)
σ_ζ	=	turbulence model closure coefficient, Eqs. (27) and (28)
σ_ω	=	turbulence model closure coefficient, Eqs. (17) and (18)
$\overline{\overline{\boldsymbol{\tau}}}$	=	turbulent Reynolds stress tensor, Eq. (29)
τ_w	=	pipe wall shear stress
$\overline{\boldsymbol{\Omega}}$	=	mean vorticity vector, $\overline{\boldsymbol{\Omega}} = \nabla \times \overline{\mathbf{V}}$
$\overline{\Omega}$	=	magnitude of the mean vorticity vector, $\overline{\Omega} = (\overline{\boldsymbol{\Omega}} \cdot \overline{\boldsymbol{\Omega}})^{1/2}$
ω	=	turbulent-energy-dissipation frequency, $\omega \equiv \varepsilon / (C_\mu k)$
ω^+	=	wall-scaled dimensionless dissipation frequency, $\omega^+ = \nu \omega / u_\tau^2$
$\widetilde{\omega}$	=	root mean square fluctuating vorticity, Eq. (68)

I. Introduction

Many of the turbulence models that are now commonly used for computational fluid dynamics (CFD) are based on the analogy between molecular and turbulent transport that was first proposed by Boussinesq.[1] The majority of these turbulence models are usually classified as either k-ε, k-ω, or k-ζ models. Conventional k-ε, k-ω, and k-ζ turbulence models are often thought of as being fundamentally different. Yet, in a larger sense, these three

model classifications could all be thought of as energy-dissipation models. This is because all such models are based on the hypothesis that Boussinesq's eddy viscosity is proportional to the product of the root mean square fluctuating velocity, or $k^{1/2}$, and the dissipation length scale $k^{3/2}/\varepsilon$. The parameters k and ε are defined in terms of the fluctuating velocity as

$$k \equiv \tfrac{1}{2}\overline{\widetilde{\mathbf{V}}\cdot\widetilde{\mathbf{V}}} = \tfrac{1}{2}\overline{\widetilde{V}^2} \tag{1}$$

$$\varepsilon \equiv \nu\,\overline{\bar{\bar{\mathbf{J}}}(\widetilde{\mathbf{V}})\cdot\bar{\bar{\mathbf{J}}}(\widetilde{\mathbf{V}})} \tag{2}$$

where $\widetilde{\mathbf{V}}$ is the fluctuating velocity vector, $\bar{\bar{\mathbf{J}}}(\widetilde{\mathbf{V}})$ is its Jacobian tensor, and the overscore denotes an ensemble mean.

The eddy-viscosity model that is the foundation for all commonly used k-ε, k-ω, and k-ζ turbulence models is

$$\nu_t = C_\mu\,k^2\big/\varepsilon \tag{3}$$

where C_μ is a dimensionless closure coefficient that is nearly universally accepted as being equal to 0.09. The k-ε turbulence models use Eq. (3) directly. The k-ω turbulence models use the change of variables $\omega \equiv \varepsilon/(C_\mu k)$ to transform Eq. (3) to the equivalent relation given by $\nu_t = k/\omega$. Similarly, the k-ζ turbulence models use the change of variables $\zeta \equiv \varepsilon/\nu$ to transform Eq. (3) to its k-ζ equivalent, $\nu_t = C_\mu k^2/(\nu\zeta)$. The commonly used k-ε, k-ω, and k-ζ turbulence models are all based on the hypothesis that the characteristic length scale for turbulent transport is proportional to the characteristic length scale for turbulent-energy dissipation.

The k-ε turbulence model that is the foundation for most modern Boussinesq-based turbulence models is that of Jones and Launder.[2] In addition to the algebraic equation for the kinematic eddy viscosity that is given by Eq. (3), the Jones-Launder turbulence model comprises the following equations for incompressible flow; the ensemble-averaged continuity equation,

$$\nabla\cdot\overline{\mathbf{V}} = 0 \tag{4}$$

the Boussinesq-based Reynolds-averaged-Navier-Stokes (RANS) equations,

$$\frac{\partial\overline{\mathbf{V}}}{\partial t} + (\overline{\mathbf{V}}\cdot\nabla)\overline{\mathbf{V}} = -\nabla\hat{\overline{p}}\big/\rho + \nabla\cdot[2(\nu+\nu_t)\bar{\bar{\mathbf{S}}}(\overline{\mathbf{V}})] \tag{5}$$

the Boussinesq-based turbulent-energy-transport equation,

$$\frac{\partial k}{\partial t} + \overline{\mathbf{V}}\cdot\nabla k = 2\nu_t\bar{\bar{\mathbf{S}}}(\overline{\mathbf{V}})\cdot\bar{\bar{\mathbf{S}}}(\overline{\mathbf{V}}) - \varepsilon + \nabla\cdot[(\nu+\nu_t/\sigma_k)\nabla k] \tag{6}$$

and a turbulent-dissipation-transport equation obtained by analogy with Eq. (6)

$$\frac{\partial\varepsilon}{\partial t} + \overline{\mathbf{V}}\cdot\nabla\varepsilon = 2C_{\varepsilon 1}\nu_t\frac{\varepsilon}{k}\bar{\bar{\mathbf{S}}}(\overline{\mathbf{V}})\cdot\bar{\bar{\mathbf{S}}}(\overline{\mathbf{V}}) - C_{\varepsilon 2}\frac{\varepsilon^2}{k} + \nabla\cdot[(\nu+\nu_t/\sigma_\varepsilon)\nabla\varepsilon] \tag{7}$$

The commonly used closure coefficients for the k-ε model are

$$C_\mu = 0.09, \quad C_{\varepsilon 1} = 1.44, \quad C_{\varepsilon 2} = 1.92, \quad \sigma_k = 1.0, \quad \sigma_\varepsilon = 1.3 \tag{8}$$

In this form, the Jones-Launder k-ε turbulence model does not exhibit the proper behavior near a solid surface. Near a no-slip boundary the turbulent velocity fluctuations and turbulent transport are suppressed by the proximity of the solid surface. Modeling this suppression accurately is critical to obtaining accurate predictions for the wall shear stress and heat transfer.

In the attempt to provide realistic results near a smooth solid surface, the Jones-Launder k-ε model is often implemented with the incorporation of what are commonly called wall damping functions. In a general form, these wall damping functions are added to Eq. (3), Eq. (7), and the definition of ε,

$$\nu_t = C_\mu f_\mu k^2 / \varepsilon_\nu \tag{9}$$

$$\frac{\partial \varepsilon_\nu}{\partial t} + \overline{\mathbf{V}} \cdot \nabla \varepsilon_\nu = 2C_{\varepsilon 1} f_1 \nu_t \frac{\varepsilon_\nu}{k} \bar{\bar{\mathbf{S}}}(\overline{\mathbf{V}}) \cdot \bar{\bar{\mathbf{S}}}(\overline{\mathbf{V}}) - C_{\varepsilon 2} f_2 \frac{\varepsilon_\nu^2}{k} + E + \nabla \cdot [(\nu + \nu_t/\sigma_\varepsilon)\nabla \varepsilon_\nu] \tag{10}$$

$$\varepsilon = \varepsilon_\nu + \varepsilon_o \tag{11}$$

A variety of k-ε turbulence models have been proposed, which differ only in the form of the wall damping functions f_μ, f_1, f_2, E, and ε_o. To complete any k-ε model of this form, the wall damping functions are specified as prescribed functions of ν, $\overline{\mathbf{V}}$, k, ε_ν, and the normal coordinate y, measured into the fluid from the wall. The wall damping functions used for the k-ε model are simply empirical corrections that are added to force the model to agree more closely with near-wall experimental data. Although several variations for the k-ε wall damping functions have been proposed,[3–7] none have been completely successful.

The k-ω turbulence models that are commonly used for CFD are built on exactly the same dissipation-based eddy-viscosity model that is given in Eq. (3), where k and ε are defined in Eqs. (1) and (2). These commonly used k-ω turbulence models are based on applying a simple change of variables to Eq. (3), i.e.,

$$\omega \equiv \varepsilon / (C_\mu k) \tag{12}$$

Examining the dimensions of k and ε from the definitions in Eqs. (1) and (2), we see that ω is a frequency, which is directly proportional to the dissipation parameter ε. The change of variables defined in Eq. (12) applied to Eq. (3) yields an algebraic equation for the kinematic eddy viscosity in terms of only the turbulent kinetic energy per unit mass, k, and the turbulent-energy-dissipation frequency, ω,

$$\nu_t = k / \omega \tag{13}$$

In addition to this algebraic equation for the kinematic eddy viscosity, the k-ω turbulence model proposed originally by Kolmogorov[8] has been refined to comprise the following equations for incompressible flow; the continuity equation combined with the Boussinesq-RANS equations,

$$\nabla \cdot \overline{\mathbf{V}} = 0 \tag{14}$$

$$\frac{\partial \overline{\mathbf{V}}}{\partial t} + (\overline{\mathbf{V}} \cdot \nabla)\overline{\mathbf{V}} = -\nabla \hat{p}/\rho + \nabla \cdot [2(\nu + \nu_t)\bar{\bar{\mathbf{S}}}(\overline{\mathbf{V}})] \tag{15}$$

the Boussinesq-based turbulent-energy-transport equation obtained by applying the change of variables defined in Eq. (12) to Eq. (6),

$$\frac{\partial k}{\partial t} + \overline{\mathbf{V}} \cdot \nabla k = 2\nu_t \bar{\bar{\mathbf{S}}}(\overline{\mathbf{V}}) \cdot \bar{\bar{\mathbf{S}}}(\overline{\mathbf{V}}) - C_\mu k\omega + \nabla \cdot [(\nu + \nu_t/\sigma_k)\nabla k] \tag{16}$$

and a dissipation-frequency-transport equation obtained by analogy with Eq. (16),

$$\frac{\partial \omega}{\partial t} + \overline{\mathbf{V}} \cdot \nabla \omega = 2C_{\omega 1} \nu_t \frac{\omega}{k} \bar{\bar{\mathbf{S}}}(\overline{\mathbf{V}}) \cdot \bar{\bar{\mathbf{S}}}(\overline{\mathbf{V}}) - C_{\omega 2}\omega^2 + \nabla \cdot [(\nu + \nu_t/\sigma_\omega)\nabla \omega] \tag{17}$$

The closure coefficients differ slightly from one version of the model to another and have changed as the model has evolved over the past six decades. In the original k-ω model, Kolmogorov[8] assumed $C_{\omega 1} = 0$ and he did not include the molecular diffusion term. The closure coefficients often used for the k-ω model[9,10] are

$$C_\mu = 0.09, \quad C_{\omega 1} = 0.52, \quad C_{\omega 2} = 0.072, \quad \sigma_k = 2.0, \quad \sigma_\omega = 2.0 \tag{18}$$

It should be noted that the turbulence variable ω, which is defined in Eq. (12) and referred to here as the *dissipation frequency*, is commonly called the *specific dissipation rate*. However, a specific property is defined traditionally to be a property per unit mass, so the expression *specific dissipation rate* could be easily confused with

the *dissipation rate per unit mass*. Hence, because ω is a frequency defined directly from the viscous dissipation parameter ε using Eq. (12), we shall continue to refer to ω as the *dissipation frequency*.

As is the case for the k-ε model, the standard k-ω model does not exhibit the proper behavior near a solid wall. By direct analogy with what has been done with the k-ε model, the k-ω model can also be implemented with the incorporation of wall damping functions. Although this terminology is not commonly used with the k-ω model, to emphasize similarities between the low-Reynolds-number corrections used for the k-ω model and those used for the k-ε model, here we will use exactly the same notation and terminology for both models. Adding wall damping functions to Eqs. (13), (16), and (17) yields

$$\nu_t = f_\mu k / \omega \tag{19}$$

$$\frac{\partial k}{\partial t} + \overline{\mathbf{V}} \cdot \nabla k = 2\nu_t \overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}}) \cdot \overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}}) - C_\mu f_k k \omega + \nabla \cdot [(\nu + \nu_t / \sigma_k) \nabla k] \tag{20}$$

$$\frac{\partial \omega}{\partial t} + \overline{\mathbf{V}} \cdot \nabla \omega = 2 C_{\omega 1} f_1 \nu_t \frac{\omega}{k} \overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}}) \cdot \overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}}) - C_{\omega 2} f_2 \omega^2 + \nabla \cdot [(\nu + \nu_t / \sigma_\omega) \nabla \omega] \tag{21}$$

To complete any k-ω turbulence model in this form, the wall damping functions f_μ, f_k, f_1, and f_2, could be specified as prescribed functions of ν, $\overline{\mathbf{V}}$, k, and ω. As an example of a k-ω turbulence model that includes such wall damping functions, consider what is commonly called the Wilcox 1998 k-ω model,[9] which is implemented in FLUENT.[10] Although Wilcox[9] uses different notation, his k-ω formulation is easily rearranged into the format of Eqs. (19)–(21), see Phillips, Hunsaker, and Spall.[11] As is the case for the k-ε model, these wall damping functions are simply empirical corrections, which are employed to force the model to agree more closely with near-wall experimental data.

For the most recent advancements in the k-ω model, including wall boundary conditions for rough and hydraulically smooth surfaces, see Wilcox.[12,13] As Wilcox points out, the ability to implement rough-wall boundary conditions is a key advantage of the k-ω parameterization over the k-ε parameterization.

There is also another change of variables that has been less commonly applied to the k-ε turbulence model. This is based on a turbulence variable called *enstrophy*, which is commonly interpreted to be the mean squared magnitude of the fluctuating vorticity. Although this was presented as the physical interpretation of ζ in their original development of the k-ζ turbulence model, Robinson et al.[14,15] defined the approximate turbulent-energy dissipation as $\varepsilon \equiv \nu \zeta$, which constitutes a simple change of variables from the k-ε model, i.e.,

$$\zeta \equiv \varepsilon / \nu \tag{22}$$

Robinson et al.[14,15] applied this change of variables to Eq. (3) to obtain an algebraic equation for the kinematic eddy viscosity in terms of the turbulent kinetic energy per unit mass, k, the molecular viscosity, ν, and their so called turbulent enstrophy, ζ,

$$\nu_t = C_\mu k^2 / (\nu \zeta) \tag{23}$$

In addition to this algebraic equation for the kinematic eddy viscosity, the Robinson-Hassan k-ζ turbulence model comprises the following equations for incompressible flow; the continuity equation combined with the Boussinesq-RANS equations,

$$\nabla \cdot \overline{\mathbf{V}} = 0 \tag{24}$$

$$\frac{\partial \overline{\mathbf{V}}}{\partial t} + (\overline{\mathbf{V}} \cdot \nabla) \overline{\mathbf{V}} = -\nabla \hat{p} / \rho + \nabla \cdot [2(\nu + \nu_t) \overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}})] \tag{25}$$

the Boussinesq-based turbulent-energy-transport equation obtained by applying the change of variables defined in Eq. (22) to Eq. (6),

$$\frac{\partial k}{\partial t} + \overline{\mathbf{V}} \cdot \nabla k = 2\nu_t \overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}}) \cdot \overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}}) - \nu \zeta + \nabla \cdot [(\nu + \nu_t / \sigma_k) \nabla k] \tag{26}$$

and a modeled turbulent-enstrophy-transport equation,

$$\frac{\partial \zeta}{\partial t} + \overline{\mathbf{V}} \cdot \nabla \zeta = [C_{\zeta 1}\zeta/\overline{\Omega}^2 - C_{\zeta 2}v_t^2\overline{\Omega}/vk - C_{\zeta 3}v_t\zeta/k\overline{\Omega}]\overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}}) \cdot (\overline{\Omega}\,\overline{\Omega})$$
$$+ C_{\zeta 2}v_t\overline{\Omega}^3/3v + C_{\zeta 3}\zeta\overline{\Omega}^2/3 - C_{\zeta 4}v\zeta^2/k + C_{\zeta 5}(v_t\zeta/k)\overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}}) \cdot \overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}})$$
$$- 4v_t\overline{\overline{\mathbf{S}}}(\overline{\Omega}) \cdot \overline{\overline{\mathbf{S}}}(\overline{\Omega}) - (\nabla\times\overline{\Omega}) \cdot \{2\nabla\cdot[v_t\overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}})] + \tfrac{1}{3}\nabla k\}$$
$$+ C_{\zeta 6}[(v_t/k)\overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}}) \cdot \overline{\Omega} - \tfrac{1}{3}\overline{\Omega}] \cdot (\nabla k \times \nabla\zeta)/\overline{\Omega}^2 + \nabla\cdot[(v + v_t/\sigma_\zeta)\nabla\zeta] \qquad (27)$$

where $\overline{\Omega} = \nabla\times\overline{\mathbf{V}}$ and $\overline{\Omega}^2 = \overline{\Omega}\cdot\overline{\Omega}$. The closure coefficients used for the k-ζ model are

$$C_\mu = 0.09, \quad C_{\zeta 1} = 1.50, \quad C_{\zeta 2} = 0.40, \quad C_{\zeta 3} = 0.84, \quad C_{\zeta 4} = 2.37,$$
$$C_{\zeta 5} = 0.70, \quad C_{\zeta 6} = 9.20, \quad 1/\sigma_k = 1.8, \quad 1/\sigma_\zeta = 1.46 \qquad (28)$$

II. The Traditional Turbulent-Energy-Transport Equation

The turbulent-energy-transport equation that is used in the traditional k-ε, k-ω, and k-ζ turbulence models is commonly developed from the Navier-Stokes equations and the definition of the *specific Reynolds stress tensor* (i.e., the Reynolds stress tensor divided by the fluid density),

$$\frac{\overline{\overline{\tau}}}{\rho} \equiv -\overline{\widetilde{\mathbf{V}}\widetilde{\mathbf{V}}} = -\begin{bmatrix} \overline{\widetilde{V}_x\widetilde{V}_x} & \overline{\widetilde{V}_x\widetilde{V}_y} & \overline{\widetilde{V}_x\widetilde{V}_z} \\ \overline{\widetilde{V}_y\widetilde{V}_x} & \overline{\widetilde{V}_y\widetilde{V}_y} & \overline{\widetilde{V}_y\widetilde{V}_z} \\ \overline{\widetilde{V}_z\widetilde{V}_x} & \overline{\widetilde{V}_z\widetilde{V}_y} & \overline{\widetilde{V}_z\widetilde{V}_z} \end{bmatrix} \qquad (29)$$

Multiplying the vector Navier-Stokes equation by the fluctuating velocity vector, taking the ensemble average, and adding the resulting tensor equation to its transpose yields a differential transport equation for the Reynolds stress tensor. From the definitions in Eqs. (1) and (29), the turbulent kinetic energy per unit mass, k, is the negative of one-half the trace of the specific Reynolds stress tensor. Hence, the turbulent-energy-transport equation is commonly obtained from the negative of one-half the trace of the specific Reynolds-stress-transport equation. For the case of compressible flow with constant dynamic viscosity, this yields

$$\frac{\partial k}{\partial t} + (\overline{\mathbf{V}}\cdot\nabla)k = \overline{\overline{\tau}}\cdot\overline{\overline{\mathbf{J}}}(\overline{\mathbf{V}})/\rho - v\overline{\overline{\overline{\mathbf{J}}}(\widetilde{\mathbf{V}}) \cdot \overline{\overline{\mathbf{J}}}(\widetilde{\mathbf{V}})} + \overline{\widetilde{p}(\nabla\cdot\widetilde{\mathbf{V}})}/\rho - \tfrac{1}{3}v\overline{(\nabla\cdot\widetilde{\mathbf{V}})^2}$$
$$+ v\nabla^2 k - \{\nabla\cdot[\tfrac{1}{2}\rho\overline{\widetilde{V}^2\widetilde{\mathbf{V}}} + \overline{\widetilde{p}\widetilde{\mathbf{V}}} - \tfrac{1}{3}\mu\overline{(\nabla\cdot\widetilde{\mathbf{V}})\widetilde{\mathbf{V}}}]\}/\rho \qquad (30)$$

The left-hand side of Eq. (30) is the mean substantial derivative of k, which is the time rate of change of the specific turbulent kinetic energy for a fluid element as it moves with the mean flow. The first term on the right-hand side of Eq. (30) is known as the production, because it is the rate at which specific kinetic energy is transferred from the mean flow to the turbulent fluctuations. The second term on the right is commonly referred to as the dissipation per unit mass, because it is usually approximated as being the rate at which specific turbulent kinetic energy is converted to thermal energy through viscous dissipation. The third and fourth terms on the right-hand side of Eq. (30) are called dilatation terms because they account for interchange between turbulent kinetic energy and thermal energy resulting from fluid expansion or compression. The fifth term on the right-hand side of Eq. (30) arises from molecular diffusion, which is the transport of specific turbulent kinetic energy resulting from the molecular motions within the fluid. The remaining terms in Eq. (30) are usually called turbulent transport terms, because they include the transport of specific turbulent kinetic energy that results from the turbulent fluctuations.

The production term in Eq. (30) can be expressed in terms of the eddy viscosity, mean velocity, and specific turbulent kinetic energy by using the Boussinesq hypothesis, which yields

$$\overline{\overline{\tau}}/\rho = 2v_t\overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}}) - \tfrac{2}{3}(k + v_t\nabla\cdot\overline{\mathbf{V}})\overline{\overline{\delta}} \qquad (31)$$

Applying Eq. (31), the first term on the right-hand side of Eq. (30) can be written as

$$\frac{\bar{\bar{\tau}}}{\rho} \cdot \bar{\bar{\mathbf{J}}}(\overline{\mathbf{V}}) = 2\nu_t \bar{\bar{\mathbf{S}}}(\overline{\mathbf{V}}) \cdot \bar{\bar{\mathbf{J}}}(\overline{\mathbf{V}}) - \tfrac{2}{3}(k + \nu_t \nabla \cdot \overline{\mathbf{V}}) \bar{\bar{\delta}} \cdot \bar{\bar{\mathbf{J}}}(\overline{\mathbf{V}})$$

$$= 2\nu_t \bar{\bar{\mathbf{S}}}(\overline{\mathbf{V}}) \cdot \bar{\bar{\mathbf{S}}}(\overline{\mathbf{V}}) - \tfrac{2}{3}(k + \nu_t \nabla \cdot \overline{\mathbf{V}}) \nabla \cdot \overline{\mathbf{V}} \qquad (32)$$

The last term in Eq. (32) is zero for incompressible flow and it is sometimes neglected even for compressible flow. The approximate dissipation term in Eq. (30) is simply the parameter ε defined in Eq. (2), which is related to ω and ζ through Eqs. (12) and (22), i.e.,

$$\nu \overline{\bar{\bar{\mathbf{J}}}(\widetilde{\mathbf{V}}) \cdot \bar{\bar{\mathbf{J}}}(\widetilde{\mathbf{V}})} \equiv \varepsilon \equiv C_\mu k\omega \equiv \nu\zeta \qquad (33)$$

The dilatation terms in Eq. (30) are zero for incompressible flow, and even for compressible flow they are commonly assumed to be negligible except for the case of flows with high supersonic mean Mach numbers. Accordingly, the third and fourth terms on the right-hand side of Eq. (30) are usually neglected for turbulence models in common use today,

$$\overline{\widetilde{p}(\nabla \cdot \widetilde{\mathbf{V}})} \big/ \rho - \tfrac{1}{3}\nu \overline{(\nabla \cdot \widetilde{\mathbf{V}})^2} \cong 0 \qquad (34)$$

The turbulent transport terms in Eq. (30) are typically combined and modeled as a pure gradient-diffusion process, which is analogous to the molecular diffusion term in Eq. (30). Thus, it is usually assumed that the terms in the square brackets on the last line of Eq. (30) can be combined and modeled as

$$\tfrac{1}{2}\rho \overline{\widetilde{V}^2 \widetilde{\mathbf{V}}} + \overline{\widetilde{p}\widetilde{\mathbf{V}}} - \tfrac{1}{3}\mu \overline{(\nabla \cdot \widetilde{\mathbf{V}})\widetilde{\mathbf{V}}} = -(\mu_t/\sigma_k)\nabla k \qquad (35)$$

where σ_k is a closure coefficient usually treated as a known scalar constant. Using Eqs. (32)–(35) in Eq. (30) yields the traditional modeled version of the turbulent-kinetic-energy-transport equation, which is used with the traditional k-ε, k-ω, and k-ζ turbulence models. For incompressible flow this results in Eqs. (6), (16), and (26).

III. An Alternate Turbulent-Energy-Transport Equation

Most turbulence models that utilize a version of the turbulent-kinetic-energy-transport equation given by Eq. (30) are based on the approximation that the turbulence parameter ε, which is defined in Eq. (2), is the dissipation of turbulent kinetic energy per unit mass. In general, this is not the case. We now consider an alternative formulation of the transport equation for turbulent kinetic energy, which leads to the exact expression for viscous dissipation.

The turbulent-energy-transport equation can be alternately developed from the mechanical energy equation, which is obtained by taking the dot product of the fluid velocity vector with the Navier-Stokes equations written in vector form. The Navier-Stokes equations can be written as

$$\rho\left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V}\right] = \nabla \cdot [2\mu \bar{\bar{\mathbf{S}}}(\mathbf{V})] - \nabla\hat{p} + g_o Z \nabla(\rho) \qquad (36)$$

Taking the dot product of Eq. (36) with the fluid velocity vector yields

$$\rho\mathbf{V} \cdot \left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V}\right] = \mathbf{V} \cdot \{\nabla \cdot [2\mu \bar{\bar{\mathbf{S}}}(\mathbf{V})] - \nabla\hat{p} + g_o Z \nabla(\rho)\} \qquad (37)$$

which is readily rearranged using the mathematical identity $VdV = d(\tfrac{1}{2}V^2)$ to give

$$\rho\left[\frac{\partial}{\partial t}(\tfrac{1}{2}V^2) + \mathbf{V} \cdot \nabla(\tfrac{1}{2}V^2)\right] = \mathbf{V} \cdot \{\nabla \cdot [2\mu \bar{\bar{\mathbf{S}}}(\mathbf{V})] - \nabla\hat{p} + g_o Z \nabla(\rho)\} \qquad (38)$$

Expanding the first term on the right-hand side of Eq. (38) in Cartesian coordinates and rearranging produces the mathematical identity

$$\mathbf{V} \cdot \{\nabla \cdot [2\mu \bar{\bar{\mathbf{S}}}(\mathbf{V})]\} \;=\; \nabla \cdot \{\mu[\nabla(\tfrac{1}{2}V^2) + (\mathbf{V}\cdot\nabla)\mathbf{V}]\} - 2\mu\bar{\bar{\mathbf{S}}}(\mathbf{V})\cdot\bar{\bar{\mathbf{S}}}(\mathbf{V}) \tag{39}$$

Hence, the dot product of the Navier-Stokes equation with the fluid velocity vector can be written as what is called the mechanical energy equation for a Newtonian fluid

$$\rho\left[\frac{\partial}{\partial t}(\tfrac{1}{2}V^2) + \mathbf{V}\cdot\nabla(\tfrac{1}{2}V^2)\right] \;=\; \nabla\cdot\{\mu[\nabla(\tfrac{1}{2}V^2) + (\mathbf{V}\cdot\nabla)\mathbf{V}]\}$$
$$-\,\mathbf{V}\cdot[\nabla\hat{p} - g_o Z\nabla(\rho)] - 2\mu\bar{\bar{\mathbf{S}}}(\mathbf{V})\cdot\bar{\bar{\mathbf{S}}}(\mathbf{V}) \tag{40}$$

The first term on the right-hand side of Eq. (40) accounts for molecular transport of mechanical energy. This divergence term describes the rate at which mechanical energy is transported from one point to another within the fluid as a result of molecular motion. The last term in Eq. (40) is exactly the viscous dissipation of mechanical energy per unit volume. This same term, with the opposite sign, also appears as a source term in the thermal-energy-transport equation. It is exactly the volumetric rate at which mechanical energy is converted to thermal energy through the process of viscous dissipation.

Writing the instantaneous velocity and pressure in Eq. (40) as the sum of the mean and fluctuating components and taking the ensemble average results in

$$\rho\left[\frac{\partial}{\partial t}(\tfrac{1}{2}\overline{V}^2 + k) + \overline{\mathbf{V}}\cdot\nabla(\tfrac{1}{2}\overline{V}^2 + k) + \overline{\tilde{\mathbf{V}}\cdot\nabla(\overline{\mathbf{V}}\cdot\tilde{\mathbf{V}})} + \overline{\tilde{\mathbf{V}}\cdot\nabla(\tfrac{1}{2}\tilde{V}^2)}\right]$$
$$=\; \nabla\cdot\{\mu[\nabla(\tfrac{1}{2}\overline{V}^2 + k) + (\overline{\mathbf{V}}\cdot\nabla)\overline{\mathbf{V}} + \overline{(\tilde{\mathbf{V}}\cdot\nabla)\tilde{\mathbf{V}}}]\} \tag{41}$$
$$-\,\overline{\mathbf{V}}\cdot[\nabla\overline{p} - g_o Z\nabla(\rho)] - \overline{\tilde{\mathbf{V}}\cdot\nabla\tilde{p}} - 2\mu[\bar{\bar{\mathbf{S}}}(\overline{\mathbf{V}})\cdot\bar{\bar{\mathbf{S}}}(\overline{\mathbf{V}}) + \overline{\bar{\bar{\mathbf{S}}}(\tilde{\mathbf{V}})\cdot\bar{\bar{\mathbf{S}}}(\tilde{\mathbf{V}})}]$$

Similarly, from the ensemble average of Eq. (36), the Reynolds-averaged Navier-Stokes equations can be written in vector notation as

$$\rho\left[\frac{\partial\overline{\mathbf{V}}}{\partial t} + (\overline{\mathbf{V}}\cdot\nabla)\overline{\mathbf{V}} + \overline{(\tilde{\mathbf{V}}\cdot\nabla)\tilde{\mathbf{V}}}\right] \;=\; \nabla\cdot[2\mu\bar{\bar{\mathbf{S}}}(\overline{\mathbf{V}})] - \nabla\overline{p} + g_o Z\nabla(\rho) \tag{42}$$

Taking the dot product of this equation with the mean velocity vector, applying the mathematical identity given in Eq. (39), and rearranging yields the mean mechanical energy equation

$$\rho\left[\frac{\partial}{\partial t}(\tfrac{1}{2}\overline{V}^2) + \overline{\mathbf{V}}\cdot\nabla(\tfrac{1}{2}\overline{V}^2) + \overline{\mathbf{V}}\cdot\overline{(\tilde{\mathbf{V}}\cdot\nabla)\tilde{\mathbf{V}}}\right] \;=\; \nabla\cdot\{\mu[\nabla(\tfrac{1}{2}\overline{V}^2) + (\overline{\mathbf{V}}\cdot\nabla)\overline{\mathbf{V}}]\}$$
$$-\,\overline{\mathbf{V}}\cdot[\nabla\overline{p} - g_o Z\nabla(\rho)] - 2\mu\bar{\bar{\mathbf{S}}}(\overline{\mathbf{V}})\cdot\bar{\bar{\mathbf{S}}}(\overline{\mathbf{V}}) \tag{43}$$

After applying the mathematical identities

$$(\tilde{\mathbf{V}}\cdot\nabla)\tilde{\mathbf{V}} \;=\; \nabla\cdot(\tilde{\mathbf{V}}\tilde{\mathbf{V}}) - \tilde{\mathbf{V}}(\nabla\cdot\tilde{\mathbf{V}}) \tag{44}$$

$$\overline{\mathbf{V}}\cdot[\nabla\cdot(\tilde{\mathbf{V}}\tilde{\mathbf{V}})] \;=\; \tilde{\mathbf{V}}\cdot\nabla(\overline{\mathbf{V}}\cdot\tilde{\mathbf{V}}) + (\overline{\mathbf{V}}\cdot\tilde{\mathbf{V}})(\nabla\cdot\tilde{\mathbf{V}}) - (\tilde{\mathbf{V}}\tilde{\mathbf{V}})\cdot\mathbf{J}(\overline{\mathbf{V}}) \tag{45}$$

to Eq. (43), we obtain

$$\rho\left[\frac{\partial}{\partial t}(\tfrac{1}{2}\overline{V}^2) + \overline{\mathbf{V}}\cdot\nabla(\tfrac{1}{2}\overline{V}^2) + \overline{\tilde{\mathbf{V}}\cdot\nabla(\overline{\mathbf{V}}\cdot\tilde{\mathbf{V}})} - \overline{\tilde{\mathbf{V}}\tilde{\mathbf{V}}}\cdot\mathbf{J}(\overline{\mathbf{V}})\right]$$
$$=\; \nabla\cdot\{\mu[\nabla(\tfrac{1}{2}\overline{V}^2) + (\overline{\mathbf{V}}\cdot\nabla)\overline{\mathbf{V}}]\} - \overline{\mathbf{V}}\cdot[\nabla\overline{p} - g_o Z\nabla(\rho)] - 2\mu\bar{\bar{\mathbf{S}}}(\overline{\mathbf{V}})\cdot\bar{\bar{\mathbf{S}}}(\overline{\mathbf{V}}) \tag{46}$$

Subtracting Eq. (46) from Eq. (41) gives

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$$\rho\left[\frac{\partial k}{\partial t}+\overline{\mathbf{V}}\cdot\nabla k+\overline{\widetilde{\mathbf{V}}\widetilde{\mathbf{V}}}\cdot\mathbf{J}(\overline{\mathbf{V}})+\overline{\widetilde{\mathbf{V}}\cdot\nabla(\tfrac{1}{2}\widetilde{V}^2)}\right]$$
$$= \nabla\cdot\{\mu[\nabla k+\overline{(\widetilde{\mathbf{V}}\cdot\nabla)\widetilde{\mathbf{V}}}]\}-\overline{\widetilde{\mathbf{V}}\cdot\nabla\widetilde{p}}-2\mu\overline{\overline{\overline{\mathbf{S}}}(\widetilde{\mathbf{V}})\cdot\overline{\overline{\mathbf{S}}}(\widetilde{\mathbf{V}})}$$

(47)

Applying the mathematical identities

$$\nabla\cdot(\rho\widetilde{V}^2\widetilde{\mathbf{V}}) = \rho\widetilde{\mathbf{V}}\cdot\nabla(\widetilde{V}^2)+\widetilde{V}^2\nabla\cdot(\rho\widetilde{\mathbf{V}})$$

(48)

$$\nabla\cdot(\widetilde{p}\widetilde{\mathbf{V}}) = \widetilde{\mathbf{V}}\cdot\nabla\widetilde{p}+\widetilde{p}(\nabla\cdot\widetilde{\mathbf{V}})$$

(49)

to Eq. (47) yields

$$\rho\left(\frac{\partial k}{\partial t}+\overline{\mathbf{V}}\cdot\nabla k\right)+\rho\overline{\widetilde{\mathbf{V}}\widetilde{\mathbf{V}}}\cdot\mathbf{J}(\overline{\mathbf{V}})+\tfrac{1}{2}\nabla\cdot(\rho\overline{\widetilde{V}^2\widetilde{\mathbf{V}}})-\tfrac{1}{2}\overline{\widetilde{V}^2\nabla\cdot(\rho\widetilde{\mathbf{V}})}$$
$$= \nabla\cdot\{\mu[\nabla k+\overline{(\widetilde{\mathbf{V}}\cdot\nabla)\widetilde{\mathbf{V}}}]\}+\overline{\widetilde{p}(\nabla\cdot\widetilde{\mathbf{V}})}-\nabla\cdot(\overline{\widetilde{p}\widetilde{\mathbf{V}}})-2\mu\overline{\overline{\overline{\mathbf{S}}}(\widetilde{\mathbf{V}})\cdot\overline{\overline{\mathbf{S}}}(\widetilde{\mathbf{V}})}$$

From the continuity equation, the definition of the Reynolds stress tensor, and the definition of the total hydrostatic pressure, we have the well known relations

$$\nabla\cdot(\rho\widetilde{\mathbf{V}}) = 0, \qquad \overline{\overline{\boldsymbol{\tau}}} \equiv -\rho\overline{\widetilde{\mathbf{V}}\widetilde{\mathbf{V}}}, \qquad \rho\overline{(\widetilde{\mathbf{V}}\cdot\nabla)\widetilde{\mathbf{V}}} = -\nabla\cdot\overline{\overline{\boldsymbol{\tau}}}, \qquad \widetilde{p} = \widetilde{p}+\tfrac{2}{3}\mu\nabla\cdot\widetilde{\mathbf{V}}$$

Hence, the **turbulent-energy-transport equation** can be rearranged as

$$\rho\left(\frac{\partial k}{\partial t}+\overline{\mathbf{V}}\cdot\nabla k\right) = \overline{\overline{\boldsymbol{\tau}}}\cdot\mathbf{J}(\overline{\mathbf{V}})-2\mu[\overline{\overline{\overline{\mathbf{S}}}(\widetilde{\mathbf{V}})\cdot\overline{\overline{\mathbf{S}}}(\widetilde{\mathbf{V}})}-\tfrac{1}{3}\overline{(\nabla\cdot\widetilde{\mathbf{V}})^2}]+\overline{\widetilde{p}(\nabla\cdot\widetilde{\mathbf{V}})}$$
$$+\nabla\cdot(\mu\nabla k-\nu\nabla\cdot\overline{\overline{\boldsymbol{\tau}}})-\nabla\cdot[\tfrac{1}{2}\rho\overline{\widetilde{V}^2\widetilde{\mathbf{V}}}+\overline{\widetilde{p}\widetilde{\mathbf{V}}}+\tfrac{2}{3}\mu\overline{(\nabla\cdot\widetilde{\mathbf{V}})\widetilde{\mathbf{V}}}]$$

(50)

The first three terms on the right-hand side of Eq. (50) are exactly the volumetric **production**, **viscous dissipation**, and **pressure dilatation** for turbulent kinetic energy, respectively. The next term is the **molecular transport** of turbulent kinetic energy per unit volume. The last term on the right-hand side of Eq. (50) accounts for the volumetric **turbulent transport** of turbulent kinetic energy. The only approximation that was made in the development of Eq. (50) is that of a Newtonian fluid.

It is important to recognize from Eq. (50) that **molecular transport of turbulent kinetic energy is not a simple gradient diffusion process**. The contribution from the first part of the molecular transport term, $\mu\nabla k$, is gradient diffusion. However, the contribution from the remaining portion of this term, $\nu\nabla\cdot\overline{\overline{\boldsymbol{\tau}}}$, is not necessarily gradient diffusion. Accordingly, even if we accept the Boussinesq analogy between molecular and turbulent transport, **we should not expect turbulent transport of kinetic energy to be a simple gradient diffusion process in general**.

Using only mathematical identities, it can be shown that Eq. (50) is mathematically equivalent to Eq. (30) under the assumption of constant dynamic viscosity, which is an assumption used in the development of Eq. (30) but not in the development of Eq. (50). However, the comparison shows that the turbulence variable ε defined in Eq. (2) is not precisely the dissipation of turbulent kinetic energy per unit mass. Using only mathematical identities and the definition of the Reynolds stress tensor, the turbulence variable ε can be written in terms of the exact dissipation per unit mass, i.e.,

$$\varepsilon \equiv \nu\overline{\overline{\overline{\mathbf{J}}}(\widetilde{\mathbf{V}})\cdot\overline{\overline{\mathbf{J}}}(\widetilde{\mathbf{V}})}$$
$$= 2\nu[\overline{\overline{\overline{\mathbf{S}}}(\widetilde{\mathbf{V}})\cdot\overline{\overline{\mathbf{S}}}(\widetilde{\mathbf{V}})}-\tfrac{1}{3}\overline{(\nabla\cdot\widetilde{\mathbf{V}})^2}]-\tfrac{1}{3}\nu\overline{(\nabla\cdot\widetilde{\mathbf{V}})^2}+\frac{\nabla\cdot[\nu\nabla\cdot\overline{\overline{\boldsymbol{\tau}}}+\mu\overline{(\nabla\cdot\widetilde{\mathbf{V}})\widetilde{\mathbf{V}}}]}{\rho}$$

(51)

The first term on the right-hand side of Eq. (51) is exactly the viscous dissipation of turbulent kinetic energy per unit mass, which is obtained by dividing the second term on the right-hand side of Eq. (50) by the fluid density, ρ. The second term on the right-hand side of Eq. (51) cancels one of the dilatation terms in Eq. (30) to produce the

pressure dilatation that appears in Eq. (50). The remaining divergence term on the right-hand side of Eq. (51) accounts for transport, not dissipation. Thus, we see that the turbulence variable ε is not exactly the viscous dissipation of turbulent kinetic energy per unit mass and the fifth term on the right-hand side of Eq. (30) does not comprise the total contribution from molecular transport.

Applying Boussinesq's analogy between molecular and turbulent transport to the turbulent transport term in Eq. (50) suggests

$$-\tfrac{1}{2}\rho\overline{\widetilde{V}^2\widetilde{\mathbf{V}}} - \overline{\widetilde{p}\widetilde{\mathbf{V}}} - \tfrac{2}{3}\mu\overline{(\nabla\cdot\widetilde{\mathbf{V}})\widetilde{\mathbf{V}}} = (\nu_t/\sigma_k)(\rho\nabla k - \nabla\cdot\bar{\bar{\tau}}) \tag{52}$$

and Eq. (50) becomes

$$\rho\left(\frac{\partial k}{\partial t} + \overline{\mathbf{V}}\cdot\nabla k\right) = \bar{\bar{\tau}}\cdot\mathbf{J}(\overline{\mathbf{V}}) - 2\mu\left[\overline{\bar{\bar{\mathbf{S}}}(\widetilde{\mathbf{V}})\cdot\bar{\bar{\mathbf{S}}}(\widetilde{\mathbf{V}})} - \tfrac{1}{3}\overline{(\nabla\cdot\widetilde{\mathbf{V}})^2}\right] + \overline{\widetilde{p}(\nabla\cdot\widetilde{\mathbf{V}})}$$
$$+ \nabla\cdot[(\nu + \nu_t/\sigma_k)(\rho\nabla k - \nabla\cdot\bar{\bar{\tau}})] \tag{53}$$

For this Boussinesq model, the Reynolds stress tensor is given by Eq. (31) and

$$\nabla\cdot\bar{\bar{\tau}} = 2\nabla\cdot[\mu_t\bar{\bar{\mathbf{S}}}(\overline{\mathbf{V}})] - \tfrac{2}{3}\nabla(\rho k + \mu_t\nabla\cdot\overline{\mathbf{V}}) \tag{54}$$

Applying Eqs. (32) and (54) to Eq. (53) results in an alternate version of the **Boussinesq-based turbulent-energy-transport equation**,

$$\rho\left(\frac{\partial k}{\partial t} + \overline{\mathbf{V}}\cdot\nabla k\right) = 2\mu_t\overline{S}^2 - \tfrac{2}{3}(\rho k + \mu_t\nabla\cdot\overline{\mathbf{V}})\nabla\cdot\overline{\mathbf{V}} - 2\mu[\widetilde{S}^2 - \tfrac{1}{3}\overline{(\nabla\cdot\widetilde{\mathbf{V}})^2}]$$
$$+ \overline{\widetilde{p}(\nabla\cdot\widetilde{\mathbf{V}})} + \nabla\cdot\left((\nu + \nu_t/\sigma_k)\{\rho\nabla k + \tfrac{2}{3}\nabla(\rho k + \mu_t\nabla\cdot\overline{\mathbf{V}}) - 2\nabla\cdot[\mu_t\bar{\bar{\mathbf{S}}}(\overline{\mathbf{V}})]\}\right) \tag{55}$$

where $\overline{S}$ and $\widetilde{S}$ are the magnitudes of the mean and fluctuating components of the strain-rate tensor, respectively,

$$\overline{S}^2 \equiv \bar{\bar{\mathbf{S}}}(\overline{\mathbf{V}})\cdot\bar{\bar{\mathbf{S}}}(\overline{\mathbf{V}}) \tag{56}$$

$$\widetilde{S}^2 \equiv \overline{\bar{\bar{\mathbf{S}}}(\widetilde{\mathbf{V}})\cdot\bar{\bar{\mathbf{S}}}(\widetilde{\mathbf{V}})} \tag{57}$$

For **incompressible flow**, Eq. (55) reduces to

$$\frac{\partial k}{\partial t} + \overline{\mathbf{V}}\cdot\nabla k = 2\nu_t\overline{S}^2 - 2\nu\widetilde{S}^2 + \nabla\cdot\left((\nu + \nu_t/\sigma_k)\{\tfrac{5}{3}\nabla k - 2\nabla\cdot[\nu_t\bar{\bar{\mathbf{S}}}(\overline{\mathbf{V}})]\}\right) \tag{58}$$

Although the third term on the right-hand side of Eq. (55) is exactly the volumetric dissipation of turbulent kinetic energy, it remains an unknown function of the turbulent velocity fluctuations. To close this formulation, we must have an additional equation to relate this dissipation to the other turbulence parameters and the mean flow.

If we were to close this formulation by following what was done in the development of the traditional k-ε turbulence model, then this volumetric dissipation could simply be written in terms of the unknown dissipation per unit mass, which is obtained by dividing the third term on right-hand side of Eq. (55) by the fluid density, ρ,

$$\widetilde{\varepsilon} \equiv 2\nu[\widetilde{S}^2 - \tfrac{1}{3}\overline{(\nabla\cdot\widetilde{\mathbf{V}})^2}] = 2\nu[\overline{\bar{\bar{\mathbf{S}}}(\widetilde{\mathbf{V}})\cdot\bar{\bar{\mathbf{S}}}(\widetilde{\mathbf{V}})} - \tfrac{1}{3}\overline{(\nabla\cdot\widetilde{\mathbf{V}})^2}] \tag{59}$$

Using Eq. (59) in Eq. (55) and neglecting the pressure dilatation term, as was done in the development of Eq. (6),

$$\overline{\widetilde{p}(\nabla\cdot\widetilde{\mathbf{V}})} \cong 0 \tag{60}$$

we obtain an alternate version of the **turbulent-energy-transport equation that could be used to replace Eq. (6) in any conventional k-ε turbulence model**,

$$\rho\left(\frac{\partial k}{\partial t}+\overline{\mathbf{V}}\cdot\nabla k\right) = 2\mu_t\overline{S}^2 - \tfrac{2}{3}(\rho k + \mu_t\nabla\cdot\overline{\mathbf{V}})\nabla\cdot\overline{\mathbf{V}} - \rho\widetilde{\varepsilon}$$
$$+\nabla\cdot\left((\nu+\nu_t/\sigma_k)\{\rho\nabla k + \tfrac{2}{3}\nabla(\rho k + \mu_t\nabla\cdot\overline{\mathbf{V}}) - 2\nabla\cdot[\mu_t\overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}})]\}\right) \tag{61}$$

The tilde over the symbol ε in Eq. (61) is simply used to indicate that this is the exact dissipation per unit mass, as defined in Eq. (59), and not the approximate dissipation per unit mass, defined in Eq. (2). Nevertheless, conventional k-ε turbulence models are based on the assumption that ε is the dissipation per unit mass. Thus, ε and $\widetilde{\varepsilon}$ could be used interchangeably in conventional k-ε turbulence models.

As an alternative to traditional k-ε turbulence models, the dissipation term that appears in Eqs. (50) and (55) can be mathematically rearranged using the mathematical identity

$$\overline{\overline{\mathbf{S}}}(\widetilde{\mathbf{V}})\cdot\overline{\overline{\mathbf{S}}}(\widetilde{\mathbf{V}}) = \tfrac{1}{2}(\nabla\times\widetilde{\mathbf{V}})\cdot(\nabla\times\widetilde{\mathbf{V}}) + \nabla\cdot[(\widetilde{\mathbf{V}}\cdot\nabla)\widetilde{\mathbf{V}}] - \widetilde{\mathbf{V}}\cdot\nabla(\nabla\cdot\widetilde{\mathbf{V}}) \tag{62}$$

The last term in Eq. (62) can be expanded using the mathematical identity

$$\nabla\cdot[(\nabla\cdot\widetilde{\mathbf{V}})\widetilde{\mathbf{V}}] = \widetilde{\mathbf{V}}\cdot\nabla(\nabla\cdot\widetilde{\mathbf{V}}) + (\nabla\cdot\widetilde{\mathbf{V}})^2 \tag{63}$$

to yield the equivalent mathematical identity

$$\overline{\overline{\mathbf{S}}}(\widetilde{\mathbf{V}})\cdot\overline{\overline{\mathbf{S}}}(\widetilde{\mathbf{V}}) = \tfrac{1}{2}(\nabla\times\widetilde{\mathbf{V}})\cdot(\nabla\times\widetilde{\mathbf{V}}) + \nabla\cdot[(\widetilde{\mathbf{V}}\cdot\nabla)\widetilde{\mathbf{V}} - (\nabla\cdot\widetilde{\mathbf{V}})\widetilde{\mathbf{V}}] + (\nabla\cdot\widetilde{\mathbf{V}})^2 \tag{64}$$

Using Eq. (64) in Eq. (59), the exact turbulent-energy dissipation per unit mass can be written as

$$\widetilde{\varepsilon} \equiv 2\nu\left[\overline{\overline{\overline{\mathbf{S}}}(\widetilde{\mathbf{V}})\cdot\overline{\overline{\mathbf{S}}}(\widetilde{\mathbf{V}})} - \tfrac{1}{3}\overline{(\nabla\cdot\widetilde{\mathbf{V}})^2}\right]$$
$$= 2\nu\left\{\tfrac{1}{2}\overline{(\nabla\times\widetilde{\mathbf{V}})\cdot(\nabla\times\widetilde{\mathbf{V}})} + \nabla\cdot\overline{[(\widetilde{\mathbf{V}}\cdot\nabla)\widetilde{\mathbf{V}} - (\nabla\cdot\widetilde{\mathbf{V}})\widetilde{\mathbf{V}}]} + \tfrac{2}{3}\overline{(\nabla\cdot\widetilde{\mathbf{V}})^2}\right\} \tag{65}$$

Using the definition of the specific Reynolds stress tensor from Eq. (29), the second ensemble mean on the right-hand side of Eq. (65) can be written as

$$\overline{(\widetilde{\mathbf{V}}\cdot\nabla)\widetilde{\mathbf{V}}} = -(\nabla\cdot\overline{\overline{\boldsymbol{\tau}}})/\rho \tag{66}$$

Using Eq. (66) in Eq. (65), the exact turbulent-energy dissipation per unit mass can be expressed as

$$\widetilde{\varepsilon} \equiv 2\nu\left[\overline{\overline{\overline{\mathbf{S}}}(\widetilde{\mathbf{V}})\cdot\overline{\overline{\mathbf{S}}}(\widetilde{\mathbf{V}})} - \tfrac{1}{3}\overline{(\nabla\cdot\widetilde{\mathbf{V}})^2}\right]$$
$$= \nu\left\{\widetilde{\omega}^2 + \tfrac{4}{3}\overline{(\nabla\cdot\widetilde{\mathbf{V}})^2} - 2\nabla\cdot\overline{[(\nabla\cdot\overline{\overline{\boldsymbol{\tau}}})/\rho + (\nabla\cdot\widetilde{\mathbf{V}})\widetilde{\mathbf{V}}]}\right\} \tag{67}$$

where $\widetilde{\omega}$ is the root mean square of the fluctuating vorticity,

$$\widetilde{\omega}^2 \equiv \overline{(\nabla\times\widetilde{\mathbf{V}})\cdot(\nabla\times\widetilde{\mathbf{V}})} \tag{68}$$

The three components of dissipation that are seen in Eq. (67) are referred to as

$$\text{solenoidal dissipation} \equiv \nu\widetilde{\omega}^2$$
$$\text{dilatational dissipation} \equiv \tfrac{4}{3}\nu\overline{(\nabla\cdot\widetilde{\mathbf{V}})^2}$$
$$\text{inhomogeneous dissipation} \equiv -2\nu\nabla\cdot\left[(\nabla\cdot\overline{\overline{\boldsymbol{\tau}}})/\rho + \overline{(\nabla\cdot\widetilde{\mathbf{V}})\widetilde{\mathbf{V}}}\right]$$

After using Eq. (67) in Eq. (50), the **turbulent-energy-transport equation** can be written as

$$\rho\left(\frac{\partial k}{\partial t}+\overline{\mathbf{V}}\cdot\nabla k\right) = \overline{\overline{\boldsymbol{\tau}}}\cdot\mathbf{J}(\overline{\mathbf{V}})-\mu\{\widetilde{\omega}^2+\tfrac{4}{3}\overline{(\nabla\cdot\widetilde{\mathbf{V}})^2}-2\nabla\cdot[(\nabla\cdot\overline{\overline{\boldsymbol{\tau}}})/\rho+\overline{(\nabla\cdot\widetilde{\mathbf{V}})\widetilde{\mathbf{V}}}]\}$$

$$+\overline{\widetilde{p}(\nabla\cdot\widetilde{\mathbf{V}})}+\nabla\cdot[(\mu\nabla k-\nu\nabla\cdot\overline{\overline{\boldsymbol{\tau}}})-\tfrac{1}{2}\rho\overline{\widetilde{V}^2\widetilde{\mathbf{V}}}-\overline{\widetilde{p}\widetilde{\mathbf{V}}}-\tfrac{2}{3}\mu\overline{(\nabla\cdot\widetilde{\mathbf{V}})\widetilde{\mathbf{V}}}] \tag{69}$$

Because only mathematical identities were used to obtain Eq. (69) from Eq. (50), these two equations are equivalent. **The only approximation that was made in the development of Eq. (69) is that of a Newtonian fluid.**

Applying Boussinesq's analogy between molecular and turbulent transport of kinetic energy to Eq. (69) gives

$$\rho\left(\frac{\partial k}{\partial t}+\overline{\mathbf{V}}\cdot\nabla k\right) = \overline{\overline{\boldsymbol{\tau}}}\cdot\mathbf{J}(\overline{\mathbf{V}})-\mu\{\widetilde{\omega}^2+\tfrac{4}{3}\overline{(\nabla\cdot\widetilde{\mathbf{V}})^2}-2\nabla\cdot[(\nabla\cdot\overline{\overline{\boldsymbol{\tau}}})/\rho+\overline{(\nabla\cdot\widetilde{\mathbf{V}})\widetilde{\mathbf{V}}}]\}$$

$$+\overline{\widetilde{p}(\nabla\cdot\widetilde{\mathbf{V}})}+\nabla\cdot[(\nu+\nu_t/\sigma_k)(\rho\nabla k-\nabla\cdot\overline{\overline{\boldsymbol{\tau}}})] \tag{70}$$

Using Eqs. (32) and (54) in Eq. (70) produces the **Boussinesq-transport equation**

$$\rho\left(\frac{\partial k}{\partial t}+\overline{\mathbf{V}}\cdot\nabla k\right) = 2\mu_t\overline{S}^2-\tfrac{2}{3}(\rho k+\mu_t\nabla\cdot\overline{\mathbf{V}})\nabla\cdot\overline{\mathbf{V}}-\mu\widetilde{\omega}^2-\tfrac{4}{3}\mu\overline{(\nabla\cdot\widetilde{\mathbf{V}})^2}$$

$$-4\mu\nabla\cdot\left(\{\tfrac{1}{3}\nabla(\rho k+\mu_t\nabla\cdot\overline{\mathbf{V}})-\nabla\cdot[\mu_t\overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}})]\}/\rho-\tfrac{1}{2}\overline{(\nabla\cdot\widetilde{\mathbf{V}})\widetilde{\mathbf{V}}}\right)$$

$$+\overline{\widetilde{p}(\nabla\cdot\widetilde{\mathbf{V}})}+\nabla\cdot\left((\nu+\nu_t/\sigma_k)\{\rho\nabla k+\tfrac{2}{3}\nabla(\rho k+\mu_t\nabla\cdot\overline{\mathbf{V}})-2\nabla\cdot[\mu_t\overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}})]\}\right) \tag{71}$$

Huang et al.[16] have shown that the dilatational dissipation rate and other terms involving the divergence of the fluctuating velocity are negligibly small, at least up to supersonic mean Mach numbers of 3. Hence, Eq. (71) is closely approximated as

$$\rho\left(\frac{\partial k}{\partial t}+\overline{\mathbf{V}}\cdot\nabla k\right) = 2\mu_t\overline{S}^2-\tfrac{2}{3}(\rho k+\mu_t\nabla\cdot\overline{\mathbf{V}})\nabla\cdot\overline{\mathbf{V}}$$

$$-\mu\widetilde{\omega}^2-4\mu\nabla\cdot\left(\{\tfrac{1}{3}\nabla(\rho k+\mu_t\nabla\cdot\overline{\mathbf{V}})-\nabla\cdot[\mu_t\overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}})]\}/\rho\right)$$

$$+\nabla\cdot\left((\nu+\nu_t/\sigma_k)\{\rho\nabla k+\tfrac{2}{3}\nabla(\rho k+\mu_t\nabla\cdot\overline{\mathbf{V}})-2\nabla\cdot[\mu_t\overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}})]\}\right) \tag{72}$$

The three lines on the right-hand side of Eq. (72) are production, dissipation, and the combination of molecular and turbulent transport, respectively.

Equation (72) is mathematically identical to Eq. (61). The difference between Eq. (61) and Eq. (72) is only the form in which the turbulent-energy dissipation per unit mass has been written. In Eq. (61), this viscous dissipation is simply treated as an unknown turbulence variable, which is defined in Eq. (59). In Eq. (72), the viscous dissipation of turbulent energy per unit mass has been related to the root-mean-square (RMS) fluctuating vorticity, through the mathematical identity in Eq. (67). Because the RMS fluctuating vorticity is also an unknown function of the turbulent fluctuations, the difference between Eq. (61) and Eq. (72) is simply a change of variables. From Eqs. (54) and (67), **after neglecting the fluctuating dilatational terms**, this change of variables is

$$\widetilde{\varepsilon} = 2\nu\widetilde{S}^2 \equiv 2\nu\overline{\overline{\overline{\mathbf{S}}}(\widetilde{\mathbf{V}})\cdot\overline{\overline{\mathbf{S}}}(\widetilde{\mathbf{V}})}$$

$$= \nu\widetilde{\omega}^2-2\nu\nabla\cdot[(\nabla\cdot\overline{\overline{\boldsymbol{\tau}}})/\rho] \equiv \nu\overline{(\nabla\times\widetilde{\mathbf{V}})\cdot(\nabla\times\widetilde{\mathbf{V}})}-2\nu\nabla\cdot[(\nabla\cdot\overline{\overline{\boldsymbol{\tau}}})/\rho] \tag{73}$$

$$= \nu\widetilde{\omega}^2+4\nu\nabla\cdot\left(\{\tfrac{1}{3}\nabla(\rho k+\mu_t\nabla\cdot\overline{\mathbf{V}})-\nabla\cdot[\mu_t\overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}})]\}/\rho\right)$$

where $\widetilde{S}$ is the RMS fluctuating strain rate and $\widetilde{\omega}$ is the RMS fluctuating vorticity.

At first, it may appear that the change of variables in Eq. (73) provides no gain whatsoever. Equation (61) was originally written in terms of three unknown functions of the turbulent fluctuations; the eddy viscosity, ν_t, the specific turbulent kinetic energy, k, and the exact turbulent dissipation per unit mass, $\widetilde{\varepsilon}$. With the application of Eq. (73), we have simply replaced the unknown turbulent dissipation with the unknown RMS turbulent vorticity, while retaining the unknown eddy viscosity and specific turbulent kinetic energy.

There are two fundamental advantages to this reparameterization of the turbulent-energy-transport equation. First, vorticity is a solenoidal function of the flow field. In other words, the divergence of vorticity is always zero, $\nabla \cdot (\nabla \times \mathbf{V}) = 0$. This can provide a significant simplifying advantage in the development of a transport equation for $\widetilde{\omega}$, which is not available for the development of an $\widetilde{\varepsilon}$ transport equation. A second and more subtle advantage to the change of variables defined by Eq. (73) is that vorticity is a well-understood function of any fluid flow field. The turbulent dissipation, on the other hand, does not have such a fundamental physical interpretation.

In the form of either Eq. (61) or Eq. (72), the turbulent-energy-transport equation provides one differential equation in three unknown turbulence variables. Completing either formulation requires the addition of two more equations. The ultimate accuracy of the complete turbulence model depends on how well the modeled equations capture the physics of the actual turbulent flow. For **incompressible flow**, the form of the turbulent-energy-transport equation given by Eq. (61) reduces to

$$\frac{\partial k}{\partial t} + \overline{\mathbf{V}} \cdot \nabla k \ = \ 2\nu_t \overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}}) \cdot \overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}}) - \widetilde{\varepsilon} + \nabla \cdot \left((\nu + \nu_t/\sigma_k)\{ \tfrac{5}{3}\nabla k - 2\nabla \cdot [\nu_t \overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}})] \} \right) \tag{74}$$

Completing this formulation requires two additional equations relating the eddy viscosity, ν_t, the specific turbulent kinetic energy, k, and the exact turbulent dissipation per unit mass, $\widetilde{\varepsilon}$. Similarly, for **incompressible flow**, the form of the turbulent-energy-transport equation given by Eq. (72) becomes

$$\begin{aligned}
\frac{\partial k}{\partial t} + \overline{\mathbf{V}} \cdot \nabla k \ = \ & 2\nu_t \overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}}) \cdot \overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}}) - \nu \left(\widetilde{\omega}^2 + 4\nabla \cdot \{ \tfrac{1}{3}\nabla k - \nabla \cdot [\nu_t \overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}})] \} \right) \\
& + \nabla \cdot \left((\nu + \nu_t/\sigma_k)\{ \tfrac{5}{3}\nabla k - 2\nabla \cdot [\nu_t \overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}})] \} \right)
\end{aligned} \tag{75}$$

Completing this formulation requires two additional equations relating the eddy viscosity, ν_t, the specific turbulent kinetic energy, k, and the RMS fluctuating vorticity, $\widetilde{\omega}$.

IV. Concerns with Traditional Dissipation-Based Turbulence Models

At this point we can identify six possible concerns with the traditional $k\text{-}\varepsilon$, $k\text{-}\omega$, and $k\text{-}\zeta$ turbulence models that are commonly used for CFD.

1. The dissipation length scale used to obtain Eqs. (3), (13) and (23) is that associated with the smaller turbulent eddies having the highest strain rates per unit kinetic energy; not that associated with the larger energy-bearing eddies, which are primarily responsible for turbulent transport.
2. The transport equations given by Eqs. (7), (17), and (27) were obtained simply from dimensional analysis and analogy with the turbulent-energy-transport equation. They were not developed in a rigorous manner from the Navier-Stokes equations.
3. The so called turbulent-energy dissipation per unit mass, ε, which is defined in Eq. (2) and used in Eq. (6) and in the development of both Eq. (16) and Eq. (26), is not equal to the true dissipation of turbulent kinetic energy per unit mass, which is specified exactly in Eq. (59).
4. Because the approximate turbulent-energy dissipation per unit mass, ε, that is used in the traditional $k\text{-}\varepsilon$, $k\text{-}\omega$, and $k\text{-}\zeta$ turbulence models includes a portion of the total molecular transport, the so called molecular transport terms that are used in Eqs. (6), (16), and (26) do not include the total molecular transport of turbulent kinetic energy per unit mass.
5. Because a part of the molecular transport was neglected in the development of Eqs. (6), (16), and (26), subsequent application of Boussinesq's analogy between molecular and turbulent transport also results in neglecting a portion of the turbulent transport of turbulent kinetic energy per unit mass.
6. Using the dissipation length scale to define the eddy viscosity as was done in Eqs. (3), (13) and (23) predicts a Reynolds stress tensor that is inversely proportional to the molecular viscosity, whereas the definition that is given in Eq. (29) shows that the Reynolds stress tensor should not depend directly on molecular viscosity.

The first three of these concerns are very straightforward and well documented in the literature.[12] The last three deserve some additional attention.

In the original development of the $k\text{-}\varepsilon$ turbulence model and in many subsequent presentations of the turbulence-energy-transport equation, the parameter ε that is defined in Eq. (2) is presented as being exactly the turbulent-energy dissipation per unit mass. With this misinterpretation, the molecular diffusion term $\nu \nabla^2 k$ is

commonly presented as being the total molecular transport of turbulent energy per unit mass. Although it is now generally recognized that ε is not precisely the turbulent-energy dissipation per unit mass, its continued use is typically justified on the grounds that the additional terms shown in Eq. (51) are small compared with the turbulent transport terms on the right-hand side of Eq. (30). On these same grounds one could also justify neglecting the additional molecular transport term that appears in Eq. (50) but not in Eq. (30). In fact, for many turbulent flows, all molecular transport can be neglected in comparison with the turbulent transport. The most significant concern with the traditional k-ε, k-ω, and k-ζ turbulence models is not the lack of precision in defining the dissipation or the molecular transport. **A more significant concern is that associated with the application of Boussinesq's analogy between molecular and turbulent transport to a molecular transport term that has been less than rigorously developed.**

The traditional k-ε, k-ω, and k-ζ turbulence models are all based on approximating the turbulent transport of turbulent kinetic energy as pure gradient diffusion. For the case of incompressible flow, these models all assume a turbulent-kinetic-energy flux given by $(\nu_t/\sigma_k)\nabla k$. The more rigorous development used to obtain Eq. (58) suggests that improved results for incompressible flow might be obtained by using a turbulent-kinetic-energy flux that is specified by

$$(\nu_t/\sigma_k)\{\tfrac{5}{3}\nabla k - 2\nabla\cdot[\nu_t\bar{\bar{\mathbf{S}}}(\overline{\mathbf{V}})]\}$$

This is based on a more direct analogy between turbulent and molecular transport.

Perhaps the greatest concern with traditional k-ε, k-ω, and k-ζ turbulence models is that all fail to exhibit proper dependence on molecular viscosity. From the definition of the Reynolds stress tensor given by Eq. (29), we see that the Reynolds stresses depend on only the fluid density and turbulent velocity fluctuations. They are independent of other natural fluid properties such as the molecular viscosity. Hence, if the Boussinesq analogy between turbulent and molecular transport is strictly followed, the dynamic eddy viscosity should be related to only the fluid density and turbulent velocity fluctuations. **The eddy viscosity should not depend directly on molecular viscosity.**

Because the turbulent-energy-dissipation parameter, ε, is directly proportional to the molecular viscosity as shown in Eq. (2), the dissipation length scale $k^{3/2}/\varepsilon$ is inversely proportional to the molecular viscosity. Thus, assuming that the transport length scale is proportional to this dissipation length scale will always result in an inverse relation between the turbulent eddy viscosity and the molecular viscosity. Applying the definition of ε from Eq. (2) to the relation for eddy viscosity given by Eq. (3), yields

$$\nu_t \;=\; C_\mu\, k^2\big/\varepsilon \;=\; C_\mu\, k^2\big/\big[\nu\,\bar{\bar{\mathbf{J}}}(\widetilde{\mathbf{V}})\cdot\bar{\bar{\mathbf{J}}}(\widetilde{\mathbf{V}})\big]$$

This violates the fundamental requirement for a consistent Boussinesq model of turbulent transport, which requires that ν_t must depend on only the turbulent velocity fluctuations, just as the molecular viscosity depends on only the molecular velocity fluctuations.

For the traditional k-ω turbulence models, the interpretation of ω has been a matter of some controversy. It is sometimes viewed simply as the ratio of the turbulence velocity scale to the turbulence length scale and is assumed to depend on only the velocity fluctuations. Similarly, we could also think of ε simply as another turbulence variable, which depends on only the velocity fluctuations. These interpretations for ε and ω resolve the issue of the eddy viscosity depending directly on the molecular viscosity when eddy viscosity is defined as either $\nu_t = C_\mu k^2/\varepsilon$ or $\nu_t = k/\omega$. However, using these same interpretations when considering the viscous dissipation terms ε and $C_\mu k\omega$, which appear in Eq. (6) and Eq. (16), respectively, leads to the even more absurd conclusion that viscous dissipation is independent of molecular viscosity.

The previous statement that, "*eddy viscosity should not depend directly on molecular viscosity*," should not be taken to imply that molecular viscosity has no effect on the eddy viscosity. The key word is *directly*. Of course, molecular viscosity has an indirect effect on the eddy viscosity, because molecular viscosity affects the velocity fluctuations through its occurrence in the transport equations. In fact, there are important turbulent flows where the production and dissipation can combine in a particular manner to produce velocity fluctuations that result in an eddy viscosity that is nearly inversely proportional to the molecular viscosity, but this is not always the case.

V. An Alternate Turbulent Frequency and Length Scale

From the kinetic theory of gases, the kinematic molecular viscosity is found to be proportional to the product of the molecular mean free path and the square root of the mean molecular kinetic energy per unit mass. Following the usual analogy with the kinetic theory of gases, the characteristic translational velocity associated with turbulent transport is assumed to be proportional to the square root of the specific turbulent kinetic energy, $k^{1/2}$. For traditional k-ε, k-ω, and k-ζ turbulence models, the turbulent length scale associated with turbulent transport is assumed to be the dissipation length scale, which is proportional to $k^{3/2}/\varepsilon$, and the frequency associated with turbulent transport is assumed to be the dissipation frequency, which is proportional to ε/k.

An alternative to traditional k-ε, k-ω, and k-ζ turbulence models can be obtained by using the RMS fluctuating vorticity, $\widetilde{\omega}$, as the characteristic frequency associated with turbulent transport. This is attractive, because the instantaneous local angular velocity of the fluid element at any point in a flow field is one-half the instantaneous local vorticity. For a fluctuating flow field that has a mean velocity of zero, the mean kinetic energy per unit mass, k, can be expressed as either one-half the mean square of the translational velocity, or alternately as one-half the mean square of the angular velocity multiplied by the square of an energy-weighted turbulent length scale, which is the *mean turbulent radius of gyration* denoted here as ℓ_k. Because the angular velocity is one-half the vorticity, we obtain the relation

$$k \equiv \tfrac{1}{2}\overline{\widetilde{\mathbf{V}}\cdot\widetilde{\mathbf{V}}} = \tfrac{1}{2}\big[\tfrac{1}{4}\overline{(\nabla\times\widetilde{\mathbf{V}})\cdot(\nabla\times\widetilde{\mathbf{V}})}\big]\ell_k^2 \tag{76}$$

Equation (76) can be simply thought of as the definition of the local mean turbulent radius of gyration, which in view of the definition of the RMS fluctuating vorticity in Eq. (68) is

$$\ell_k \equiv \sqrt{8k}\big/\widetilde{\omega} \tag{77}$$

There are other possible ways to define a local turbulence length scale from only the turbulent kinetic energy and the RMS fluctuating vorticity. However, these will differ from the length scale defined in Eq. (77) only by a constant.

Continuing to follow the analogy with the kinetic theory of gases, the kinematic eddy viscosity should be proportional to the characteristic turbulent length scale multiplied by the characteristic translational velocity. Because the energy-weighted turbulent length scale is proportional to $k^{1/2}/\widetilde{\omega}$ and the characteristic translational velocity is proportional to $k^{1/2}$, the kinematic eddy viscosity should be proportional to $k/\widetilde{\omega}$. This produces the foundation for an **energy-vorticity turbulence model**, which is based on the following two equations for **incompressible flow**; an algebraic equation for the **kinematic eddy viscosity**,

$$\nu_t = C_\nu\, k\big/\widetilde{\omega} \tag{78}$$

and the **turbulent-energy-transport equation** from Eq. (75),

$$\begin{aligned}
\frac{\partial k}{\partial t}+\overline{\mathbf{V}}\cdot\nabla k &= 2\nu_t\overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}})\cdot\overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}})-\nu\Big(\widetilde{\omega}^2+4\nabla\cdot\{\tfrac{1}{3}\nabla k-\nabla\cdot[\nu_t\overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}})]\}\Big)\\
&\quad +\nabla\cdot\Big((\nu+\nu_t/\sigma_k)\{\tfrac{5}{3}\nabla k-2\nabla\cdot[\nu_t\overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}})]\}\Big)
\end{aligned} \tag{79}$$

where C_ν and σ_k are dimensionless closure coefficients. Completing this formulation requires one additional equation relating the eddy viscosity, ν_t, the specific turbulent kinetic energy, k, the RMS fluctuating vorticity, $\widetilde{\omega}$, and the mean velocity vector, $\overline{\mathbf{V}}$. The formulation could be completed by including a vorticity-transport equation for the RMS fluctuating vorticity, $\widetilde{\omega}$.

Equations (78) and (79) are easily recast in terms of the *mean fluctuating enstrophy*, which is commonly denoted as ζ and defined to be the mean squared magnitude of the fluctuating vorticity,

$$\zeta \equiv \widetilde{\omega}^2 \equiv \overline{(\nabla\times\widetilde{\mathbf{V}})\cdot(\nabla\times\widetilde{\mathbf{V}})} \tag{80}$$

Using Eq. (80) in Eqs. (78) and (79) produces the foundation for a k-ζ **turbulence model**, which is based on the following two equations for **incompressible flow**; an algebraic equation for the **kinematic eddy viscosity**,

$$\nu_t = C_\nu\, k\big/\zeta^{1/2} \tag{81}$$

and the **turbulent-energy-transport equation** obtained by applying Eq. (80) to Eq. (79),

$$\frac{\partial k}{\partial t}+\overline{\mathbf{V}}\cdot\nabla k = 2\nu_t\overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}})\cdot\overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}})-\nu\left(\zeta+4\nabla\cdot\{\tfrac{1}{3}\nabla k-\nabla\cdot[\nu_t\overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}})]\}\right)$$
$$+\nabla\cdot\left((\nu+\nu_t/\sigma_k)\{\tfrac{5}{3}\nabla k-2\nabla\cdot[\nu_t\overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}})]\}\right) \tag{82}$$

Completing this formulation requires one additional equation relating the eddy viscosity, ν_t, the specific turbulent kinetic energy, k, the mean fluctuating enstrophy, ζ, and the mean velocity vector, $\overline{\mathbf{V}}$. This formulation could be completed by including Eq. (27) from the traditional Robinson-Hassan k-ζ turbulence model.[14,15] Perhaps a more promising enstrophy-transport equation for closing the proposed k-ζ turbulence model can be obtained from the DNS-based solenoidal-dissipation model of Kreuzinger, Friedrich, and Gatski.[17]

Equations (78) and (79) can also be reparameterized using a turbulent-transport length scale in place of the RMS fluctuating vorticity. This turbulent-transport length scale can be defined conveniently so that the kinematic eddy viscosity is equal to the product of the length scale and the velocity scale. In view of the definitions in Eqs. (1) and (68), this suggests the change of variables

$$\lambda \equiv C_\nu\,\frac{k^{1/2}}{\widetilde{\omega}} = C_\nu\sqrt{\frac{\overline{\widetilde{\mathbf{V}}\cdot\widetilde{\mathbf{V}}}}{2\overline{(\nabla\times\widetilde{\mathbf{V}})\cdot(\nabla\times\widetilde{\mathbf{V}})}}} \tag{83}$$

where the turbulent-transport length scale, λ, will be referred to here as the *mean vortex wavelength*. Applying this change of variables to Eqs. (78) and (79) produces the foundation for a **k-λ turbulence model**, which is based on the following two equations for **incompressible flow**; an algebraic equation for the **kinematic eddy viscosity**,

$$\nu_t = \lambda k^{1/2} \tag{84}$$

and the **turbulent-energy-transport equation** obtained by applying Eq. (83) to Eq. (79),

$$\frac{\partial k}{\partial t}+\overline{\mathbf{V}}\cdot\nabla k = 2\nu_t\overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}})\cdot\overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}})-\nu\left(C_\lambda\, k/\lambda^2+4\nabla\cdot\{\tfrac{1}{3}\nabla k-\nabla\cdot[\nu_t\overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}})]\}\right)$$
$$+\nabla\cdot\left((\nu+\nu_t/\sigma_k)\{\tfrac{5}{3}\nabla k-2\nabla\cdot[\nu_t\overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}})]\}\right) \tag{85}$$

where C_λ and σ_k are dimensionless closure coefficients, $C_\lambda \equiv C_\nu^2$. Completing this formulation requires one additional equation relating the eddy viscosity, ν_t, the specific turbulent kinetic energy, k, the mean vortex wavelength, λ, and the mean velocity vector, $\overline{\mathbf{V}}$.

VI. The Energy-Vorticity Turbulence Variables in Fully Rough Pipe Flow

To complete a two-equation energy-vorticity turbulence model, a second transport equation like that suggested by Kreuzinger, Friedrich, and Gatski[17] is required; and the closure coefficients in the turbulent-energy-transport equation must be evaluated. Closure coefficients are typically evaluated from well established experimental data. One case that is very well documented is fully developed flow in a pipe. Because the friction factor becomes independent of the Reynolds number for fully rough pipe flow, estimating the closure coefficients from data for rough surfaces might be more straightforward than using data for smooth surfaces. In this section, the mean-vortex-wavelength and RMS-fluctuating-vorticity profiles for fully developed, fully rough pipe flow are inferred from Eqs. (84) and (85) combined with experimental data and well established empirical correlations. This will provide some insight into the nature of the energy-vorticity turbulence variables and establish certain relations between the closure coefficients.

A. Friction Factor

The foundation for what is known today about turbulent flow in rough pipes is the semi-empirical *mixing-length theory* developed by Ludwig Prandtl and his students. The rough-wall version of Prandtl's mixing-length theory is

based on experimental data collected by Prandtl's famous student Johann Nikuradse[18] using pipes roughened artificially with uniform grains of sand. Nikuradse's data for fully rough pipe flow is also the foundation for the empirical relations commonly use today for predicting pressure losses in rough pipes; including the Colebrook equation,[19] which was used to generate the well known Moody chart.[20] Based on his experimental results for the *Darcy friction factor*, which is four times the *Fanning friction factor* often called the *skin-friction coefficient*, Nikuradse[18] proposed using the following empirical correlation for **fully rough pipe flow**:

$$Darcy\ friction\ factor \equiv 4C_f \equiv 4\frac{\tau_w}{\frac{1}{2}\rho V_m^2} = [2.00 \log_{10}(R/k_s) + 1.74]^{-2} \tag{86}$$

where R is the pipe radius, V_m is the bulk velocity, and k_s is the roughness element size defined to be the screen mesh size of the sieve that Nikuradse used to sift the sand. Equation (86) provided a key result in the development of our current capability to predict pressure losses for turbulent flow through rough pipes, and it is referred to herein as the *Nikuradse equation*. Equation (86) was the starting point for the development the Colebrook equation[19] and the associated Moody chart.[20] Thus, **the Colebrook equation and the Moody chart assume the validity of the Nikuradse equation.**

The Nikuradse equation is often presented in a form that differs slightly from Eq. (86). In the original work by Nikuradse[18] and its subsequent presentation by another of Prandtl's famous students, Hermann Schlichting,[21] the pipe roughness was characterized using the dimensionless *roughness ratio*, R/k_s. When Colebrook[19] applied the Nikuradse equation to his work, he chose to characterize pipe roughness using the dimensionless *relative roughness*, k_s/D, where D is the pipe diameter. Thus, the Nikuradse equation was rearranged by Colebrook into the form

$$4C_f = \left[2.00 \log_{10}\left(\frac{10^{1.74/2.00}/2}{k_s/2R}\right)\right]^{-2} \tag{87}$$

Rounding the constants to two significant digits yields the **most widely accepted form of the Nikuradse equation**,

$$4C_f = \left[2.0 \log_{10}\left(\frac{3.7}{k_s/D}\right)\right]^{-2} \tag{88}$$

The validity of Eq. (88) is so widely accepted that today it has become the definition of surface roughness. The roughness of any surface is typically defined in terms of the *equivalent sand-grain roughness* first introduced by Schlichting,[22] which is defined to be the value of k_s that gives the correct fully rough limit for the friction factor when inserted into Eq. (88). It should be noted that when Moody[20] used the relation developed by Colebrook[19] to generate the well known Moody chart, he used the symbol ε to denote Nikuradse's sand-grain roughness. Here we will continue to use k_s to signify the equivalent sand-grain roughness, as was done by Nikuradse[18] and Schlichting.[21]

The Nikuradse equation expressed equivalently in Eqs. (86) and (88) provides an accurate means for predicting the Darcy friction factor when the Reynolds number is large enough so that the friction factor becomes independent of the molecular viscosity. However, the Nikuradse equation alone provides no information regarding how large the Reynolds number must be to make this empirical correlation valid. From Nikuradse's data[18] on artificially roughened pipes, it is commonly accepted that this correlation for fully rough flow is valid whenever the Reynolds number based on the shear velocity u_τ and the equivalent sand-grain roughness k_s, usually called the roughness Reynolds number, is greater than about 70. The Nikuradse equation can be used as a reference when calculating the Darcy friction factor for fully rough pipe flows.

Of the conventional k-ε, k-ω, and k-ζ turbulence models, only k-ω models are capable of implementing rough-wall boundary conditions without employing wall functions. However, even the current k-ω models are not capable of predicting friction factors that agree with experimental data at very high roughness Reynolds numbers, where the molecular viscosity is negligible compared to the eddy viscosity throughout the pipe. The effects of surface roughness are incorporated into conventional k-ω turbulence models by altering the surface boundary condition on ω. For example, with his 1998 model, Wilcox[23] suggests using the rough-wall boundary condition

$$\omega^{+}\Big|_{y^{+}=0} = \begin{cases} (50/k_s^{+})^2, & k_s^{+} \leq 25 \\ 100/k_s^{+}, & k_s^{+} \geq 25 \end{cases} \tag{89}$$

For his 2006 model, Wilcox[24] recommends using

$$\omega^{+}\Big|_{y^{+}=0} = \begin{cases} (200/k_s^{+})^2, & k_s^{+} \leq 5 \\ 100/k_s^{+} + [(200/k_s^{+})^2 - 100/k_s^{+}]\exp(5-k_s^{+}), & k_s^{+} \geq 5 \end{cases} \tag{90}$$

A comparison between experimental data[18,25] and results obtained from the Wilcox 1998 k-ω model[23] is shown in Fig. 1 for roughness Reynolds numbers greater than 70. The Darcy friction factors obtained from the Wilcox 2006 k-ω model[24] are shown in Fig. 2. To assure that the results shown in Figs. 1 and 2 are grid resolved, converged solutions were obtained on coarse, medium, and fine grids, containing 401, 801, and 1601 nodes, respectively. The values of y^{+} at the first node from the pipe wall in these coarse, medium, and fine grids were 0.4, 0.2, and 0.1, respectively. The Richardson[26] extrapolation was then used with these three solutions to obtain the results shown in Figs. 1 and 2. The maximum difference observed between the Richardson[26] extrapolation and the solution obtained on the fine grid was 1.3 percent for the 1998 model and 0.04 percent for the 2006 model.

The 1998 and 2006 codes provided by Wilcox would not converge for high roughness Reynolds numbers, which encompasses the region to the right of the right-hand dashed curves in Figs. 1 and 2. Along these curves the ratio of the eddy viscosity near the pipe wall to the molecular viscosity is nearly constant at about 1.7 for the 1998 model and 1.3 for the 2006 model. As a result, the Wilcox 1998 and 2006 k-ω models cannot be used to predict the distributions of the turbulence variables in the fully rough region, where the molecular viscosity is negligible compared to the eddy viscosity throughout the pipe. One important objective for an improved rough-wall turbulence model should be the capability to accurately predict the Darcy friction factor at high roughness numbers.
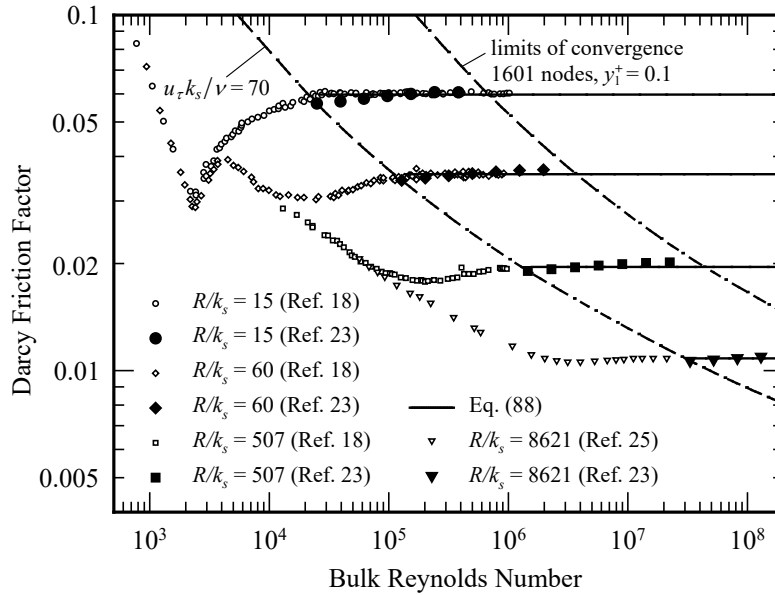

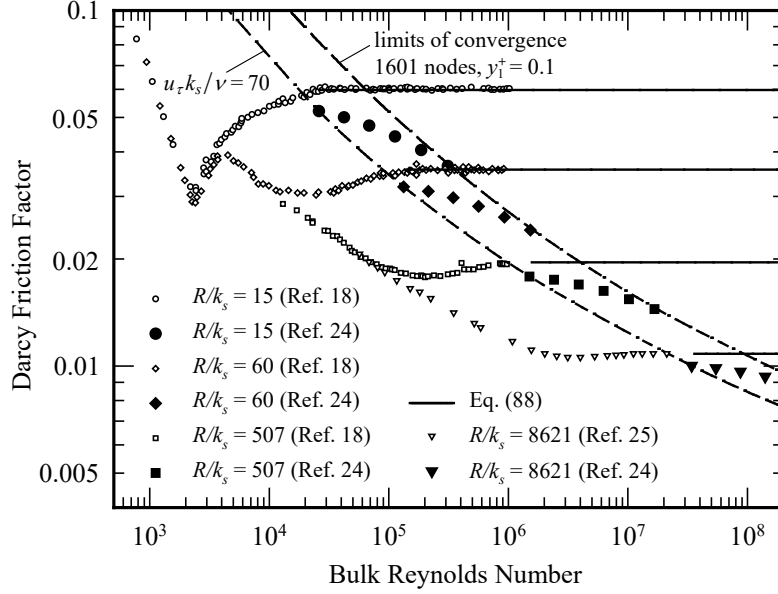**Figure 1. Darcy friction factor, as predicted from the Wilcox 1998 k-ω model.**[23]

Figure 2. Darcy friction factor, as predicted from the Wilcox 2006 k-ω model.[24]

B. Mean-Velocity Distribution

A sensitive indicator for the effect of surface roughness in a pipe is given by the behavior of the mean velocity profile. Mean-velocity-profile measurements taken by Nikuradse[18] in the fully rough limit were found to be in excellent agreement with the empirical correlation

$$\frac{\overline{V}_z}{u_\tau} = 2.5 \ln\left(\frac{y}{k_s}\right) + 8.5 \tag{91}$$

where y is the normal coordinate measured into the fluid from the pipe wall. Equation (91) is commonly referred to as the *law of the wall* for fully rough pipe flow. A comparison between Eq. (91) and experimental data collected by Nikuradse[18] is shown in Fig. 3.
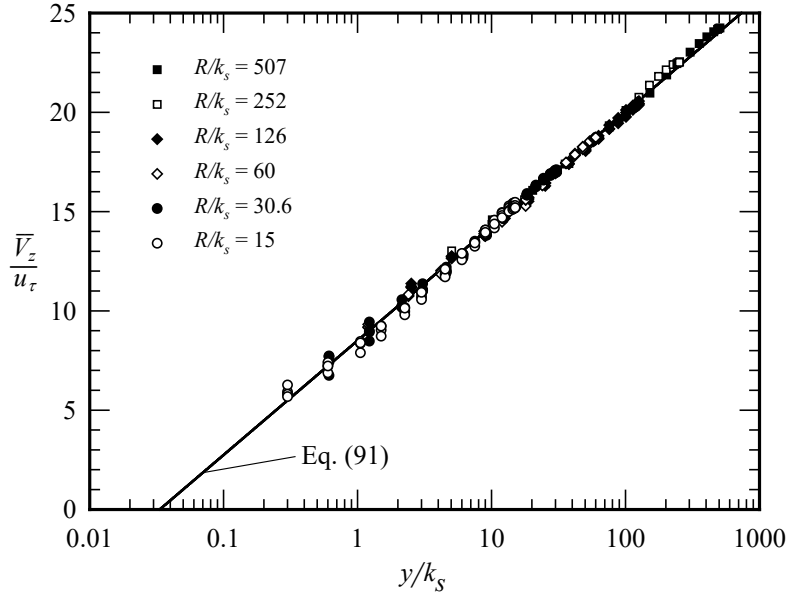
Figure 3. Experimental velocity profiles in rough pipes at high Reynolds numbers.[18]

Obviously, Eq. (91) does not apply over the entire flow field from the pipe wall to the centerline, because it satisfies neither the no-slip boundary condition at the pipe wall nor the symmetry boundary condition at the pipe centerline. In order to satisfy these two boundary conditions and provide better agreement with Nikuradse's experimental data while leaving the integral of the velocity profile unchanged, a correction is applied to the law of the wall that is given by Eq. (91),[27]

$$\frac{\overline{V}_z}{V_m} = \frac{\hat{R}_s^2 \ln(\hat{R}_s \hat{y} + 1)}{D_0} + \delta \tag{92}$$

or

$$\frac{\overline{V}_z}{u_\tau} = \frac{1}{\kappa} \ln(\hat{R}_s \hat{y} + 1) + \frac{D_0 \delta}{\kappa \hat{R}_s^2} \tag{93}$$

A seventh-order corrective function δ is given by[27]

$$\delta = C_0 + C_1 \hat{r} + C_2 \hat{r}^2 + C_3 \hat{r}^3 + C_4 \hat{r}^4 + C_5 \hat{r}^5 + C_6 \hat{r}^6 + C_7 \hat{r}^7 \tag{94}$$

where

$$
\begin{aligned}
& D_0 = (\hat{R}_s + 1)^2 \left[\ln(\hat{R}_s + 1) - \frac{3}{2} \right] + 2\hat{R}_s + \frac{3}{2}, \quad C_0 = 1.2 \left(\frac{k_s}{R} \right)^{1.4} \\
& C_1 = \frac{\hat{R}_s^3}{(\hat{R}_s + 1) D_0}, \quad C_2 = \frac{\hat{R}_s^2}{2 D_0} \left[\frac{\hat{R}_s^2}{(\hat{R}_s + 1)^2} - \frac{\kappa}{0.056} \right], \quad C_3 = \frac{\hat{R}_s^5}{3(\hat{R}_s + 1)^3 D_0} \\
& D_1 = -C_0 - C_1 - C_2 - C_3 - C_7, \quad D_2 = -C_1 - 2C_2 - 3C_3 - 7C_7 \\
& D_3 = -C_0/2 - C_1/3 - C_2/4 - C_3/5 - C_7/9 \\
& C_4 = -39 D_1 + 3 D_2 + 168 D_3, \quad C_5 = 84 D_1 - 7 D_2 - 336 D_3 \\
& C_6 = -44 D_1 + 4 D_2 + 168 D_3, \quad C_7 = -0.65 \\
& \hat{R}_s = \frac{1}{\gamma(k_s/R)}, \quad \hat{y} = y/R, \quad \hat{r} = 1 - \hat{y}, \quad \kappa = 0.403, \quad \gamma = 0.0324
\end{aligned}
\tag{95}
$$

The eight coefficients in Eq. (94) that are given in Eq. (95) were adjusted to optimize the fit with Nikuradse's velocity profile data for fully rough flow while satisfying the physical constraints imposed on the empirical function δ by the symmetry and no-slip boundary conditions. The values used for the von Kármán constant, κ, and the Nikuradse constant, γ, were also adjusted to optimize the fit with Nikuradse's velocity profile data. Hence, we are using slightly different values for these constants than the values $\kappa = 0.40$ and $\gamma = 0.033$, which were used by Nikuradse[18] and Schlichting.[21] The resulting velocity profiles were found to be in good agreement with Nikuradse's experimental data on fully rough pipe flow. As an example, Fig. 4 shows the velocity obtained from Eqs. (92)–(95) compared to the law of the wall and Nikuradse's experimental data for a roughness ratio, R/k_s, of 15. Because the velocity profiles obtained from Eqs. (92)–(95) satisfy the physical boundary conditions and show better agreement with experimental data, they can be used in place of Eq. (91) as an improved reference for the mean velocity profiles when evaluating turbulence model closure coefficients for fully rough pipe flows.
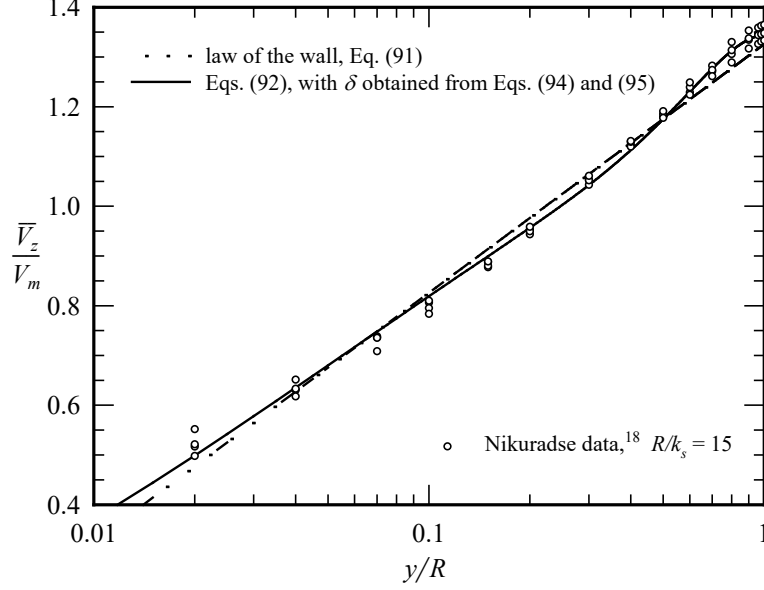
Figure 4. Velocity profiles compared to the law of the wall and Nikuradse's experimental data.[18]

C. Mean-Vortex-Wavelength Distribution

In order to complete the formulation given by Eqs. (84) and (85), an equation is needed for the mean vortex wavelength, λ. To provide some insight into the nature of the mean vortex wavelength and the RMS fluctuating vorticity, Eqs. (84) and (85) can be used in combination with empirical correlations for the friction-factor and the mean-velocity profiles, to infer the mean-vortex-wavelength and RMS-fluctuating-vorticity distributions for fully developed, fully rough pipe flow.

From Eq. (85), the turbulent-kinetic-energy transport equation for axisymmetric fully developed pipe flow is given by

$$\frac{1}{r}\frac{d}{dr}\left[-\left(\frac{\nu}{3}+\frac{5\nu_t}{3\sigma_k}\right)r\frac{dk}{dr}\right] = \nu_t\left(\frac{u_\tau^2 r}{(\nu+\nu_t)R}\right)^2 - C_\lambda \frac{\nu k}{\lambda^2} \tag{96}$$

where $\nu_t = \lambda k^{1/2}$. This equation contains two unknown coefficients σ_k and C_λ. These two coefficients must be dimensionless universal constants. Equation (96) requires a wall boundary condition for the turbulent kinetic energy. The specific turbulent kinetic energy at a rough wall should be proportional to the shear velocity squared,

$$k\big|_{r=R} = k_{\text{wall}}^+ u_\tau^2 \tag{97}$$

where k_{wall}^+ is an unknown dimensionless proportionality coefficient. For incompressible flow, the dimensionless parameter, k_{wall}^+, should be a unique function of the roughness Reynolds number. As the roughness Reynolds number approaches zero, k_{wall}^+ should also approach zero. When the roughness Reynolds number becomes large enough, the solution must be independent of molecular viscosity. Hence, for fully rough flow, the dimensionless parameter, k_{wall}^+, must approach a universal constant.

An algebraic function for the mean vortex wavelength in fully rough pipe flow is given by[27]

$$\lambda = (A_{\lambda 0} k_s + A_{\lambda 1} y)\left(1 - \frac{1}{2}\frac{y}{R}\right)\left[B_{\lambda 0} + B_{\lambda 1}\left(\frac{r}{R}\right)^2 + B_{\lambda 2}\left(\frac{r}{R}\right)^4 + (1 - B_{\lambda 0} - B_{\lambda 1} - B_{\lambda 2})\left(\frac{r}{R}\right)^6\right] \tag{98}$$

where $y = R - r$. This equation contains five unknown coefficients $A_{\lambda 0}$, $A_{\lambda 1}$, $B_{\lambda 0}$, $B_{\lambda 1}$, and $B_{\lambda 2}$. At the wall, the mean-vortex-wavelength equation reduces to

$$\lambda\big|_{r=R} = A_{\lambda 0}k_s \tag{99}$$

For fully rough pipe flow, the value of the mean vortex wavelength at the wall should depend only on the surface roughness k_s. Therefore, the coefficient $A_{\lambda 0}$ should be a constant. The remaining four coefficients in the mean-vortex-wavelength profile, $A_{\lambda 1}$, $B_{\lambda 0}$, $B_{\lambda 1}$, and $B_{\lambda 2}$, need not be constants, but can be functions of the flow parameters such as the Reynolds number and roughness ratio. Eight coefficients are associated with Eqs. (96)–(98); the two closure coefficients σ_k and C_λ from the turbulent-kinetic-energy transport equation, the proportionality coefficient k_{wal}^+ from the wall boundary condition, and the five coefficients $A_{\lambda 0}$, $A_{\lambda 1}$, $B_{\lambda 0}$, $B_{\lambda 1}$, $B_{\lambda 2}$ from the empirical algebraic relation for the mean vortex wavelength.

The eight coefficients σ_k, C_λ, k_{wal}^+, $A_{\lambda 0}$, $A_{\lambda 1}$, $B_{\lambda 0}$, $B_{\lambda 1}$, $B_{\lambda 2}$ were evaluated using a computer optimization program.[28] This optimization program minimizes a fitness parameter that quantifies how close the solution is to a target solution. The target solution is a weighted function based on the friction factor obtained from Eq. (88) and the velocity profile given by Eqs. (92)–(95). This fitness parameter was minimized over the Reynolds-number range starting at a roughness Reynolds number of 1000 and continuing up to a bulk Reynolds number as large as 2×10^9. The optimization program used to find the coefficients implements the BFGS algorithm, named after the work of Broyden,[29] Fletcher,[30] Goldfarb,[31] and Shanno.[32] The resulting algebraic relation for the mean vortex wavelength was found to give good agreement with experimental data, provided that σ_k is in the range 2.0 to 6.0, k_{wal}^+ is in the range 0.05 to 1.0, and the following relations between the coefficients are maintained;

$$C_\lambda = (1.143\times10^{-4}\sigma_k^4 - 2.157\times10^{-3}\sigma_k^3 + 1.545\times10^{-2}\sigma_k^2 - 5.315\times10^{-2}\sigma_k + 8.365\times10^{-2})^2 \tag{100}$$

$$A_{\lambda 0} = -5.481\times10^{-6}\sigma_k^4 + 1.083\times10^{-4}\sigma_k^3 - 5.882\times10^{-4}\sigma_k^2 + 6.427\times10^{-5}\sigma_k + 7.056\times10^{-3} \tag{101}$$

$$A_{\lambda 10} = 6.055\times10^{-6}\sigma_k^4 - 1.746\times10^{-4}\sigma_k^3 + 1.708\times10^{-3}\sigma_k^2 - 7.802\times10^{-3}\sigma_k + 2.362\times10^{-2} \tag{102}$$

$$A_{\lambda 11} = 1.643\times10^{-4}\sigma_k^4 - 3.288\times10^{-3}\sigma_k^3 + 2.367\times10^{-2}\sigma_k^2 - 7.482\times10^{-2}\sigma_k + 1.374\times10^{-1} \tag{103}$$

$$
\begin{aligned}
C_4 &= -3.625\times10^{-4}k_{\mathrm{wall}}^{+2} - 2.377\times10^{-4}k_{\mathrm{wall}}^+ + 3.181\times10^{-4}\\
C_3 &= 6.746\times10^{-3}k_{\mathrm{wall}}^{+2} + 3.503\times10^{-3}k_{\mathrm{wall}}^+ - 7.604\times10^{-3}\\
C_2 &= -4.306\times10^{-2}k_{\mathrm{wall}}^{+2} - 2.005\times10^{-2}k_{\mathrm{wall}}^+ + 5.737\times10^{-2}\\
C_1 &= 1.110\times10^{-1}k_{\mathrm{wall}}^{+2} + 5.732\times10^{-2}k_{\mathrm{wall}}^+ - 1.577\times10^{-1}\\
C_0 &= -9.789\times10^{-2}k_{\mathrm{wall}}^{+2} - 7.677\times10^{-2}k_{\mathrm{wall}}^+ + 1.640\times10^{-1}\\
A_{\lambda 12} &= C_4\,\sigma_k^4 + C_3\,\sigma_k^3 + C_2\,\sigma_k^2 + C_1\,\sigma_k + C_0
\end{aligned}
\tag{104}
$$

$$A_{\lambda 13} = 2.178\times10^{-4}\sigma_k^4 - 4.613\times10^{-3}\sigma_k^3 + 3.200\times10^{-2}\sigma_k^2 - 7.606\times10^{-2}\sigma_k + 2.679\times10^{-1} \tag{105}$$

$$B_{\lambda 00} = 1.539\times10^{-3}\sigma_k^4 - 3.768\times10^{-2}\sigma_k^3 + 3.054\times10^{-1}\sigma_k^2 - 9.981\times10^{-1}\sigma_k + 1.123 \tag{106}$$

$$B_{\lambda 01} = -1.761\times10^{-3}\sigma_k^4 + 3.347\times10^{-2}\sigma_k^3 - 2.156\times10^{-1}\sigma_k^2 + 4.948\times10^{-1}\sigma_k + 4.408\times10^{-1} \tag{107}$$

$$C_4 = 4.596 \times 10^{-3} k_{\text{wall}}^{+2} + 2.083 \times 10^{-3} k_{\text{wall}}^{+} - 1.328 \times 10^{-2}$$

$$C_3 = -9.312 \times 10^{-2} k_{\text{wall}}^{+2} - 2.857 \times 10^{-2} k_{\text{wall}}^{+} + 2.759 \times 10^{-1}$$

$$C_2 = 6.040 \times 10^{-1} k_{\text{wall}}^{+2} + 1.876 \times 10^{-1} k_{\text{wall}}^{+} - 2.008$$

$$C_1 = -1.495 \, k_{\text{wall}}^{+2} - 7.378 \times 10^{-1} k_{\text{wall}}^{+} + 6.034 \tag{108}$$

$$C_0 = 1.214 \, k_{\text{wall}}^{+2} + 1.270 \, k_{\text{wall}}^{+} - 7.287$$

$$B_{\lambda 02} = C_4 \, \sigma_k^4 + C_3 \, \sigma_k^3 + C_2 \, \sigma_k^2 + C_1 \, \sigma_k + C_0$$

$$B_{\lambda 03} = 6.345 \times 10^{-4} \sigma_k^4 - 1.716 \times 10^{-2} \sigma_k^3 + 1.490 \times 10^{-1} \sigma_k^2 - 5.319 \times 10^{-1} \sigma_k + 7.978 \times 10^{-1} \tag{109}$$

$$B_{\lambda 10} = -1.013 \times 10^{-3} \sigma_k^4 + 1.915 \times 10^{-2} \sigma_k^3 - 1.222 \times 10^{-1} \sigma_k^2 + 2.553 \times 10^{-1} \sigma_k + 1.436 \tag{110}$$

$$B_{\lambda 11} = -6.621 \times 10^{-3} \sigma_k - 13.83 \tag{111}$$

$$C_4 = -2.020 \times 10^{-3} k_{\text{wall}}^{+2} + 4.328 \times 10^{-3} k_{\text{wall}}^{+} - 1.395 \times 10^{-3}$$

$$C_3 = 2.657 \times 10^{-2} k_{\text{wall}}^{+2} - 6.414 \times 10^{-2} k_{\text{wall}}^{+} + 2.362 \times 10^{-2}$$

$$C_2 = -1.199 \times 10^{-1} k_{\text{wall}}^{+2} + 3.337 \times 10^{-1} k_{\text{wall}}^{+} - 1.386 \times 10^{-1}$$

$$C_1 = 2.194 \times 10^{-1} k_{\text{wall}}^{+2} - 7.081 \times 10^{-1} k_{\text{wall}}^{+} + 3.032 \times 10^{-1} \tag{112}$$

$$C_0 = -1.407 \times 10^{-1} k_{\text{wall}}^{+2} + 4.584 \times 10^{-1} k_{\text{wall}}^{+} - 8.085 \times 10^{-1}$$

$$B_{\lambda 20} = C_4 \, \sigma_k^4 + C_3 \, \sigma_k^3 + C_2 \, \sigma_k^2 + C_1 \, \sigma_k + C_0$$

$$B_{\lambda 21} = -2.682 \times 10^{-3} \sigma_k + 40.04 \tag{113}$$

The twelve constants $A_{\lambda 10}$, $A_{\lambda 11}$, $A_{\lambda 12}$, $A_{\lambda 13}$, $B_{\lambda 00}$, $B_{\lambda 01}$, $B_{\lambda 02}$, $B_{\lambda 03}$, $B_{\lambda 10}$, $B_{\lambda 11}$, $B_{\lambda 20}$, and $B_{\lambda 21}$ are related to the four coefficients $A_{\lambda 1}$, $B_{\lambda 0}$, $B_{\lambda 1}$, $B_{\lambda 2}$ according to

$$A_{\lambda 1} = A_{\lambda 11} + A_{\lambda 12} \hat{k}_s + (A_{\lambda 10} - A_{\lambda 11} - A_{\lambda 12} \hat{k}_s) \exp(-\hat{k}_s^{A_{\lambda 13}}) \tag{114}$$

$$B_{\lambda 0} = B_{\lambda 01} + B_{\lambda 02} \hat{k}_s + (B_{\lambda 00} - B_{\lambda 01} - B_{\lambda 02} \hat{k}_s) \exp(-\hat{k}_s^{B_{\lambda 03}}) \tag{115}$$

$$B_{\lambda 1} = B_{\lambda 10} + B_{\lambda 11} \hat{k}_s \tag{116}$$

$$B_{\lambda 2} = B_{\lambda 20} + B_{\lambda 21} \hat{k}_s \tag{117}$$

The mean-vortex-wavelength equation, given by Eq. (98), depends on the five coefficients $A_{\lambda 0}$, $A_{\lambda 1}$, $B_{\lambda 0}$, $B_{\lambda 1}$, and $B_{\lambda 2}$, which are obtained from Eqs. (101)–(117).

Computational results obtained from this algebraic relation agree with Nikuradse's experimental data on flow in artificially roughened pipes[18] as well as more recent experimental data presented by Shockling, Allen, and Smits.[25] For example, Fig. 5 shows a comparison between experimental data for the Darcy friction factor[18,25] and results obtained from the proposed algebraic relation. Notice that the current algebraic relation predicts a friction factor that becomes independent of Reynolds number as the Reynolds number becomes large. This is shown in Fig. 5 for bulk Reynolds numbers as large as 2×10^9.

Fully rough flow is defined to be the asymptotic high-Reynolds-number limit as the turbulent eddy viscosity ν_t becomes large compared to the molecular viscosity ν throughout the flow field. Because ν_t is smallest near the wall, the limit for application of the fully rough flow approximation can be evaluated by examining the near-wall behavior of ν/ν_t. For comparison, three curves of constant $(\nu/\nu_t)_{\text{wall}}$ are superimposed on the friction-factor results shown in Fig. 5. The dashed curve on the left is the locus of points having an eddy viscosity at the wall equal to the molecular

viscosity. The roughness Reynolds number for all points along this curve is approximately 77. Along the second dashed curve, the eddy viscosity at the wall is 10 times the molecular viscosity and the roughness Reynolds number is 766. The dashed curve on the right is the locus of points where the eddy viscosity at the wall is 100 times the molecular viscosity and the roughness Reynolds number is 7658. At a roughness Reynolds number of 1000, the molecular viscosity is nearly 8% of the eddy viscosity at the wall. For roughness Reynolds numbers below 1000, the molecular viscosity becomes more significant, and the fully rough flow approximation breaks down near the pipe wall. For fully rough flow, the velocity profiles compare well with the log law given by Eq. (91), as shown in Figs. 6 and 7. Results similar to those shown in Figs. 5–7 are obtained for any value of σ_k in the range 2.0 to 6.0 and any value of k_{wal}^+ in the range 0.05 to 1.0, provided that the relations given by Eqs. (100)–(117) are maintained.
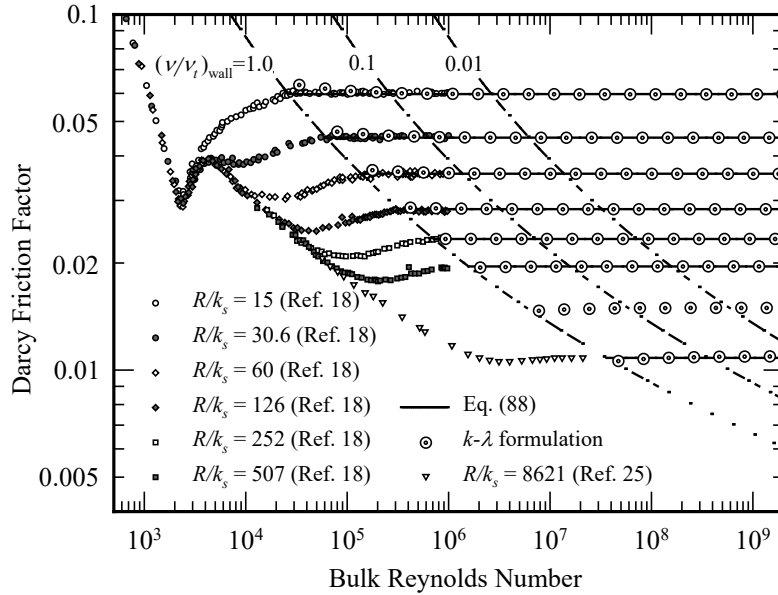
Figure 5. Darcy friction factor in rough pipes, predicted from the k-λ formulation with the algebraic relation for λ, using $\sigma_k = 4.0$ and $k_{wal}^+ = 0.1$.
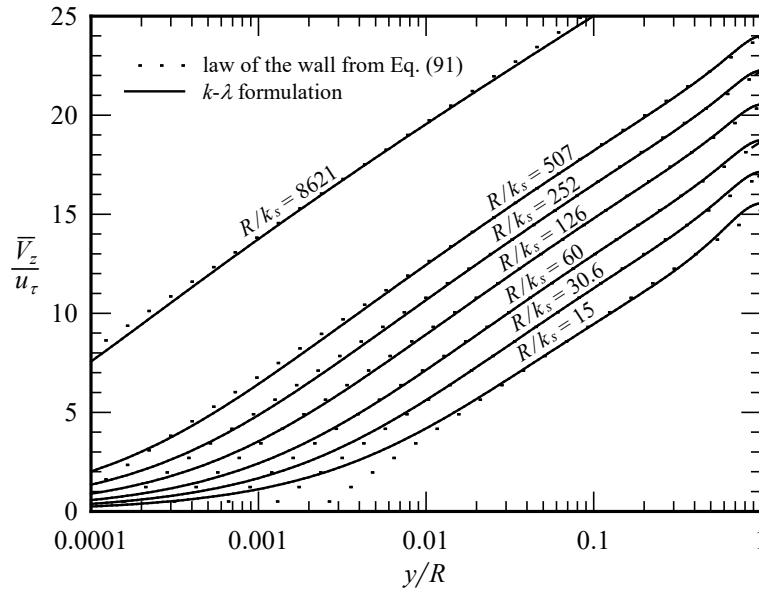

Figure 6. Velocity profiles in fully rough pipe flow, predicted from the k-λ formulation with the algebraic relation for λ, using $\sigma_k = 4.0$ and $k_{wal}^+ = 0.1$.
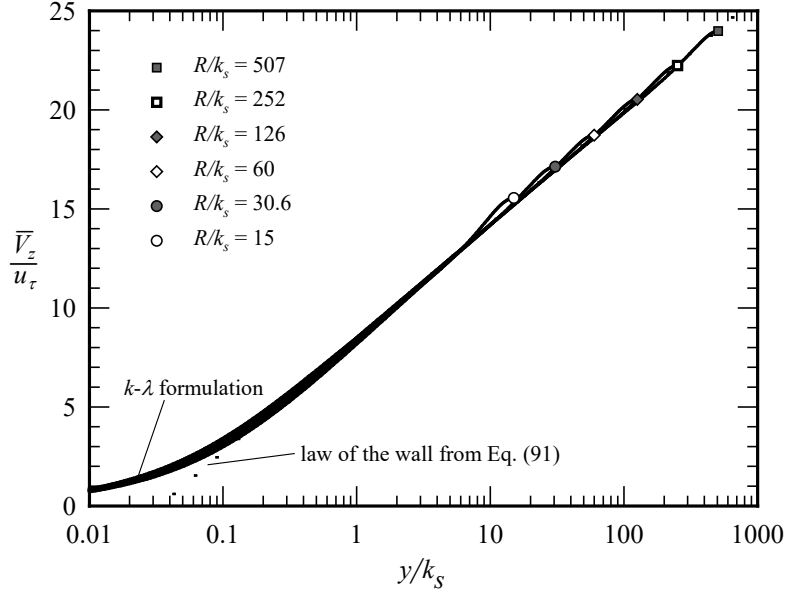
25

Figure 7. Velocity profiles in rough pipes at a roughness Reynolds number of 1000, predicted from the k-λ formulation with the algebraic relation for λ, using $\sigma_k = 4.0$ and $k_{wal}^+ = 0.1$.

Because results obtained from Eqs. (84) and (85) combined with Eqs. (97)–(117) agree well with experimental data obtained for the velocity profile and friction factor at high Reynolds numbers, they can be used to predict the distributions of the turbulence variables in fully rough pipe flow. Example profiles for the turbulent kinetic energy, mean vortex wavelength, kinematic eddy viscosity, and RMS fluctuating vorticity are presented in Figs.8–11, respectively. Very small variations in the turbulence variable distributions can be seen between the roughness Reynolds numbers of 1,000 and 80,000. This is because, at these high roughness Reynolds numbers, the molecular viscosity is negligible when compared to the eddy viscosity throughout the pipe.
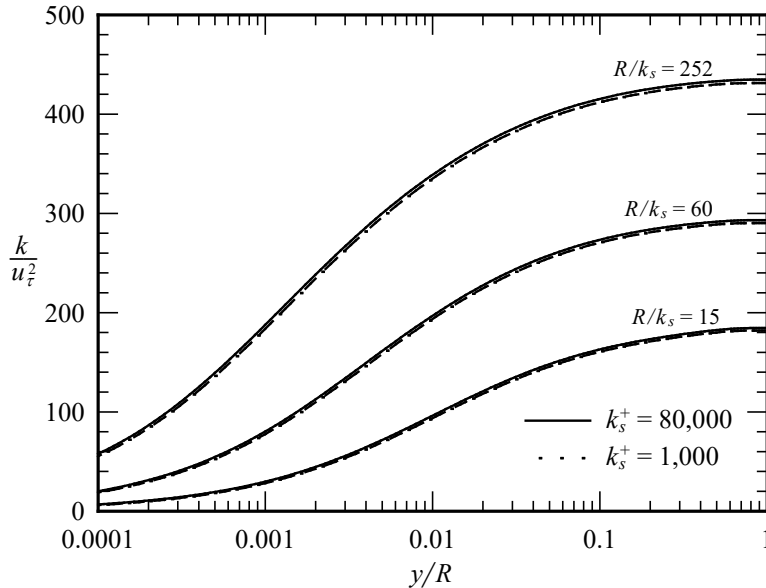

Figure 8. Turbulent-kinetic-energy profiles, predicted from the k-λ formulation with the algebraic relation for λ, using $\sigma_k = 4.0$ and $k_{wal}^+ = 0.1$.
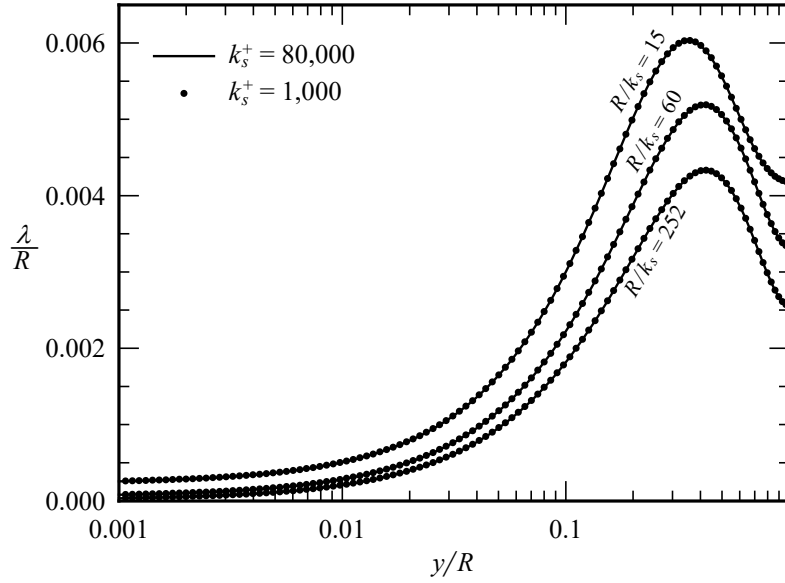
26

Figure 9. Mean-vortex-wavelength profiles, predicted from the *k-λ* formulation with the algebraic relation for *λ*, using $\sigma_k = 4.0$ and $k_{wal}^+ = 0.1$.
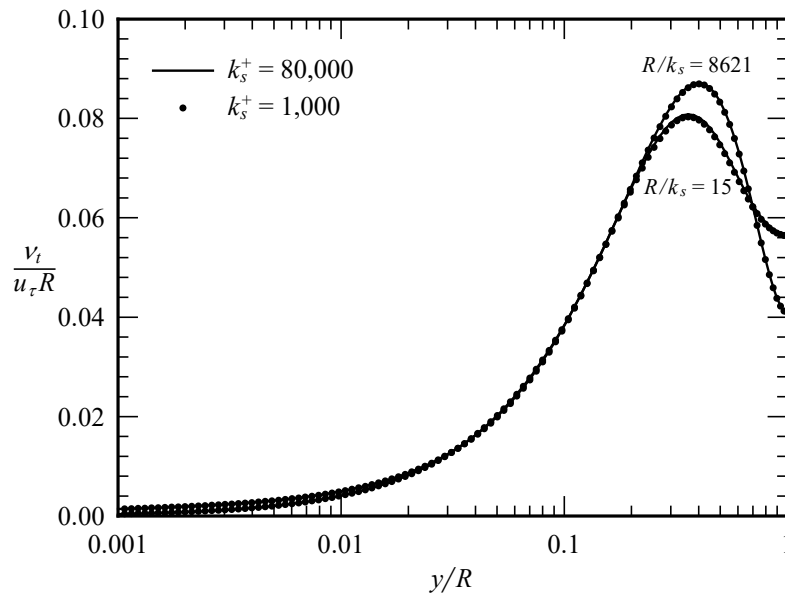

Figure 10. Turbulent-eddy-viscosity profiles, predicted from the *k-λ* formulation with the algebraic relation for *λ*, using $\sigma_k = 4.0$ and $k_{wal}^+ = 0.1$.
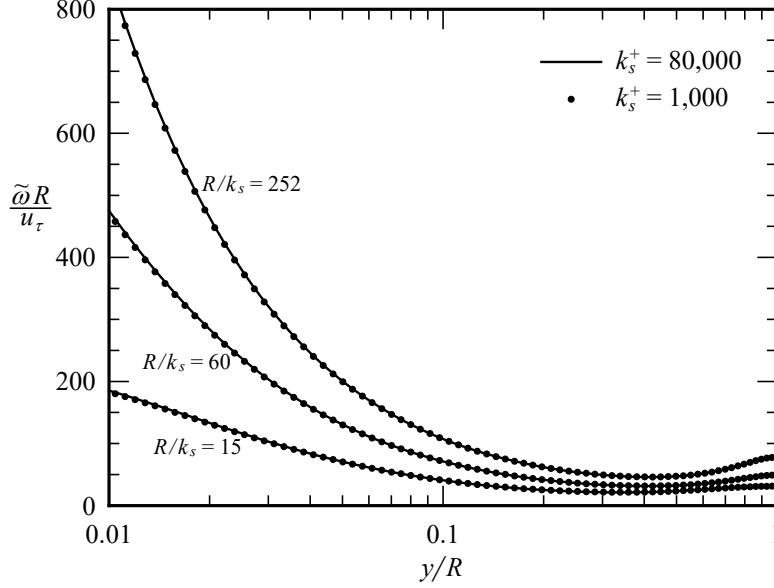
Figure 11. RMS-fluctuating-vorticity profiles, predicted from the k-λ formulation with the algebraic relation for λ, using $\sigma_k = 4.0$ and $k_{\mathrm{wal}}^+ = 0.1$.

VIII. Conclusions

For the case of incompressible flow, Eqs. (84) and (85) provide the foundation for an *energy-vorticity* RANS turbulence model that differs significantly from traditional *energy-dissipation* models, which include the established k-ε, k-ω, and k-ζ models commonly used for CFD. Following an analogy with the kinetic theory of gases, the development of Eqs. (84) and (85) is based on a more direct analogy between turbulent and molecular transport.

Whereas traditional k-ε, k-ω, and k-ζ turbulence models relate the kinematic eddy viscosity to a dissipation-based length scale associated with the smaller turbulent eddies having the highest strain rates per unit kinetic energy, Eq. (84) relates the kinematic eddy viscosity to an energy-weighted *mean vortex wavelength* associated with the larger energy-bearing eddies that are primarily responsible for turbulent transport. As can be seen from the definitions in Eqs. (1) and (83), the kinematic-eddy-viscosity model hypothesized in Eq. (84) depends on only the turbulent velocity fluctuations, just as the molecular viscosity depends on only the molecular velocity fluctuations. In contrast, the eddy-viscosity model used in traditional k-ε, k-ω, and k-ζ turbulence models results in a kinematic eddy viscosity that is inversely proportional to the molecular viscosity, which violates a fundamental requirement for a Boussinesq model of turbulent transport that is consistent with the definition of the specific Reynolds stress tensor given in Eq. (29).

The formulation of the turbulent-energy-transport equation that is given in Eq. (85) was developed rigorously from the Navier-Stokes equations and includes exact expressions for the viscous dissipation and molecular transport of turbulent kinetic energy. This development shows that molecular transport of turbulent kinetic energy is not a simple gradient diffusion process. Hence, subsequent application of the Boussinesq analogy between turbulent and molecular transport led to the modeled turbulent-transport term in Eq. (85), which is not simple gradient diffusion. In contrast, the turbulent-energy-transport equation that is used in traditional k-ε, k-ω, and k-ζ turbulence models is based on approximations for both the viscous dissipation and molecular transport of turbulent kinetic energy. These approximations led to the traditional turbulent-energy-transport equation, which models both the molecular and turbulent transport of turbulent kinetic energy as pure gradient diffusion.

The eddy-viscosity model and turbulent-energy-transport equation that are proposed here alleviate 5 of the 6 concerns with traditional RANS-based turbulence models, which are described in Section IV. The foundational equations can be written in terms of the RMS fluctuating vorticity [Eqs. (78) and (79)], the mean fluctuating enstrophy [Eqs. (81) and (82)], or the mean vortex wavelength [Eqs. (84) and (85)]. The final concern described in Section IV could be addressed by closing the model with a more rigorously developed transport equation such as that developed by Kreuzinger, Friedrich, and Gatski.[17]

The turbulent-energy-transport equation that is given in Eq. (85) contains two unknown closure coefficients; the viscous-dissipation coefficient, C_λ, and the turbulent-transport coefficient, σ_k. These coefficients should both be dimensionless universal constants. It has been shown here that excellent agreement with experimental data for velocity profiles and friction factors in fully rough pipe flow can be attained over the range of about $2 < \sigma_k < 6$ and $0.00001 < C_\lambda < 0.00056$, provided that the relation between C_λ and σ_k that is given in Eq. (100) is maintained.

In addition, the turbulent-energy transport equation requires a wall boundary condition for the specific turbulent kinetic energy, k. The specific turbulent kinetic energy at a rough surface should be proportional to the square of the friction velocity, u_τ, as shown in Eq. (97). For incompressible flow, the dimensionless proportionality coefficient, k_{wal}^+, is expected to be a unique function of the roughness Reynolds number, $k_s u_\tau / \nu$. As the roughness Reynolds number approaches zero, k_{wal}^+ should approach zero as well. By definition, fully rough flow occurs when the roughness Reynolds number is high enough so that the solution becomes independent of molecular viscosity. Hence, for fully rough flow, the dimensionless parameter, k_{wal}^+, must approach another universal constant associated with the turbulence model. It has been shown here that excellent agreement with experimental data for fully rough pipe flow can be attained over the range of about $0.05 < k_{\mathrm{wal}}^+ < 1.0$, provided that certain relations are maintained between the mean vortex wavelength, λ, and the dimensionless parameter, k_{wal}^+. Additionally, results show that the formulation can be used to predict the Darcy friction factor for fully rough pipe flow down to a roughness Reynolds number as low as about 100, which is much lower than the ratio of molecular viscosity to eddy viscosity at the wall would imply.

It is important to recognize from the discussion above that excellent agreement with experimental data for fully rough pipe flow can be attained over a range of the model constants, which include C_λ, σ_k, and the fully rough limit for k_{wal}^+. In terms of future development, this is fortunate, because it provides a great deal of flexibility that can be used when tuning the model to agree with experimental data for other turbulent flows.

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