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Ranking Score Vectors of Tournaments

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ABSTRACT

Ranking Score Vectors

of Tournaments

by

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Given T, a tournament on n vertices, Landau derived a method to determine how close T is to being transitive or regular. This comparison is based on the tournament's hierarchy number, $h_{\bar{x}}$, a value derived from its score vector $\bar{x} = (x_1, \ldots, x_n)$. Let X be the set of all score vectors of tournaments on n vertices with the entries listed in non-decreasing order. A partial order, poset, exists on the set X using the following binary relation. Given $\bar{x}, \bar{y} \in X$ such that $\bar{x} \neq \bar{y}$, let $\bar{x} \leq \bar{y}$ if $\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i$ for $k = 1, \ldots, n$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$. Let this poset be represented as $\mathcal{T}(n) = (X, P)$ where $P = \{(\bar{x}, \bar{y}) \in X \times X \ s.t. \bar{x} \leq \bar{y}\}$. The value $h_{\bar{x}}$ can also be used to define a partial order on the set X where $\bar{x} \leq \bar{y}$ if $h_{\bar{x}} > h_{\bar{y}}$. I propose that this new poset is an extension of the poset $\mathcal{T}(n)$. This can be proven using a method of comparing score vectors algebraically equivalent to Landau's hierarchy method. Specifically, $\sum_{i=1}^{n} {x_i}^2 = \sum_{i=1}^{n} {y_i}^2$ if and only if $h_{\bar{x}} = h_{\bar{y}}$. Furthermore, I conjecture that extending this method to compare $\bar{x}$ and $\bar{y}$ using $\sum_{i=1}^{n} {x_i}^m$ for $m \geq 2$ will always yield an extension of $\mathcal{T}(n)$, and that there exists some integer m dependent on n which will result in a linear extension of $\mathcal{T}(n)$.

(70 pages)

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CHAPTER 1

INTRODUCTION TO TOURNAMENTS

A Graph is a way to describe a collection of objects V and a binary relation $\sim$ between the objects. It is typically written as $G = (V, E)$ where V is the set of objects or *vertices* and $E = \{(x, y) \in V \times V \; s.t. x \sim y\}$ is the set of *edges* present. See the examples below.

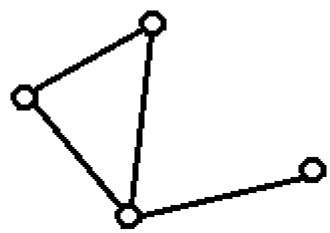


Figure 1. Simple graph on 4 vertices

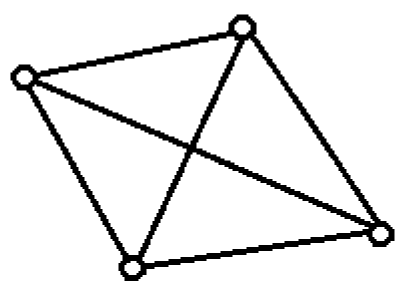


Figure 2. Simple, complete graph on 4 vertices

Both are examples of *simple graphs* on 4 vertices, where no more than one edge exists between any two vertices, and where $(x, x) \notin E$, or there are no loops in the graph. Figure 2 is an example of a *complete graph*, where an edge exists between every pair of vertices. A complete graph on n vertices is typically written as K_n and has $\binom{n}{2}$ edges.

Directed Graphs, or *Digraphs*, are graphs such that the binary relation $\rightarrow$ implies an ordering of the two vertices or a beating of one vertex by another. The notation is similar with $D = (V, A)$ being the digraph, V being the set of vertices, and $A = \{(x, y) \in V \times V\ s.t. x \rightarrow y\}$ being the set of *arcs*. Verbally, when $(x, y) \in A$, x is said to 'beat' y. A Tournament T is a simple, complete, digraph, getting its name from the idea of a Round-Robin Tournament and the relation among its players. It is *simple* as only one game result between any two players is recorded and no player competes with itself, *complete* as every pair of players compete against each other, and *directed* as there are no ties allowed, there is always a 'winner' given any two players.

Tournaments contain a vast amount of structure. As such, when discussing various topics relevant to graphs in general, we find certain patterns emerging once they are applied to tournaments. *Paths* and *cycles* are two such topics. A *path* of length m in a graph is a sequence of m unique vertices $x_1, x_2, \dots, x_m$ such that $(x_i, x_{i+1}) \in E$ for $i = 1, \dots, m-1$. Similarly, a *directed path* of length m in a digraph is such a sequence where $x_i \rightarrow x_{i+1}$ for $i = 1, \dots, m-1$. There are as many paths of length one as there are edges (arcs) in the graph (digraph), so paths of a specified or a maximal length are typically sought for. A path or directed path of length n, where all n elements in the graph are represented, is called a *Hamiltonian path*. A graph containing such a path is said to be, not surprisingly, *Hamiltonian*. When it comes to tournaments, every tournament on n vertices contains at least one Hamiltonian path. That being the case, for any two vertices x and y, there is always a directed path such that

$x = x_1, \dots, x_m = y$ or $y = x_1, \dots, x_m = x$, being some subset of the elements on the Hamiltonian path with $2 \le m \le n$.

What other significance do paths play in tournaments? If, for any two vertices x and y in a tournament T, there not only is a directed path from x to y, but there also exists a directed path from y back to x, then T is said to be *strongly connected* or *strong*. Though all tournaments contain at least one Hamiltonian path, not all tournaments are strong. That leads us to the next topic. If we can 'travel' from x to y then back to x again, we've completed a *cycle*. A cycle of length m in an arbitrary graph is a sequence of unique vertices $x_1, x_2, \dots, x_m$ such that $(x_i, x_{i+1}) \in E$ for $i = 1, \dots, m$ where $m + 1 = 1$. A directed cycle of length m follows similarly along the implied direction. Discussing the length of a cycle as we did for a path leads to the possibility of a cycle of length n or a *Hamiltonian cycle*.

Theorem 1. A Hamiltonian cycle exists on a tournament T if and only if T is strong [1].

Specific types of vertices begin to stand out in a tournament as we consider possible paths and/or cycles. A *source* is a vertex v such that all $n - 1$ arcs connected to it originate from v, or it beats every other vertex in the tournament. Note it is impossible to have more than one source as one will have to beat the other. A vertex v is a *sink* if all $n - 1$ arcs connected to it end with v, or it is beaten by every other vertex in the tournament. Similarly, there can be no more than one sink in a tournament. If a tournament has a source (sink), then any path it is on will begin (end) there, with no possibility of the said vertex being contained in a cycle. Such a tournament cannot have a Hamiltonian cycle, hence the tournament is not strong.

The converse of this statement is not true as it is possible to have a tournament that is *not* strong that contains neither a source nor a sink. The smallest n for which this occurs is $n = 6$, and this tournament is represented in the figure below.

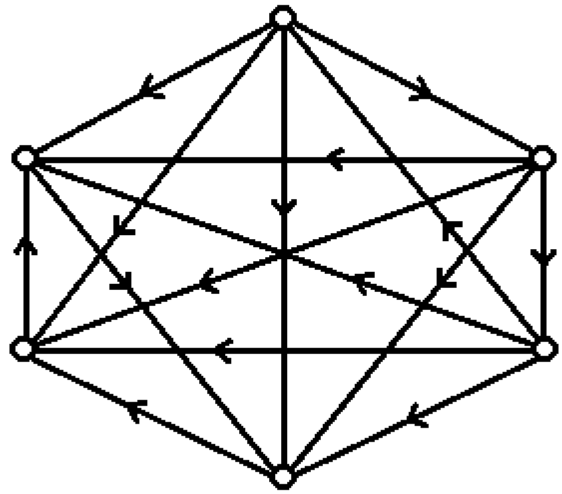


Figure 3. Tournament on six vertices that is not strong and contains no source or sink

The fact that this tournament is not strong can be verified in different ways. Consider the three vertices on the bottom left of the tournament, which constitute a 3-cycle. It is impossible for those 3 vertices to be on a cycle of length 4 or greater as none of them beat any of the other vertices in the tournament. Since there is no Hamiltonian cycle on this tournament, it is not strong.

After considering a source, which beats every other vertex directly in exactly one step, it may be natural to consider a vertex v that beats all other vertices in one *or* two steps. That is, let v be a vertex such that for all other $x \in V$ either $v \to x$ or $\exists w \in X$ s.t. $v \to w \to x$. Landau was one of the original pioneers in studying tournaments in the 1950's and defined several key concepts, including this one [10]. Such a vertex is often described as a *king* in a tournament. This slight change in the number of steps necessary to take in order to beat all other vertices makes a huge difference as to whether or not such a vertex exists. As stated previously, a source may or may not be present in a tournament, whereas it can be verified that every tournament contains at least one king [10].

In generalizing the idea of the number of vertices a given vertex beats directly, other definitions and notations may become helpful. Given a vector v, its *outset* $\{v\}^+$consists of all $x \in V$ s.t. $v \to x$ and its *inset* $\{v\}^-$ is all $x \in V$ s.t. $x \to v$. The cardinality of these sets is described as the vertex's *outdegree* $d^+(v)$ and *indegree* $d^-(v)$, respectively, and sums up to the total degree of the vertex, $n-1$. The outdegree of a vertex v is often referred to as its *score*.

Considering the score of each vertex separately, we can write that information as a vector called none other than the tournament's *score vector* $\bar{x} = (x_1, \dots, x_n)$. For the purposes of this paper, the scores will be listed in non-decreasing order. A score vector can be constructed for any digraph; but with a tournament, we benefit with additional information given the structure present. As a tournament is a complete digraph on n vertices, the total number of arcs is $\binom{n}{2}$. Using the notation above, that gives $\sum_{i=1}^{n} x_i = \binom{n}{2}$. Clearly, the values of a score range between 0 and $(n-1)$. Landau gave further constraints for a score vector on a tournament [10].

Theorem 2. A vector of non-negative integers, $\bar{x} = (x_1, \dots, x_n)$, is a score vector for a tournament on n vertices if and only if $\sum_{i=1}^{k} x_i \geq \binom{k}{2}$, $k = 1, \dots, n$, with equality for $k = n$.

Score vectors have been studied extensively and varying results have come of it. One of note gives constraints on a score vector representing a strong tournament, hereafter referred to as a *strong score vector*. This was done by Harary and Moser in 1966 [5].

Theorem 3. A score vector is strong if and only if $\sum_{i=1}^{k} x_i \geq \binom{k}{2} + 1$ for $k = 1, \dots., n-1$, with Landau's requirement of $\sum_{i=1}^{k} x_i = \binom{k}{2}$ for $k = n$.

We can now verify that the tournament given in Figure 3 is not strong using this requirement. The tournament's score vector is $\bar{x} = (1,1,1,4,4,4)$. The summation of the first three terms, $\sum_{i=1}^{3} x_i = 3$, is clearly less than the necessary $\binom{3}{2} + 1 = 4$. This result coincides with the reasoning given above for it not being strong. At least one of the 3 vertices in question would need to beat an additional vertex out of the 3-cycle, increasing its score by one value, and allowing a Hamiltonian cycle to occur.

We will now turn our attention to a different type of relation given a set X.

CHAPTER 2

INTRODUCTION TO POSETS

A *Partially Ordered Set*, or a *Poset,* consists of a set of elements X coupled with a binary relation $\leq$ which is *reflexive*, *anti-symmetric*, and *transitive*. Given $x, y, z \in X$, the binary relation $\leq$ is *reflexive* if $x \leq x, \forall x \in X$. It is *anti-symmetric* if for $x \neq y$, $x \leq y \Rightarrow y \nleq x$. It is *transitive* if $x \leq y, y \leq z \Rightarrow x \leq z.$ The poset is typically expressed as $\mathcal{P} = (X, P)$, where $P = \{(x, y) \in X \times X\ s.t.\, x \leq y\}$. When $(x, y) \in P$, $x \leq y$ or $y \geq x$ is written. Two elements are said to be *comparable*, $x \perp y$, if they relate to each other, and *incomparable*, $x \parallel y$, if they do not. There are some sources which use the definition of an *irreflexive* rather than reflexive binary relation. That is, $x \nleq x, \forall x \in X$. For the purposes of this paper, we will use definitions and notations consistent with [16]¸ and specify whether the poset in question is reflexive or irreflexive. We will also assume throughout that x and y are elements of X such that $x \neq y$.

Given an element $x \in X$, it is a *maximal element* if $\nexists y \in X$ such that $x \leq y$. Similarly, $x \in X$ is a *minimal element* if $\nexists y \in X$ such that $y \leq x$. Neither a maximal element nor a minimal element is necessarily unique, but they will always exist in a finite poset. An element $x \in X$ is a *maximum element* if for all other $y \in X$, $y \leq x$; and it is a *minimum element* if for all other $y \in X$, $x \leq y$. Note that for maximum and minimum elements, uniqueness is implied by definition, and they may or may not exist in a given poset.

Given $Y \subseteq X$, a non-empty subset of the elements of the poset, the poset relation can be restricted to the elements in Y. This results in the *subposet* $\big(Y, P(Y)\big)$. Being a subposet, ideas relevant to posets can similarly be talked about in $\big(Y, P(Y)\big)$. In addition to the topics of maximal, minimal, maximum and minimum elements, *upper bounds* and *lower bounds* can be discussed. An element $x \in X$ is an *upper bound* for a non-empty subset $Y \subseteq X$ if $y \leq x, \forall y \in Y$. It is considered the *least upper bound* if $x \leq x'$ where x' is any upper bound for the subset.

Likewise, $x \in X$ is a *lower bound* for Y if $x \leq y, \forall y \in Y$ and it the *greatest lower bound* if for all other lower bounds x', $x' \leq x$. Similar to maximum and minimum elements, least upper bounds and greatest lower bounds are by definition unique. A poset is called a *lattice* if every subset $Y \subseteq X$ has both a least upper bound and a greatest lower bound.

Looking at the structure of the poset as a whole, it is called *connected* if there is a finite sequence of comparable elements between any two given elements. That is, $\forall x, y \in X$, there exists a sequence of distinct elements $x_1, \dots, x_n$ such that $x_1 = x$, $x_n = y$, and $x_i \perp x_{i+1}$ for $i = 1, \dots, n-1$. There need only be a comparability between two adjacent elements. This sequence of elements is not necessarily a transitive sequence between x and y.

Given two posets (X, P) and (Y, Q), how can they be compared to one another? If there exists a function $f: X \to Y$ such that $x_1 \leq x_2$ in (X, P) implies that $f(x_1) \leq f(x_2)$ in (Y, Q), then f is said to be an *order preserving function*. Similarly, the two posets are said to be *isomorphic* if there exists a bijection $f: X \to Y$ such that $x_1 \leq x_2$ in (X, P) if and only if $f(x_1) \leq f(x_2)$ in (Y, Q).

Visually representing a poset as a graph is an important tool and can be done by having the vertices of the graph represent the elements of X and the edges between them illustrate their comparability, or lack thereof. The most common method is given two elements x and y where $(x, y) \in P$, an edge exists between them in the graph if and only if $\nexists z \in X$ such that $x \leq z \leq y$. That is to say, x is *covered* by y. Since P is a transitive relation, this minimizes the amount of edges necessary to represent in the graph. Orienting the graph so the relation is pointed upwards with the minimal elements towards the bottom and the maximal elements towards the top allows the relation to be seen more clearly. This is referred to as its Hasse diagram. Below are some examples of posets with their accompanying Hasse diagrams.

Example 1. $\mathcal{P} = (X, P)$ is a poset given that X is a finite set of integers with $(x, y) \in P$ if x is a divisor of y. This fits the definition of a poset that is reflexive (every integer is a divisor of itself), anti-symmetric (given any two distinct integers x and y such that x is a divisor of y, y is certainly not a divisor of x), and transitive (if x is a divisor of y and y is a divisor of z, then by virtue of division, x is a divisor of z as well). Below is a specific example of such a poset.

Let $X = \{1,2,3,6,9,12,18,36\}$, then $P = \{(x, x)\ \forall x \in X, (1, x)\ \forall x \in X, (2,6), (2,12),$ $(2,18), (2,36), (3,6), (3,9), (3,12), (3,18), (3,36), (6,12), (6,18), (6,36), (9,18),$ $(9,36), (12,36), (18,36)\}$

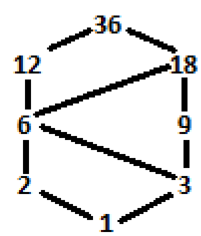


Figure 4. Hasse diagram for Example 1

Example 2. Given an n element set X, the family of all possible subsets of X coupled with the relation of set inclusion is a poset. This also satisfies the reflexive, anti-symmetric, and transitive properties of posets.

Let $X = \{1,2\}$, then $P = \{(\emptyset, \emptyset),\ (\{1\}, \{1\}),\ (\{2\}, \{2\}),\ (\{1,2\}, \{1,2\}),\ (\emptyset, \{1\}),$ $(\emptyset, \{2\}),\ (\emptyset, \{1,2\}), (\{1\}, \{1,2\}),\ (\{2\}, \{1,2\})\}$

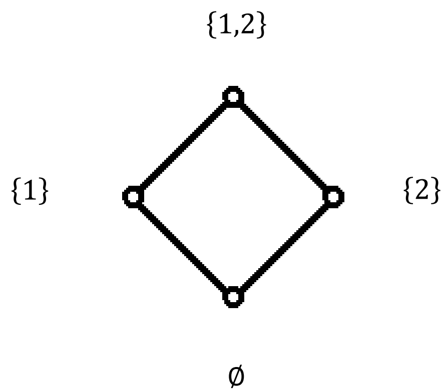


Figure 5. Hasse diagram for Example 2

Chains and Antichains

Chains and *Antichains* are fundamental structures in posets. A poset or a subposet is called a chain if all the elements are pair-wise comparable. Similarly, a poset or a subposet is called an antichain if all the elements are pair-wise incomparable. Certain types of chains and antichains are worth mentioning. Given a poset $\mathcal{P} = (X, P)$, the set of all its chains coupled with the relation of set inclusion is itself a poset. A chain C is considered a *maximal chain* if it is a maximal element in this new poset, i.e. it is not contained in any other chain by set inclusion. Different than the maximum element of a poset, C is a *maximum chain* if its cardinality if maximum. That is, if it contains the maximum amount of elements found in any chains. A maximum chain is not necessarily unique.

Maximal and *maximum antichains* are defined similarly. Given a poset $\mathcal{P} = (X, P)$, A is a maximal antichain if it is a maximal element in the poset consisting of all antichains with the relation of set inclusion. It is a *maximum antichain* if it contains the maximum amount of elements represented in any antichain. Again, neither is necessarily unique. Note that minimal chains and minimal antichains are not specifically defined as any individual element can be considered as such. The *height* of a poset is defined to be the cardinality of a maximum chain while its *width* is the cardinality of a maximum antichain. When a poset is represented using its Hasse diagram, these definitions have structural meaning as well. Two important theorems regarding chains and antichains are given below [3].

Theorem 4. Given a poset $\mathcal{P} = (X, P)$ with height n, there exists a partition of the elements of X into n antichains $A_1, A_2, \dots, A_n$.

The number of antichains into which X can be partitioned is certainly no smaller than n and is no larger than the cardinality of X itself. This theorem shows the existence of exactly n antichains and can be proved by construction. Let A_1 be the set of maximal elements in the poset. These constitute an antichain as no two are comparable. Next, define A_2 to be the set of maximal elements in the subposet $X - A_1$. This process continues down the height of the entire poset, until only the antichain A_n remains.

Theorem 5. Given a poset (X, P) with width n, there exists a partition of the elements of X into n chains, $C_1, C_2, \dots, C_n$.

The number of chains can be no fewer than the width n and is bounded above by the number of elements in X. This theorem proves the existence of a partition into exactly n chains. The proof is by induction on the cardinality of the set X.

Linear Extensions, Realizers, and Dimension

Given a poset $\mathcal{P} = (X, P)$, the poset $\mathcal{Q} = (X, Q)$ is an *extension* of $\mathcal{P}$ given that if $x \leq y$ in P, then $x \leq y$ in Q. In other words, $P \subseteq Q$. Notice it is possible to have a relation $x' \leq y'$ in Q without the same relation being present in P. Construction of a new poset is possible where the elements are all the extensions of $\mathcal{P} = (X, P)$, including $\mathcal{P}$ itself. $\mathcal{P}$ is the minimum element, and the maximal elements of this poset are called *linear extensions*. Equivalently, one may also see written that Q is an extension of P.

Linear extensions are of great interest when studying posets for the following reason. Consider the set of all linear extensions, $\mathcal{E}(P)$. The intersection of all these linear extensions of P is exactly P itself. A fairly simple conclusion, but it has far reaching applications. By definition, if you were given a family of t linear orders on the set X, $\mathcal{R} = \{L_1, \dots, L_t\}$, $\mathcal{R}$ is said to be a *realizer* of the poset relation P if $P = \cap\, \mathcal{R}$. Note that this can only happen if $\mathcal{R} \subseteq \mathcal{E}(P)$. Therefore, we can talk of some subset of linear extensions which realize the entire poset relation P. Linear extensions that realize P are of great interest as they are fairly easy to store and convey all the relations for the entire poset. The smallest value of t for which this occurs is called the *dimension* of the poset, often written as $\dim(\mathcal{P})$. Below are two theorems regarding dimension of a poset [16].

Theorem 6. Given a poset $\mathcal{P} = (X, P)$ and $\mathcal{R} = \{L_1, \dots, L_t\}$, a family of linear extensions of P, the following statements are equivalent.

i. $\mathcal{R}$ is a realizer of P

ii. $P = \cap\, \mathcal{R}$

iii. $\forall x, y \in X$ such that $x \parallel y$, $\exists i, j\ (i \neq j)$ with $1 \leq i, j \leq t$ such that $x < y$ in L_i and $y < x$ in L_j

iv. $\forall x, y \in X$ such that $x \parallel y$, $\exists j$ with $1 \leq j \leq t$ such that $y < x$ in L_j

Showing the existence of such a linear extension described in part iv. is a way often used to show that a list of linear extensions is in fact a realizer.

Theorem 7. Given a poset $\mathcal{P} = (X, P)$ with a non-empty subset $Y \subseteq X$, the dimensions of the poset and subposet are such that $\dim\big(Y, P(Y)\big) \leq \dim(X, P)$.

This is easily verifiable as given a set of t linear extensions that realize (X, P), it must also be a realizer of the subposet $\big(Y, P(Y)\big)$ once it is restricted to the elements of Y. There may be fewer linear extensions of Y which realize $P(Y)$, but the minimum number of them can be no more than t.

The dimension of a poset is, not surprisingly, dependent on the internal structure of the poset. Before concluding, some basic examples of posets and their dimensions may be of interest to note. A poset has a dimension of 1 if and only if it is a chain. An antichain with 2 or more elements has a dimension of 2. This can be seen as any linear ordering of the elements in the antichain can serve as one of the linear extensions in the realizer. An exact reversal of this ordering can serve as the second. There have been numerous methods proposed to determine upper or lower bounds on the dimension of a poset. One of significance is that the dimension of a poset $\mathcal{P} = (X, P)$ must be less than or equal to the width of the poset [3].

CHAPTER 3

INTERNAL RANKING OF A TOURNAMENT

Since we are looking at ‘tournaments’ with various real-life examples and the rich social use of the word in sporting events, etc., it may not be too surprising to find the question asked, “So, who won?” People tend to rate objects to better understand information that may be present in a complex system, or even to compare it with other similar structures. The answer to the question, however, is not so simply put and requires a context to compare the objects in. We will discuss some of the methods that have been explored in an attempt to rank the vertices of a tournament.

Kings

Let T be a tournament on n vertices. From what we know of tournaments so far, a source would certainly be an ideal ‘winner’ of T having beaten all other elements in the set. However, as not all tournaments contain a source, it may fall short of being ideal in a general setting, depending on how often a source is present in a tournament. Maurer gives an excellent paper [11] which summarizes basic theorems that have been developed regarding kings and touches on the question at hand. The answer to how often a source exists in a tournament can be found using probability. If we make the assumption that given any two vertices v_1 and v_2 the probability of $v_1 \to v_2$ is equally as likely as $v_2 \to v_1$, that is the probability of either event occurring is equal to $\frac{1}{2}$, then all $2^{\binom{n}{2}}$ possible orientations of arcs in a tournament are equally likely. This allows us to refer to a *random tournament*, a tournament on n vertices with any one of these equally possible orientations. The proof of the following theorem uses probability and is also given in [11].

Theorem 8. Given T, a random tournament on n vertices, the probability of T containing a source goes to 0 as n goes to ∞.

Let T be a random tournament with the vertex set $V = \{v_1, \dots, v_n\}$, and let E_i be the event that v_i is a source for $i = 1, \dots, n$. As a source beats all other $n-1$ vertices, the probability of E_i occurring, $P(E_i)$, is equal to the probability of having all $n-1$ of these arcs oriented in such a manner. That is, $P(E_i) = \left(\frac{1}{2}\right)_1 \left(\frac{1}{2}\right)_2 \dots \left(\frac{1}{2}\right)_{n-1} = \left(\frac{1}{2}\right)^{n-1}$ for $i = 1, \dots, n$. As a tournament can have no more than one source, the probability that T contains a source is equal to $P(E_1 \cup E_2 \cup \dots \cup E_n) = \left(\left(\frac{1}{2}\right)^{n-1}\right)_1 + \left(\left(\frac{1}{2}\right)^{n-1}\right)_2 + \cdots + \left(\left(\frac{1}{2}\right)^{n-1}\right)_n = n\left(\frac{1}{2}\right)^{n-1}$ or equivalently $\frac{n}{2^{n-1}}$. The limit of this expression as $n \to \infty$ is equal to 0.

As sources do not occur often enough, finding the 'next best object' seems a reasonable solution, and a king fits the bill. Recall that a king beats all others vertices in one or two steps, and every tournament contains at least one king. Given this information, a logical approach to rank the players in a tournament $T = T_1$ would be to find the said king, proclaim it the overall winner, and then consider the remaining $n-1$ vertices. The structure we are left with after deleting the initial vertex is itself a tournament on $n-1$ vertices. Let's call it T_2. Being a tournament, it also contains a king. Find that king, label it as the 2nd place winner, and then consider the tournament on the remaining $n-2$ vertices, and so on. This process continues until we are left with 1 remaining element, vacuously the 'winner' of its own tournament T_n, but last in the ranking of the entire vertex set.

Not a terribly complicated algorithm except for one detail. What if there is more than one king in a tournament? If the number of kings is relatively small, a tweaking of the above algorithm could be applied. Pull out all the kings of T_1 at once and lump them together as the first 'winning class' k_1. Consider the tournament $T_2 = T_1 - k_1$ and find all of its kings, labeling that class k_2, etc. This continues until all the elements are partitioned into m winning classes, $k_1, \dots, k_m$ with $1 \leq m \leq n$. This would not result in a linearization of the elements, but may still be a significant ranking scheme if m is close enough to n. To linearly rank *all* the elements, another method would need to be devised to distinguish the individual kings in an otherwise equivalent winning class. We now need to determine if the number of kings for a given tournament is small enough so this ranking of the vertices has relevance in a general setting. Before addressing it directly, note the following theorem.

Theorem 9. If a vertex v is beat in a tournament, it must be beat by a king [11].

For the proof, consider a vertex v which is beat and its inset, $\{v\}^- = \{x \in X \text{ s.t. } x \to v\}$. Since $\{v\}^-$ is not empty, it is a sub-tournament which contains a king, call it v^*. This vertex by definition beats every other element in $\{v\}^-$ in one or two steps, by construction beats v directly and beats every element in the original vertex's outset $\{v\}^+$ either directly or through v, again in one or two steps. Hence, v^* is also a king for the original tournament, and v is beaten by a king.

The above theorem allows us to begin to determine the number of kings possible in a tournament. A tournament may certainly contain just one king. That is the case if and only if the said vertex is also the source. A rare occurrence by Theorem 8. Can a tournament contain exactly *two* kings?

Theorem 10. There can never be exactly two kings in any given tournament [11].

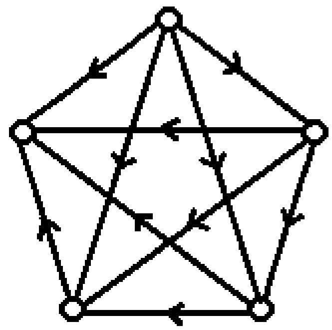
The proof can be done by contradiction. Assuming there exist exactly two kings in a tournament, v_1 and v_2, we know one must beat the other. Let $v_1 \to v_2$. Since v_2 is a king, it must beat v_1 in two steps. That is there exists another vertex v_3 such that $v_3 \to v_1$. Since v_1 is beaten, it must be beaten by a king, contradiction as v_2 is the only other king. Therefore, if there is no source, there must be at least three kings.

How many kings *can* exist in a tournament? Or better yet, what is the most common number of kings to exist in a tournament on n vertices? This answer to this can also be found using probability. Maurer makes use of it in [11] by showing that the probability of a vertex v *not* being a king in a random tournament goes to 0 as n goes to ∞. Hence, the probability of all vertices in a random tournament being kings goes to 1 as n goes to ∞. This clearly damages the relevance of ranking the vertices of a tournament into m distinct classes of kings. More than likely, m will very nearly be equal to 1, especially for large values of n. If a ranking of the vertices is still desired, another method will have to be derived.

Score Vectors

Another context we have in which to compare vertices is their respective scores. Comparing the vertices' individual scores seems a reasonable way to determine a winner among them. Interestingly, this way of comparing players is rarely sufficient to produce a linear ranking of the elements in any given tournament. There may be two or more of them who share the same score. In fact, the only score vector where each vertex has a unique score value is when $\bar{x} = (0,1,2, \dots, n-2, n-1)$. This score vector implies a linear ranking of the elements from its sink all the way to its source (or vice versa).

The tournament this vector corresponds to is called the *transitive tournament*. Similar to the transitive relation described for posets, if $x, y, z \in X$ such that $x \to y$ and $y \to z$, then $x \to z$ in the tournament. Below is a representation of the transitive tournament on $n = 5$ vertices, along with its score vector.



$\bar{x} = (0,1,2,3,4)$

Figure 6. The transitive tournament on 5 vertices

Theorem 11. Given a tournament T, the following are equivalent [1],[4],[5]:

i. T is transitive

ii. T is acyclic (no cycles of any length)

iii. T has a unique Hamiltonian path

iv. The score vector for T is $(0,1,2,\ldots,n-2,n-1)$

v. T contains $\frac{n(n-1)(n-2)}{6}$ transitive triples

The last statement refers to *transitive triples*. That is simply a set of 3 vertices v_1, v_2 and v_3 such that their relation is transitive, v_1 beats both v_2 and v_3, v_2 beats v_3, and v_3 beats neither v_1 nor v_2. The number of transitive triples is equal to $\binom{n}{3}$, the total number of ways to consider 3 vertices at a time. An equivalent version of this statement is to say that there are no cycles of length 3 in the tournament.

The transitive tournament is unique in many aspects. It is the only tournament with a linearization of the vertices using the score vector, the only tournament with just *one* Hamiltonian path, and the only tournament structure without cycles of any length present. This is some fairly detailed information regarding *one* score vector and the tournament it corresponds to. What other characteristics exist among the set of all possible score vectors? Are there any extremes or classifications among them? There have been algorithms used to numerically generate all possible score vectors for a specific n, but let's consider the varying score vectors generally.

The transitive score vector $\bar{x} = (0,1,2, \dots, n-2, n-1)$ can be considered as one extreme on the continuum of score vectors as each vertex in the tournament has a unique score. These unique score values are the values necessary to attain the lower bound of $\sum_{i=1}^{k} x_i = \binom{k}{2}$ for $k = 1, \dots n$ in Landau's Theorem. For all other score vectors, there will always be at least two vertices with the same score. As the multiplicities of a given score value changes, the remaining score values and their multiplicities will have to vary as well in order to keep with the restrictions of being a score vector. For example, $\bar{x} = (n-2, n-1, \dots, n-1)$ is not a score vector for $n \geq 3$ as it exceeds the number of arcs possible, $\binom{n}{2}$. These varying score values and their multiplicities continue to fluctuate until we reach the other end of the spectrum. Here we find each of the vertices having the same score, or nearly the same score as the average possible score value $\frac{n-1}{2}$. This is the *regular score vector* $\bar{x} = \left(\frac{n-1}{2}, \frac{n-1}{2}, \dots, \frac{n-1}{2}\right)$ when n is an odd integer, and the *nearly regular score vector* $\bar{x} = \left(\frac{n-2}{2}, \frac{n-2}{2}, \dots, \frac{n-2}{2}, \frac{n}{2}, \frac{n}{2}, \dots, \frac{n}{2}\right)$ when n is an even integer. The tournaments which correspond to these vectors are similarly called *regular* and *nearly regular*.

Between these two extremes sit the vast majority of score vectors. Among them, there is one other type of score vector worth mentioning now, those score vectors which correspond to an *upset tournament*. A tournament is considered upset if a vertex v_1 beats another vertex v_2 such that $d^+(v_1) < d^+(v_2)$. A classic example of an upset tournament is found by taking the transitive tournament and creating an arc-reversal so that the former 'sink' is now beating the former 'source'. This tournament is given below for $n = 5$ with the reversed arc bolded.

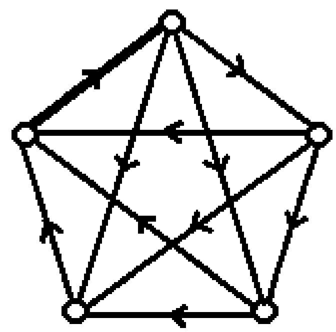


Figure 7. An upset tournament on 5 vertices

The score vector for this specific tournament on 5 vertices is $\bar{x} = (1,1,2,3,3)$ and can be written as $\bar{x} = (1,1,2,3, \ldots, n-3, n-2, n-2)$ for n in general. This score vector has a significant place in the set of all score vectors. Recall that a score vector is strong if and only if $\sum_{i=1}^{k} x_i \geq \binom{k}{2} + 1$ for $k = 1, \ldots., n-1$, and $\sum_{i=1}^{k} x_i = \binom{k}{2}$ for $k = n$. This score vector contains the values necessary to attain the lower bound of $\sum_{i=1}^{k} x_i = \binom{k}{2} + 1$ for $k = 1, \ldots, n-1$ with the other necessary requirement of $\sum_{i=1}^{k} x_i = \binom{k}{2}$ when $k = n$. Hence, it is often referred to as the 'smallest' strong score vector.

So, how many tournaments exist on n vertices? That depends on how you count. Is the interest in the number of distinct labelings possible given a unique tournament structure, or is it in the number of structurally unique tournaments for a given score vector? For the transitive tournament given above, there are $5! = 120$ ways to choose a labeling on its vertices. Similarly, there are $n!$ ways to label any given tournament on n vertices. Considering the vast number of labeled tournaments on n vertices, if we were given any two labeled tournaments, how can we go about determining if they actually have the same underlying structure? Two tournaments on n vertices, $T_1 = (V_1, A_1)$ and $T_2 = (V_2, A_2)$, are said to be *isomorphic* if there exists a function $f: V_1 \rightarrow V_2$ such that $x \rightarrow y$ in T_1 if and only if $f(x) \rightarrow f(y)$ in T_2. In other words, the vertices relate together the same way, call them what you will.

For the purposes of this paper, only *non-isomorphic* tournaments will be considered as the interest is ultimately in comparing two tournaments which are structurally distinct. As to the other method of counting, it has already been established in the above theorem that the transitive score vector represents a structurally unique tournament for any value of n. For $n = 1, \ldots, 4$, every score vector relates to a unique non-isomorphic tournament. For $n > 4$, there can be multiple non-isomorphic tournaments with the same score vector. As an example, when $n = 5$, there are 9 unique score vectors representing a total of 12 non-isomorphic tournaments. The smallest strong score vector $\bar{x} = (1,1,2,3,3)$ represents the tournament in Figure 7 as well as the tournament given in Figure 8 below. The score vector $\bar{y} = (1,2,2,2,3)$ corresponds to 3 non-isomorphic tournaments.

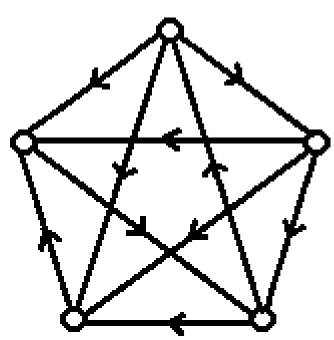


Figure 8. Second non-isomorphic tournament with score vector $\bar{x} = (1,1,2,3,3)$

As n increases, the number of non-isomorphic tournaments grows quite rapidly compared to the number of score vectors that correspond to them. Below is a table of values listing the number of score vectors and non-isomorphic tournaments for small values of n [14],[15].

n	score vectors	non-isomorphic tournaments
3	2	2
4	4	4
5	9	12
6	22	56
7	59	456
8	167	6,880
9	490	191,536
10	1,486	9,733,056
11	4,639	903,753,248
12	14,805	154,108,311,168

Table 1. Score vectors and non-isomorphic tournaments for $n = 1, \ldots, 12$

For $n = 7$, the regular score vector $\bar{x} = (3,3,3,3,3,3,3)$ corresponds to 3 non-isomorphic tournaments. This is the first value of n for which more than one structurally unique tournament is represented by the regular score vector. Despite this rapid growth in the number of non-isomorphic tournaments versus the number of possible score vectors, the transitive score vector and its accompanying tournament remains unique for all values of n.

In considering these two extremes on the continuum of score vectors and their tournament structures, the transitive score vector gives a linear ordering of all the vertices directly, whereas the regular and nearly regular score vectors show us that trying to order the vertices based on their individual scores will partition the vertices into, at most, two classes. This method of ranking vertices by their scores will not be beneficial for the (nearly) regular tournaments, especially for large values of n. How do the remaining score vectors fair? Moon proved that, considering this spectrum of tournaments from transitive to regular, most tournaments are actually strong [12]. That is, most tournaments are more closely related to the regular tournament(s) as compared to the transitive one. That clearly implies less and less variety in the individual scores and less relevance in ranking the vertices by their scores. Yet again, a method to rank the individual vertices falls short at being successful for a random tournament as there can be several, if not all, vertices with the same score.

Interestingly, the concept of kings relates to scores of the vertices. Any vertex with maximum score value is also a king [7]. That doesn't necessarily imply that every king has the highest score value possible. The upset tournament in Figure 7 is an example where the vertex with the minimum score value of 1 is also a king. (It beats one vertex directly and the remaining vertices in two steps.) Despite this, there is a connection between the two concepts, so perhaps it is not too surprising that they have similar draw backs as a linear ranking mechanism.

Other Methods

There have been various other methods proposed to rank the vertices in a tournament, the detail of which goes beyond the scope of this paper. Some attempt to take into account attributes that the score values cannot convey. For example, two vertices may have the same score, but perhaps one of them beats higher ranking vertices than the other. That could imply they are not as equally ranked as the score vector might imply. *Weighted score vectors* have been used to compensate for such differences among the vertices. See [2], [8], and [13] for further details.

Though these varying methods of comparing vertices have drawbacks when studying a random tournament on n vertices, they can be beneficial for tournaments that are closely related to the transitive tournament. Landau explored possible connections of tournaments manifest in social structures with mathematical models and conclusions about them. The most popular real-life example of a tournament is that of a natural chicken coop where a pecking-order exists. Given any two hens, one has the 'authority' to peck the other, and not vice-versa. He found these pecking orders to be very nearly hierarchical, i.e. transitive. This leads us to the next topic of how close a given tournament, or more specifically a given score vector, comes to being hierarchical.

CHAPTER 4

LANDAU'S HIERARCHY NUMBER

In Chapter 3, a common issue that continues to surface for a random tournament on n vertices is that a linearization of the vertices themselves becomes highly unlikely, especially for large values of n. Given a container filled with all random tournaments on n vertices, the probability of pulling out one that has all kings, is strong, or has individual scores that are closer to $\frac{n-1}{2}$ rather than either of the extremes of 0 or $n-1$ is extremely high as n goes to ∞. Rather than being able to determine a 'winner' or a ranking of some kind within a random tournament, we see an equality emerging among the vertices in the very topics we were hoping to rank them by. However, there is a different type of classification emerging, and that is among the tournaments themselves. I propose this classification can be seen as a poset structure on the set of all score vectors, and can be used to compare tournaments with differing score vector representations. This poset will be studied in depth in the remaining chapters. In the meantime, let's begin by expanding our view to look at the collection of all non-isomorphic tournaments on n vertices.

As n increases, there remain the extremes of two types, the transitive tournament and the (nearly) regular tournaments. For the remaining variety of tournaments in between, the vast majority of them are 'closer' to the regular tournaments than the transitive one. What type of relations exists among them all? What kind of ranking scheme can be used to determine how close a given tournament is to either extreme? We will now look at one well established ordering scheme based on the tournament's score vectors.

Landau derived a method to determine how close a given score vector is to being transitive on a scale from 0 to 1 [9]. It is called the score vector's *hierarchy index* or *hierarchy number*. Given a score vector $\bar{x} = (x_1, \dots, x_n)$, its hierarchy number, which we shall write as $h_{\bar{x}}$, is defined to be $\frac{12}{n(n^2-1)}\sum_{i=1}^{n}\left(x_i - \frac{n-1}{2}\right)^2$. This value is minimized when the score vector is regular, or nearly-regular, and maximized when it is transitive. Let's verify these values algebraically.

Regular and Nearly Regular Score Vectors

When n is odd and $\bar{x}$ is a regular score vector, $x_i = \frac{n-1}{2}$ for all $i = 1, \dots, n$, hence its hierarchy number is found to be equal to zero. When n is even, the nearly regular score vector is equal to $\left(\frac{n-2}{2}, \frac{n-2}{2}, \dots, \frac{n-2}{2}, \frac{n}{2}, \frac{n}{2}, \dots, \frac{n}{2}\right)$. Half of its scores are $\frac{n-1}{2}$ rounded down to the nearest integer while the other half are $\frac{n-1}{2}$ rounded up. Since $x_i - \frac{n-1}{2} = \frac{1}{2}$ for all $i = 1, \dots n$, its hierarchy number $h_{\bar{x}}$ is equal to $\frac{12}{n(n^2-1)} n \left(\frac{1}{2}\right)^2 = \frac{3}{n^2-1}$. This number approaches zero as n goes to infinity.

Transitive Score Vector

For the other extreme when $\bar{x}$ is transitive, let's consider the two cases of n being odd and even separately. When $n = 2k + 1$ for some integer k, $\frac{n-1}{2}$ is an integer and is represented in the middle of the vector $\bar{x} = (0,1,2, \dots, n-2, n-1)$, so that $x_{k+1} = \frac{n-1}{2}$ with exactly k terms before and after it. From this symmetry and the fact the distance between any two consecutive terms is 1, it can be concluded that for every term after $x_{k+1} = \frac{n-1}{2}$ in the vector, there exists a term preceding it in the vector such that they are equidistant to the middle term x_{k+1}. In other words, for $j = 1, \dots, k$, $\left|x_{(k+1)+j} - \frac{n-1}{2}\right| = \left|x_{(k+1)-j} - \frac{n-1}{2}\right|$. Thus, $\sum_{i=1}^{n}\left(x_i - \frac{n-1}{2}\right)^2$ simplifies as follows:

$$2\left[\left(\frac{n+1}{2} - \frac{n-1}{2}\right)^2 + \left(\frac{n+3}{2} - \frac{n-1}{2}\right)^2 + \cdots + \left((n-2) - \frac{n-1}{2}\right)^2 + \left((n-1) - \frac{n-1}{2}\right)^2\right]$$

$$= 2\left[1^2 + 2^2 + \cdots + \left(\frac{n-3}{2}\right)^2 + \left(\frac{n-1}{2}\right)^2\right]$$

$$= 2\sum_{i=1}^{\left(\frac{n-1}{2}\right)} i^2 = 2\left[\frac{(n-1)(n+1)(n)}{24}\right] = \frac{(n-1)(n+1)(n)}{12}$$

$$\Rightarrow h_{\bar{x}} = \frac{12}{n(n^2-1)} \cdot \frac{n(n^2-1)}{12} = 1$$

The hierarchy number for the transitive score vector when $n = 2k$ for some integer k can be found similarly. Since n is even, $\frac{n-1}{2}$ is not an integer in the score vector. However, it is still a number such that k of the terms in the score vector are less than it and k are greater than it with the same equidistant property as stated above. So $\sum_{i=1}^{n}\left(x_i - \frac{n-1}{2}\right)^2$ simplifies to:

$$2\left[\left(\frac{n}{2} - \frac{n-1}{2}\right)^2 + \left(\frac{n+2}{2} - \frac{n-1}{2}\right)^2 + \cdots + \left((n-2) - \frac{n-1}{2}\right)^2 + \left((n-1) - \frac{n-1}{2}\right)^2\right]$$

$$= 2\left[\left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \cdots + \left(\frac{n-3}{2}\right)^2 + \left(\frac{n-1}{2}\right)^2\right]$$

$$= 2\sum_{i=1}^{\left(\frac{n}{2}\right)}\left(i - \frac{1}{2}\right)^2 = 2\left[\sum_{i=1}^{\left(\frac{n}{2}\right)} i^2 - \sum_{i=1}^{\left(\frac{n}{2}\right)} i + \sum_{i=1}^{\left(\frac{n}{2}\right)} \frac{1}{4}\right]$$

$$= 2\left[\frac{(n)(n+2)(n+1)}{24} - \frac{(n)(n+2)}{8} + \frac{n}{8}\right] = 2\left[\frac{n^3-n}{24}\right]$$

$$\Rightarrow h_{\bar{x}} = \frac{12}{n(n^2-1)} \cdot \frac{n(n^2-1)}{12} = 1$$

Smallest Strong Score Vector

After verifying the hierarchy number for the two extremes in the set of score vectors, it is interesting to note the hierarchy number for the smallest strong score vector described earlier, namely $\bar{x} = (1,1,2,3,\ldots,n-3,n-2,n-2)$. In order to do this, the two cases when n

is odd and when it is even still need to be considered separately. As this score vector is essentially the transitive score vector with the exception of the first and last terms, similar steps and assumptions can be made in determining its hierarchy number.

When $n = 2k + 1$ for some integer k, $\frac{n-1}{2}$ is an integer and is represented in the middle of the score vector so that $x_{k+1} = \frac{n-1}{2}$. The expression $\sum_{i=1}^{n}\left(v_i - \frac{n-1}{2}\right)^2$ simplifies as follows:

$$2\left[\left(\frac{n+1}{2} - \frac{n-1}{2}\right)^2 + \left(\frac{n+3}{2} - \frac{n-1}{2}\right)^2 + \cdots + \left((n-3) - \frac{n-1}{2}\right)^2\right] + 4\left((n-2) - \frac{n-1}{2}\right)^2$$

$$= 2\left[1^2 + 2^2 + \cdots + \left(\frac{n-5}{2}\right)^2\right] + 4\left(\frac{n-3}{2}\right)^2$$

$$= 2\left[\sum_{i=1}^{(n-5)/2} i^2\right] + 4\frac{(n-3)^2}{4}$$

$$= 2\left[\frac{(n-5)(n-3)(n-4)}{24}\right] + (n-3)^2$$

$$= \frac{n^3 - 25n + 48}{12}$$

$$\Rightarrow h_{\bar{x}} = \frac{n^3 - 25n + 48}{n(n^2 - 1)}$$

When n is even, $\frac{n-1}{2}$ is not represented in the score vector. As the distances between the first k terms in the score vector and $\frac{n-1}{2}$ are the same for the remaining k terms, the expression $\sum_{i=1}^{n}\left(v_i - \frac{n-1}{2}\right)^2$ simplifies in the following manner:

$$2\left[\left(\frac{n}{2} - \frac{n-1}{2}\right)^2 + \left(\frac{n+2}{2} - \frac{n-1}{2}\right)^2 + \cdots + \left((n-3) - \frac{n-1}{2}\right)^2\right] + 4\left((n-2) - \frac{n-1}{2}\right)^2$$

$$= 2\left[\left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \cdots + \left(\frac{n-5}{2}\right)^2\right] + 4\left(\frac{n-3}{2}\right)^2$$

$$= 2\left[\sum_{i=1}^{(n-4)/2}\left(i-\frac{1}{2}\right)^2\right] + (n-3)^2$$

$$= 2\left[\sum_{i=1}^{(n-4)/2} i^2 - \sum_{i=1}^{(n-4)/2} i + \sum_{i=1}^{(n-4)/2}\frac{1}{4}\right] + (n-3)^2$$

$$= 2\left[\frac{(n-4)(n-2)(n-3)}{24} - \frac{(n-4)(n-2)}{8} + \frac{n-4}{8}\right] + (n-3)^2$$

$$= \frac{n^3 - 25n + 48}{12}$$

$$\Rightarrow h_{\bar{x}} = \frac{n^3 - 25n + 48}{n(n^2-1)}$$

Thus, the hierarchy number for the smallest strong score vector is the same whether n is even or odd and is equal to $\frac{n^3-25n+48}{n(n^2-1)}$. As n tends to infinity, this hierarchy number approaches 1. It is interesting that using the hierarchy number, the smallest strong score vector, whose tournaments contain a Hamiltonian cycle, hence cycles of all length, is more and more comparable to the transitive score vector, the acyclic tournament, as n increases. Despite the structural difference in the tournaments, this is due to the similarities in the score vectors themselves.

Exploring Landau's Hierarchy Number Further

Although comparisons can be made between two distinct score vectors using Landau's hierarchy number, there are obstacles present if the aim is to linearly rank the score vectors and/or the tournaments they represent. The function of mapping a score vector to a hierarchy number is not one-to-one. This begins to be manifest for small values of n. When $n = 4$, two of the four score vectors, $\bar{x} = (1,1,1,3)$ and $\bar{y} = (0,2,2,2)$, have the same hierarchy number of $\frac{3}{5}$ as $h_{\bar{x}} = \frac{1}{5}\left[3\left(-\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2\right] = \frac{3}{5} = \frac{1}{5}\left[\left(-\frac{3}{2}\right)^2 + 3\left(\frac{1}{2}\right)^2\right] = h_{\bar{y}}$. When $n = 5$, the score vectors

$\bar{x} = (1,1,1,3,4), \bar{y} = (0,2,2,2,4)$ and $\bar{z} = (0,1,3,3,3)$ all have the same hierarchy number of $\frac{4}{5}$ while $\bar{v} = (1,1,2,2,4)$ and $\bar{w} = (0,2,2,3,3)$ both have a hierarchy number of $\frac{3}{5}$. This trend of having multiple score vectors share the same hierarchy number continues as n increases. Although the hierarchy number does not provide a linear ordering among the score vectors, it does present some sort of ordering and has a significant relation to an additional ordering method that will be discussed later. For now, we will take it as it is and present an alternative, yet algebraically equivalent way of comparing score vectors.

Consider again the hierarchy number for a given score vector $\bar{x} = (x_1, \dots, x_n)$.

$$h = \frac{12}{n(n^2-1)} \sum_{i=1}^{n} \left(x_i - \frac{n-1}{2}\right)^2$$

Given two score vectors $\bar{x} = (x_1, x_2, \dots, x_n)$ and $\bar{y} = (y_1, y_2, \dots, y_n)$ such that $\bar{x} \neq \bar{y}$ and their hierarchy numbers are equivalent, that is to say that $h_{\bar{x}} = h_{\bar{y}}$, we can rewrite this relation and see an equivalent one that exists between the two score vectors.

$$h_{\bar{x}} = \frac{12}{n(n^2-1)} \sum_{i=1}^{n} \left(x_i - \frac{n-1}{2}\right)^2 = \frac{12}{n(n^2-1)} \sum_{i=1}^{n} \left(y_i - \frac{n-1}{2}\right)^2 = h_{\bar{y}}$$

$$\Rightarrow \sum_{i=1}^{n} \left(x_i - \frac{n-1}{2}\right)^2 = \sum_{i=1}^{n} \left(y_i - \frac{n-1}{2}\right)^2$$

$$\Rightarrow \sum_{i=1}^{n} \left(x_i^2 - 2x_i\left(\frac{n-1}{2}\right) + \left(\frac{n-1}{2}\right)^2\right) = \sum_{i=1}^{n} \left(y_i^2 - 2y_i\left(\frac{n-1}{2}\right) + \left(\frac{n-1}{2}\right)^2\right)$$

$$\Rightarrow \sum_{i=1}^{n} (x_i^2) - (n-1)\sum_{i=1}^{n} x_i + n\left(\frac{n-1}{2}\right)^2 = \sum_{i=1}^{n} (y_i^2) - (n-1)\sum_{i=1}^{n} y_i + n\left(\frac{n-1}{2}\right)^2$$

Clearly, the last terms on both sides of the equation are equivalent and can be canceled out. Similarly, as $\bar{x}$ and $\bar{y}$ are both score vectors for the same n, we know by definition that

$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i = \binom{n}{2}$. So the second terms on both sides of the equation are likewise equivalent and can be canceled out, leaving the following relation:

$$\sum_{i=1}^{n} {x_i}^2 = \sum_{i=1}^{n} {y_i}^2$$

Theorem 12. Two score vectors for any given n have the same hierarchy number if and only if their respective inner products are equivalent. That is, their magnitudes are equivalent.

For purposes later on, it may be helpful to specifically note the following implication.

Corollary to Theorem 12.

$$\sum_{i=1}^{n} {x_i}^2 > \sum_{i=1}^{n} {y_i}^2 \Leftrightarrow h_{\bar{x}} > h_{\bar{y}}$$

CHAPTER 5

POSET STRUCTURE ON SCORE VECTORS

In 1934, Hardy, Littlewood and Polya introduced the notion of *majorization* [6]. That is, given two vectors $\bar{x}$ and $\bar{y}$ such that they consist of elements of $\mathbb{R}^n$, $\bar{x}$ is said to be *majorized* by $\bar{y}$ if $\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i$ for $1 \leq k \leq n$ with $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$.

This method of comparing vectors can be applied to score vectors. Let X be the set of all score vectors on n vertices. Given two score vectors $\bar{x}, \bar{y} \in X$ such that they are listed in non-decreasing order, let $\bar{x} \leq \bar{y}$ if $\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i$ for $k = 1, \dots, n$ with equality when $k = n$. Restricting this relation to score vectors such that $\bar{x} \neq \bar{y}$, and letting $\bar{x}$ and $\bar{y}$ be incomparable otherwise, will define a poset that has an irreflexive, anti-symmetric, and transitive binary relation. Let $\mathcal{T}(n) = (X, P)$ be the poset where $P = \{(\bar{x}, \bar{y}) \in X \times X \ s.t. \bar{x} \leq \bar{y}\}$. For notational convenience, let $\Sigma_{\bar{x}} = \left(\sum_{i=1}^{1} x_i, \sum_{i=1}^{2} x_i, \dots, \sum_{i=1}^{n} x_i\right)$ be the *summation vector* for the score vector $\bar{x} = (x_1, x_2, \dots, x_n)$. Let's explore some properties of this poset.

Recall Landau's Theorem stating that a vector $\bar{x} = (x_1, \dots, x_n)$ is a score vector for a tournament on n vertices if and only if $(\Sigma_{\bar{x}})_k \geq \binom{k}{2}$ for $k = 1, \dots, n$, with equality for $k = n$. For the transitive score vector $\bar{x} = (0,1, \dots, n-2, n-1)$, its summation vector is equal to $\big(0, (0+1), \dots, (0+1+2+\cdots+(n-2)), (0+1+2+\cdots+(n-2)+(n-1))\big)$, which simplifies nicely to $\left(\binom{1}{2}, \binom{2}{2}, \dots, \binom{n-1}{2}, \binom{n}{2}\right)$. By Landau's Theorem, every other score vector $\bar{y} \in \mathcal{T}(n)$ is such that $\binom{k}{2} = \sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i$ for all $k = 1, \dots, n$. Therefore, $\bar{x} \leq \bar{y}$ and the transitive score vector is the minimum element in $\mathcal{T}(n)$.

Similarly, the maximum element in the poset can be found. When n is odd, consider the regular score vector $\bar{x} = \left(\frac{n-1}{2}, \frac{n-1}{2}, \dots, \frac{n-1}{2}\right)$. The ith term attains the maximal score value possible for each $i = 1, \dots, n$, namely $\frac{n-1}{2}$. This implies that $\sum_{i=1}^{k} x_i$ is maximized for each value of $k = 1, \dots, n$. Thus, for all other score vectors $\bar{y} \in \mathcal{T}(n)$, $\bar{y} \leq \bar{x}$ in the poset. When n is even, the nearly regular score vector $\bar{x} = \left(\frac{n-2}{2}, \dots, \frac{n-2}{2}, \frac{n}{2}, \dots, \frac{n}{2}\right)$ likewise has the maximal score value possible for a score vector in each term. This implies that $\sum_{i=1}^{k} x_i$ is maximized for each k, and $\bar{y} \leq \bar{x}$ for all other score vectors $\bar{y} \in \mathcal{T}(n)$. Hence, the maximum element in the poset is the regular score vector when n is odd and the nearly regular score vector when n is even.

The score vector $\bar{x} = (1,1,2,3, \dots, n-3, n-2, n-2)$ also has an interesting place in the poset. As stated previously, this is the strong score vector whose hierarchy number tends to 1 as n tends to ∞. It was also referred to as the 'smallest' as $\sum_{i=1}^{k} x_i$ was equal to the minimum value possible for a strong score vector. Looking at this in the context of the poset $\mathcal{T}(n)$, for every other strong score vector $\bar{y} \in \mathcal{T}(n)$, $\binom{k}{2} + 1 = \sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i$ for $k = 1, \dots, n-1$. Therefore, in the subposet consisting of all strong score vectors, $\bar{x}$ is the minimum element and is well-named as the smallest strong score vector.

Constructing $\mathcal{T}(n)$ for small values of n may be helpful to determine other general characteristics. Below are figures of $\mathcal{T}(n)$ for $n = 3,4,5$, and 6 with the score vectors.

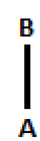


Figure 9. Poset on the score vectors with $n = 3$

$A = (0,1,2) \Rightarrow \Sigma_A = (0,1,3)$ $\qquad$ $B = (1,1,1) \Rightarrow \Sigma_B = (1,2,3)$

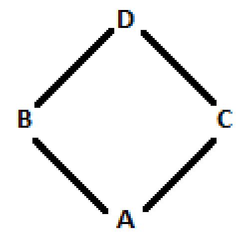


Figure 10. Poset on the score vectors with $n = 4$

$A = (0,1,2,3) \Rightarrow \Sigma_A = (0,1,3,6)$ $\quad$ $B = (1,1,1,3) \Rightarrow \Sigma_B = (1,2,3,6)$

$C = (0,2,2,2) \Rightarrow \Sigma_C = (0,2,4,6)$ $\quad$ $D = (1,1,2,2) \Rightarrow \Sigma_D = (1,2,4,6)$

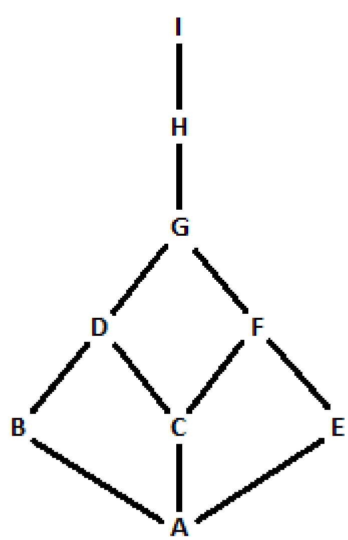


Figure 11. Poset on the score vectors for $n = 5$

$A = (0,1,2,3,4) \Rightarrow \Sigma_A = (0,1,3,6,10)$ $\quad$ $B = (1,1,1,3,4) \Rightarrow \Sigma_B = (1,2,3,6,10)$

$C = (0,2,2,2,4) \Rightarrow \Sigma_C = (0,2,4,6,10)$ $\quad$ $D = (1,1,2,2,4) \Rightarrow \Sigma_D = (1,2,4,6,10)$

$E = (0,1,3,3,3) \Rightarrow \Sigma_E = (0,1,4,7,10)$ $\quad$ $F = (0,2,2,3,3) \Rightarrow \Sigma_F = (0,2,4,7,10)$

$G = (1,1,2,3,3) \Rightarrow \Sigma_G = (1,2,4,7,10)$ $\quad$ $H = (1,2,2,2,3) \Rightarrow \Sigma_H = (1,3,5,7,10)$

$I = (2,2,2,2,2) \Rightarrow \Sigma_I = (2,4,6,8,10)$

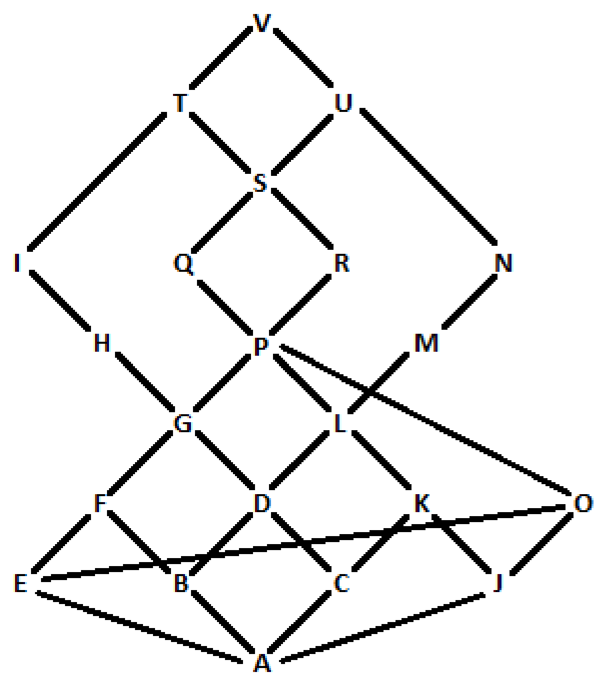


Figure 12. The poset on the score vectors for $n = 6$

$A = (0,1,2,3,4,5) \Rightarrow \Sigma_A = (0,1,3,6,10,15)$ $\quad$ $B = (0,2,2,2,4,5) \Rightarrow \Sigma_B = (0,2,4,6,10,15)$

$C = (0,1,3,3,3,5) \Rightarrow \Sigma_C = (0,1,4,7,10,15)$ $\quad$ $D = (0,2,2,3,3,5) \Rightarrow \Sigma_D = (0,2,4,7,10,15)$

$E = (1,1,1,3,4,5) \Rightarrow \Sigma_E = (1,2,3,6,10,15)$ $\quad$ $F = (1,1,2,2,4,5) \Rightarrow \Sigma_F = (1,2,4,6,10,15)$

$G = (1,1,2,3,3,5) \Rightarrow \Sigma_G = (1,2,4,7,10,15)$ $\quad$ $H = (1,2,2,2,3,5) \Rightarrow \Sigma_H = (1,3,5,7,10,15)$

$I = (2,2,2,2,2,5) \Rightarrow \Sigma_I = (2,4,6,8,10,15)$ $J = (0,1,2,4,4,4) \Rightarrow \Sigma_J = (0,1,3,7,11,15)$

$K = (0,1,3,3,4,4) \Rightarrow \Sigma_K = (0,1,4,7,11,15)$ $L = (0,2,2,3,4,4) \Rightarrow \Sigma_L = (0,2,4,7,11,15)$

$M = (0,2,3,3,3,4) \Rightarrow \Sigma_M = (0,2,5,8,11,15)$ $N = (0,3,3,3,3,3) \Rightarrow \Sigma_N = (0,3,6,9,12,15)$

$O = (1,1,1,4,4,4) \Rightarrow \Sigma_O = (1,2,3,7,11,15)$ $P = (1,1,2,3,4,4) \Rightarrow \Sigma_P = (1,2,4,7,11,15)$

$Q = (1,2,2,2,4,4) \Rightarrow \Sigma_Q = (1,3,5,7,11,15)$ $R = (1,1,3,3,3,4) \Rightarrow \Sigma_R = (1,2,5,8,11,15)$

$S = (1,2,2,3,3,4) \Rightarrow \Sigma_S = (1,3,5,8,11,15)$ $T = (2,2,2,2,3,4) \Rightarrow \Sigma_T = (2,4,6,8,11,15)$

$U = (1,2,3,3,3,3) \Rightarrow \Sigma_U = (1,3,6,9,12,15)$ $V = (2,2,2,3,3,3) \Rightarrow \Sigma_V = (2,4,6,9,12,15)$

The transitive and (nearly) regular score vectors are placed appropriately as the minimum and maximum elements in each poset. There is much symmetry to be observed elsewhere. Consider the placement of the non-strong score vectors in the posets, not including the transitive score vectors. That consists of elements B and C in Figure 8, elements B through F in Figure 9, and elements B through O in Figure 10. There seems to be a polarization in the posets between score vectors with a source and no sink (the chain along the far left-hand side of the posets) and those with a sink and no source (the chain along the far right-hand side of the posets). For the values of n being represented, any two such vectors are incomparable. This is always the case, regardless of n.

Theorem 13. Let $\bar{x}$ and $\bar{x}'$ be score vectors on n vertices where $\bar{x}$ has a source and no sink, and $\bar{x}'$ has a sink and no source. Then $\bar{x} \parallel \bar{x}'$ in the poset $\mathcal{T}(n)$.

Let $\bar{x} = (a, \dots, n-1)$ and $\bar{x}' = (0, \dots, b)$ where $0 < a$ and $b < n-1$. Consider their respective summation vectors $\Sigma_{\bar{x}} = \left(a, \dots, \binom{n}{2} - (n-1), \binom{n}{2}\right)$ and $\Sigma_{\bar{x}'} = \left(0, \dots, \binom{n}{2} - b, \binom{n}{2}\right)$. Since $0 < a$ and $\left(\binom{n}{2} - (n-1)\right) < \left(\binom{n}{2} - b\right)$, we know $\bar{x} \parallel \bar{x}'$.

Let's turn out attention to the structure around the smallest strong score vector $\bar{x} = (1,1,2,3, \dots, n-3, n-2, n-2)$. It was previously determined that $\bar{x}$ is the minimum element in the subposet of strong score vectors. This can be verified visually for the elements B, D, G, and P in Figures 9 through 12 respectively. For $n = 3$ and 4, B and D also happen to be the maximum elements of the posets. A consistent pattern occurs directly beneath these elements for $n = 4{,}5$, and 6. In each case, there is no unique score vector covered by them. D and G both cover two distinct score vectors while P covers three. Is this always the case?

Two distinct score vectors can be made from $\bar{x} = (1,1,2,3 \dots, n-3, n-2, n-2)$ for $n \geq 4$ by switching a score value of 1 between two adjacent entries in the score vector. Let $\overline{y_1} = (1,1,2,3, \dots, n-3, n-1)$ be made by switching a score value of between the last two entries of $\bar{x}$, creating a source; and let $\overline{y_2} = (0,2,2,3, \dots, n-2, n-2)$ be made by switching a score value between the first two entries of $\bar{x}$, creating a sink. Depending on which direction the score is 'switched', a reordering of the entries may be necessary. Note that $\overline{y_1}$ and $\overline{y_2}$ represent two distinct score vectors directly beneath $\bar{x}$ in $\mathcal{T}(4)$, $\mathcal{T}(5)$, and $\mathcal{T}(6)$.

Theorem 14. For $n \geq 4$, the two vectors $\overline{y_1} = (1,1,2,3, \dots, n-3, n-1)$ and $\overline{y_2} = (0,2,2,3, \dots, n-2, n-2)$ are score vectors by definition and are less than the smallest strong score vector $\bar{x} = (1,1,2,3, \dots, n-3, n-2, n-2)$ in the poset $\mathcal{T}(n)$.

To prove this, consider the two summation vectors $\Sigma_{\overline{y_1}} = \left(1,2,4,7, \dots, \binom{n-1}{2}, \binom{n}{2}\right)$ and $\Sigma_{\overline{y_2}} = \left(0,2,4,7, \dots, \binom{n-1}{2} + 1, \binom{n}{2}\right)$. Both have exactly one entry that is strictly less than an entry in $\Sigma_{\bar{x}} = \left(1,2,4,7, \dots, \binom{n-1}{2} + 1, \binom{n}{2}\right)$, the $(n-1)^{\text{st}}$ and 1^{st} respectively. These entries are exactly equal to $\binom{k}{2}$, the minimal value necessary to be a score vector. The remaining entries are equal to $\binom{k}{2} + 1$, hence Landau's criteria for score vectors is met and they are both less than the original score vector $\bar{x}$.

The question remains whether $\bar{x}$ *covers* both $\overline{y_1}$ and $\overline{y_2}$ for any value of n, not just for those visually verifiable. This can be seen as it is impossible for another score vector $\overline{y_3}$ to exist whose summation vector can be placed with strict inequality between $\overline{y_1}$ and $\bar{x}$ or $\overline{y_2}$ and $\bar{x}$, as there is a difference of only 1 between the entries where the inequalities occur. Therefore, we can conclude the following:

Corollary to Theorem 14. For $n \geq 4$, the score vectors $\overline{y_1} = (1,1,2,3, \dots, n-3, n-1)$ and $\overline{y_2} = (0,2,2,3, \dots, n-2, n-2)$ are both covered by the smallest strong score vector $\bar{x} = (1,1,2,3, \dots, n-3, n-2, n-2)$. Hence, $\bar{x}$ covers at least two score vectors.

For $\mathcal{T}(4)$ and $\mathcal{T}(5)$, $\overline{y_1}$ and $\overline{y_2}$ are the *only* score vectors covered by $\bar{x}$. As n increases, this is not necessarily the case. For $n = 6$, the score vector $O = (1,1,1,4,4,4)$ is also covered by $\bar{x} = P = (1,1,2,3,4,4)$. Interestingly, O can also be constructed from P by switching a score value of 1 between two adjacent entries in P. A switch between the third and fourth entries will result in either $(1,1,3,2,4,4) = (1,1,2,3,4,4) = \bar{x}$ or $(1,1,1,4,4,4) = O$. For $n = 7$, the smallest strong score vector $\bar{x} = (1,1,2,3,4,5,5)$ covers the score vectors $\bar{u} = (1,1,2,2,5,5,5)$ and $\bar{v} = (1,1,1,4,4,5,5)$ in addition to $\overline{y_1}$ and $\overline{y_2}$ constructed earlier. These vectors can also be found by switching a score value of 1 between adjacent entries in $\bar{x}$. The fact that they are score vectors and are covered by $\bar{x}$ can be seen by comparing their summation vectors $\Sigma_{\bar{x}} = (1,2,4,7,11,16,21)$, $\Sigma_{\bar{u}} = (1,2,4,6,11,16,21)$ and $\Sigma_{\bar{v}} = (1,2,3,7,11,16,21)$, similar to the proof of Theorem 14 and its corollary above.

It is tempting to generalize and say that such a switch in scores between two adjacent entries in the smallest strong score vector, $\bar{x}$, will result in either $\bar{x}$, or a new score vector covered by $\bar{x}$; but this is not always the case. Looking again at the smallest strong score vector $P = (1,1,2,3,4,4)$ when $n = 6$, such a switch between the second and third entries in the vector will result in $(1,2,1,3,4,4) = (1,1,2,3,4,4) = P$ or the vector $(1,0,3,3,4,4) = (0,1,3,3,4,4) = K$.

In the poset relation, $K \leq P$, but it is not covered by P as $K \leq L \leq P$. The score vector $F = (1,1,2,2,4,5)$ can similarly be found by switching between the fourth and fifth entries while in the poset $F \leq G \leq P$. It does, however, seem consistent that such a switch in score values will result in either the original score vector or one that is less than it in the poset structure.

Theorem 15. Switching a score value of 1 between x_k and x_{k+1} for $k = 1, \dots, n-1$ in the smallest strong score vector $\bar{x} = (1,1,2,3, \dots, n-3, n-2, n-2)$ will either result in $\bar{x}$ or another score vector $\bar{y}$ such that $\bar{y} \leq \bar{x}$ in $\mathcal{T}(n)$ for $n \geq 3$.

For $k = n-1$ or 1, the resulting vector will be $\overline{y_1} = (1,1,2,3, \dots, n-3, n-1)$ or $\overline{y_2} = (0,1,2,3, \dots, n-2, n-2)$, respectively. By Theorem 14 and its corollary, we know both are covered by $\bar{x}$. For $k = n-2$ or 2, if x_k increases in value by 1, the resulting vector will be $\bar{x}$. If x_k decreases in value by 1, the resulting vector will be $\overline{y_3} = (1,1,2,3, \dots, n-4, n-2, n-1)$ or $\overline{y_4} = (0,1,3,3, \dots, n-3, n-2, n-2)$, respectively. Comparing the summation vectors, we see $\Sigma_{\overline{y_3}} = \left(1,2,4,7, \dots, \binom{n-2}{2}, \binom{n-1}{2}, \binom{n}{2}\right)$ and $\Sigma_{\overline{y_4}} = \left(0,1,4,7, \dots, \binom{n-2}{2}+1, \binom{n-1}{2}+1, \binom{n}{2}\right)$ are both score vectors by definition, and are less than $\bar{x}$ in $\mathcal{T}(n)$. For $k = 3, \dots, n-3$, if x_k increases in value by 1, the resulting vector will be $\bar{x}$. If x_k decreases in value by 1, the resulting vector $\bar{y}$ has a summation vector $\Sigma_{\bar{y}}$ such that $\left(\Sigma_{\bar{y}}\right)_k = \binom{k}{2}$, while $\left(\Sigma_{\bar{y}}\right)_j = (\Sigma_{\bar{x}})_j$ for $j \neq k$. Thus, $\bar{y}$ is a score vector such that $\bar{y} \leq \bar{x}$ in the poset, and similar to $\overline{y_1}$ and $\overline{y_2}$, is covered by $\bar{x}$.

As a side note, this has to maintain the strictness of dealing with adjacent entries in the score vector x_k and x_{k+1} such that $k = 1, \dots, n-1$. For the smallest strong score vector $\bar{x} = (1,1,2,3, \dots, n-3, n-2, n-2)$, a switch in values between the first and last entries will result in $(2,1,2,3, \dots, n-2, n-3) = (1,2,2,3, \dots, n-4, n-3, n-3, n-2) = \bar{z}$ or the transitive score vector $(0,1,2,3, \dots, n-2, n-1)$. While the transitive score vector is the minimum element in the poset, $\bar{z}$ is a strong score vector and is greater than $\bar{x}$ in the poset. When $n = 6$, this score vector is $S = (1,2,2,3,3,4)$.

The symmetry of these posets, and the great deal of structure inherent in the score vectors will undoubtedly lead to several other generalizations of $\mathcal{T}(n)$. We will end this chapter with a conjecture, based on what is visibly happening in $\mathcal{T}(n)$ for $n = 5$ and 6.

Conjecture 1. The subposet, $\big(Y\ P(Y)\big)$, where Y is the set of all strong score vectors, is a lattice.

CHAPTER 6

$\mathcal{T}(n)$ AND LANDAU'S HIERARCHY NUMBER

We will now turn our attention to the connection between Landau's hierarchy number and the poset $\mathcal{T}(n)$ for any n. Rather than comparing two score vectors $\bar{x} = (x_1, \dots, x_n)$ and $\bar{y} = (y_1, \dots, y_n)$ using Landau's hierarchy number exactly, we will compare them using the algebraically equivalent method derived in Chapter 4, namely their respective inner products $\sum_{i=1}^{n} x_i{}^2$ and $\sum_{i=1}^{n} y_i{}^2$. Visually, the structure of $\mathcal{T}(n)$ has a remarkable correlation to the score vector's inner products which we can see below for $n = 3,4,5$, and 6.

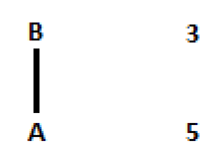


Figure 13. $\mathcal{T}(3)$ with its corresponding inner products

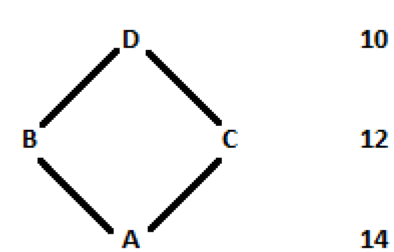


Figure 14. $\mathcal{T}(4)$ with its corresponding inner products

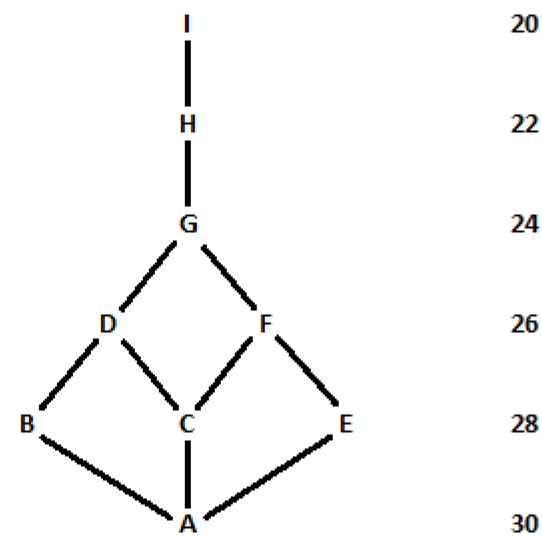


Figure 15. $\mathcal{T}(5)$ with its corresponding inner products

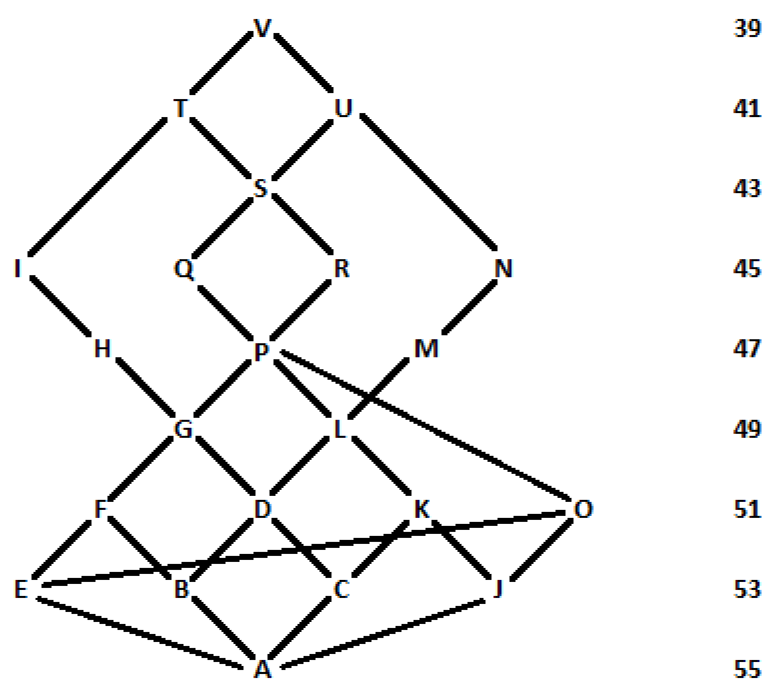


Figure 16. $\mathcal{T}(6)$ with its corresponding inner products

The inner products are certainly smallest for the (nearly) regular score vectors and largest for the transitive, corresponding to the 0 to 1 scale of the hierarchy number. For $n = 3,4,5$, and 6, we see that the score vectors in each of the horizontal antichains can be represented by the same inner product value. Is this always the case?

Theorem 16. Given that $\bar{x}$ and $\bar{y}$ are two distinct score vectors in the poset $\mathcal{T}(n)$ for $n \geq 3$, if $\bar{x} \leq \bar{y}$ in $\mathcal{T}(n)$, then $\sum_{i=1}^{n} x_i{}^2 > \sum_{i=1}^{n} y_i{}^2$.

To begin the proof, consider two score vectors for tournaments on n vertices $\bar{x}$ and $\bar{y}$ such that $\bar{x} \neq \bar{y}$ and $\bar{x} \leq \bar{y}$ in the poset $\mathcal{T}(n)$. Recall $\bar{x} \leq \bar{y}$ in $\mathcal{T}(n)$ if $\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i$ for all values of $k = 1, \dots, n$ with equality for the summations when $k = n$. Since $\bar{x} \neq \bar{y}$, we know there must exist at least one value k^* in $\{1, \dots, n-1\}$ such that there is strict inequality or $\sum_{i=1}^{k^*} x_i < \sum_{i=1}^{k^*} y_i$. Let k^* be the *first* value and l^* be the *last* value in $\{1, \dots, n-1\}$ where this strict inequality occurs. As $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i = \binom{n}{2}$ by virtue of $\bar{x}$ and $\bar{y}$ being score vectors, $l^* \neq n$. Let's compare the remaining values in the two summation vectors $\Sigma_{\bar{x}}$ and $\Sigma_{\bar{y}}$.

$$\Sigma_{\bar{x}} = \left(x_1, \sum_{i=1}^{2} x_i, \dots, \sum_{i=1}^{k^*-1} x_i, \sum_{i=1}^{k^*} x_i, \dots, \sum_{i=1}^{l^*} x_i, \sum_{i=1}^{l^*+1} x_i, \dots, \sum_{i=1}^{n} x_i \right)$$

$$\Sigma_{\bar{y}} = \left(y_1, \sum_{i=1}^{2} y_i, \dots, \sum_{i=1}^{k^*-1} y_i, \sum_{i=1}^{k^*} y_i, \dots, \sum_{i=1}^{l^*} y_i, \sum_{i=1}^{l^*+1} y_i, \dots, \sum_{i=1}^{n} y_i \right)$$

As $k = k^*$ and $k = l^*$ are defined as the first and last terms in the summation vectors with strict inequality, this implies equality in the summation vectors for$k = 1, \dots, k^* - 1$ and $k = l^* + 1, \dots, n$. Equalities for specific terms in the score vectors $\bar{x}$ and $\bar{y}$ will result as well. Let's determine the values of i for which $x_i = y_i$. Starting at $i = 1$ (assuming $k^* > 1$), since $x_1 = y_1$ and $x_1 + x_2 = y_1 + y_2$, we know $x_2 = y_2$. This relation continues up until we reach $i = k^* - 1$. Now starting at $i = n$ (assuming $l^* + 1 < n$), since $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$ and

$\sum_{i=1}^{n-1} x_i = \sum_{i=1}^{n-1} y_i$, the resulting equation $\sum_{i=1}^{n} x_i - \sum_{i=1}^{n-1} x_i = \sum_{i=1}^{n} y_i - \sum_{i=1}^{n-1} y_i$ implies that $x_n = y_n$. This relation continues from $i = n$ down until we reach $i = l^* + 2$. Therefore, $x_i = y_i$ for $i = 1, \dots, k^* - 1$ and for $i = (l^* + 2), \dots, n$.

Certain inequalities in the score vectors are also apparent. Since $\sum_{i=1}^{k^*} x_i < \sum_{i=1}^{k^*} y_i$ with equality for terms prior to $k = k^*$ in the summation vectors, $i = k^*$ is also the first term in the score vectors such that $x_i < y_i$. The last inequality between the score vectors can also be determined. As $\sum_{i=1}^{l^*} x_i < \sum_{i=1}^{l^*} y_i$ with equality for all terms after $k = l^*$ in the summation vectors, we know that $\sum_{i=1}^{l^*+1} x_i - \sum_{i=1}^{l^*} x_i > \sum_{i=1}^{l^*+1} y_i - \sum_{i=1}^{l^*} y_i \Rightarrow x_{l^*+1} > y_{l^*+1}$. This coupled with the fact that the terms in the score vectors are listed in non-decreasing order, we can conclude that:

$$x_{k^*} < y_{k^*} \leq y_{l^*+1} < x_{l^*+1} \tag{1}$$

Finally, $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$ and $x_i = y_i$ for $i = 1, \dots, k^*$ and $i = l^* + 2, \dots, n$ implies:

$$\sum_{i=k^*}^{l^*+1} x_i = \sum_{i=k^*}^{l^*+1} y_i \tag{2}$$

Now, let's compare the score vectors' respective inner products, $\sum_{i=1}^{n} {x_i}^2$ and $\sum_{i=1}^{n} {y_i}^2$. Since $x_k = y_k \Rightarrow {x_k}^2 = {y_k}^2$ for $k = 1, \dots, k^* - 1$ and $k = l^* + 2, \dots, n$, if the expressions $\sum_{i=1}^{n} {x_i}^2$ and $\sum_{i=1}^{n} {y_i}^2$ have any possibility of *not* being equivalent, it will result from values of i for which $i = k^*, \dots, l^* + 1$. In comparing the summations $\sum_{i=k^*}^{l^*+1} {x_i}^2$ and $\sum_{i=k^*}^{l^*+1} {y_i}^2$, we will consider two cases: (1) $k^* = l^*$, that is there is only one term in the summation vectors where strict inequality exists, and (2) $k^* < l^*$.

Case (1): $k^* = l^*$

$$x_{k^*} + x_{k^*+1} = y_{k^*} + y_{k^*+1}$$
$$\Rightarrow (x_{k^*} + x_{k^*+1})^2 = (y_{k^*} + y_{k^*+1})^2$$

$$\Rightarrow {x_{k^*}}^2 + 2x_{k^*}x_{k^*+1} + {x_{k^*+1}}^2 = {y_{k^*}}^2 + 2y_{k^*}y_{k^*+1} + {y_{k^*+1}}^2$$

In comparing the two products $x_{k^*}x_{k^*+1}$ and $y_{k^*}y_{k^*+1}$, if it can be shown that $x_{k^*}x_{k^*+1} < y_{k^*}y_{k^*+1}$, then we can conclude ${x_{k^*}}^2 + {x_{k^*+1}}^2 > {y_{k^*}}^2 + {y_{k^*+1}}^2$. From equation (1) above, we know that $y_{k^*} - x_{k^*} = x_{k^*+1} - y_{k^*+1} = a > 0$. This leads to rewriting y_{k^*} as $x_{k^*} + a$ and x_{k^*+1} as $y_{k^*+1} + a$. Let's compare the products in question for (a) $y_{k^*} = y_{k^*+1}$ and for (b) $y_{k^*} < y_{k^*+1}$. That is, when there exists some $b > 0$ such that $y_{k^*+1} = y_{k^*} + b$.

(a) $y_{k^*} = y_{k^*+1}$

$$x_{k^*}x_{k^*+1} = x_{k^*}(x_{k^*} + 2a) = {x_{k^*}}^2 + 2ax_{k^*}$$

$$y_{k^*}y_{k^*+1} = (x_{k^*} + a)(x_{k^*} + a) = {x_{k^*}}^2 + 2ax_{k^*} + a^2$$

Since $a > 0$, we now know that $x_{k^*}x_{k^*+1} < y_{k^*}y_{k^*+1}$. This inequality between the two products implies that ${x_{k^*}}^2 + {x_{k^*+1}}^2 > {y_{k^*}}^2 + {y_{k^*+1}}^2$. Therefore $\sum_{i=1}^{n} {x_i}^2 > \sum_{i=1}^{n} {y_i}^2$.

(b) $y_{k^*} < y_{k^*+1}$. So there exists some $b > 0$ such that $y_{k^*+1} = y_{k^*} + b$.

$$x_{k^*}x_{k^*+1} = x_{k^*}\big(x_{k^*} + (2a + b)\big) = {x_{k^*}}^2 + (2a + b)x_{k^*}$$

$$y_{k^*}y_{k^*+1} = (x_{k^*} + a)\big(x_{k^*} + (a + b)\big) = {x_{k^*}}^2 + (2a + b)x_{k^*} + (a^2 + ab)$$

Again, since a and b are both greater than zero, $x_{k^*}x_{k^*+1} < y_{k^*}y_{k^*+1}$. This inequality implies that ${x_{k^*}}^2 + {x_{k^*+1}}^2 > {y_{k^*}}^2 + {y_{k^*+1}}^2$. Therefore, $\sum_{i=1}^{n} {x_i}^2 > \sum_{i=1}^{n} {y_i}^2$.

Case (2): $k^* < l^*$

This case will also compare terms in the expansion of $\left(\sum_{i=k^*}^{l^*+1} x_i\right)^2 = \left(\sum_{i=k^*}^{l^*+1} y_i\right)^2$. All values which are *not* squares of individual terms in $\bar{x}$ or $\bar{y}$ will again have a strict inequality, leading to the expected relation in the hierarchy numbers.

$$\left(\sum_{i=k^*}^{l^*+1} x_i\right)^2 = \sum_{i=k^*}^{l^*+1} {x_i}^2 + 2\left(\sum_{p,q=k^*}^{l^*+1} x_p x_q\right)$$

$$\left(\sum_{i=k^*}^{l^*+1} y_i\right)^2 = \sum_{i=k^*}^{l^*+1} {y_i}^2 + 2\left(\sum_{p,q=k^*}^{l^*+1} y_p y_q\right)$$

Consider the summation of all the pair-wise products of the terms in the score vectors from $k = k^*$ up to $k = l^* + 1$. Equation (1) still holds true; however, we do not know the relation between x_k and y_k for any given $k = k^* + 1, \dots, l^*$. Rather than considering all possible cases of inequalities, let us consider the case which *maximizes* the values in $\sum_{p,q=k^*}^{l^*+1} x_p x_q$ and *minimizes* the values in $\sum_{p,q=k^*}^{l^*+1} y_p y_q$. If we attain the result of $\sum_{p,q=k^*}^{l^*+1} x_p x_q < \sum_{p,q=k^*}^{l^*+1} y_p y_q$ under these conditions, then any possible case of inequalities would attain such a result as well.

In the first part of case (1), we have already seen $\sum_{p,q=k^*}^{l^*+1} y_p y_q$ being minimized when $k^* = l^*$. That is, $y_{k^*} y_{k^*+1}$ attained its smallest possible value when y_{k^*} equaled y_{k^*+1}.

Similarly, $\sum_{p,q=k^*}^{l^*+1} y_p y_q$ will be minimized when:

$$y_{k^*} = y_{k^*+1} = y_{k^*+2} = \cdots = y_{l^*} = y_{l^*+1} \tag{3}$$

The summation $\sum_{p,q=k^*}^{l^*+1} x_p x_q$ will be maximized when:

$$x_{k^*} < x_{k^*+1} = x_{k^*+2} = \cdots = x_{l^*} = x_{l^*+1} \tag{4}$$

In this extreme setting, since $y_{l^*+1} < x_{l^*+1}$, it follows that y_i must be strictly less than x_i for all $i = k^* + 1, \dots, l^*$. What is the actual difference between any two such terms? Assuming again that $y_{k^*} - x_{k^*} = a > 0$ and that $x_{l^*+1} - y_{l^*+1} = b > 0$, we can rewrite the necessary equivalence of $y_{k^*} - x_{k^*} = \sum_{i=k^*+1}^{l^*+1} (x_i - y_i)$ as $a = (l^* - k^*)b$. Thus $b = \frac{a}{(l^* - k^*)}$.

Note that this relation may not necessarily exist between any two given score vectors; b must be an integer, and $\bar{x}$ and $\bar{y}$ will still have to pass all criteria for being score vectors. However, considering this extreme case where $\sum_{p,q=k^*}^{l^*+1} x_p x_q$ is as large as possible and $\sum_{p,q=k^*}^{l^*+1} y_p y_q$ is as small as possible, we will be able to make a generalization for *all* score vectors, so long as $\sum_{p,q=k^*}^{l^*+1} x_p x_q < \sum_{p,q=k^*}^{l^*+1} y_p y_q$.

Consider the following sum of all pairwise multiples resulting from $\left(\sum_{i=k^*}^{l^*+1} x_i\right)^2$.

$$\sum_{p,q=k^*}^{l^*+1} x_p x_q = (l^* - k^*)\left[x_{k^*}\left(x_{k^*} + (a+b)\right)\right] + \binom{l^* - k^*}{2}\left[\left(x_{k^*} + (a+b)\right)^2\right]$$

Simplifying this value will be much easier after making a few substitutions. Let $l^* - k^* = c > 0$. This implies that $a + b = a + \frac{a}{(l^* - k^*)} = \frac{a(c+1)}{c}$. Allowing for these substitutions, we can rewrite the above as:

$$c\left[x_{k^*}\left(x_{k^*} + \frac{a(c+1)}{c}\right)\right] + \binom{c}{2}\left[\left(x_{k^*} + \frac{a(c+1)}{c}\right)^2\right]$$

$$= c\left[{x_{k^*}}^2 + \frac{a(c+1)}{c}x_{k^*}\right] + \left(\frac{c(c-1)}{2}\right)\left[{x_{k^*}}^2 + \frac{2a(c+1)}{c}x_{k^*} + \left(\frac{a(c+1)}{c}\right)^2\right]$$

$$= \left(c + \frac{c(c-1)}{2}\right){x_{k^*}}^2 + \left(a(c+1) + a(c^2-1)\right)x_{k^*} + \left(\frac{a^2(c+1)^2(c-1)}{2c}\right)$$

$$= \left(\frac{c(c+1)}{2}\right){x_{k^*}}^2 + \left(ac(c+1)\right)x_{k^*} + \left(\frac{a^2(c+1)^2(c-1)}{2c}\right)$$

Now, consider the sum of all pairwise multiples resulting from $\left(\sum_{i=k^*}^{l^*+1} y_i\right)^2$.

$$\sum_{p,q=k^*}^{l^*+1} y_p y_q = \binom{l^* - k^* + 1}{2}\left[(x_{k^*} + a)^2\right]$$

Using the same substitution for $l^* - k^*$, this simplifies to:

$$\binom{c+1}{2}[(x_{k^*}+a)^2]$$

$$= \left(\frac{(c+1)(c)}{2}\right)[x_{k^*}{}^2 + 2ax_{k^*} + a^2]$$

$$= \left(\frac{c(c+1)}{2}\right)x_{k^*}{}^2 + \big(ac(c+1)\big)x_{k^*} + \left(\frac{a^2c(c+1)}{2}\right)$$

Note that the coefficients for $x_{k^*}{}^2$ and x_{k^*} are equivalent in both products. If $\sum_{p,q=k^*}^{l^*+1} x_p x_q$ is strictly less than $\sum_{p,q=k^*}^{l^*+1} y_p y_q$, the proof of it will be found in comparing the two constant terms $\frac{a^2(c+1)^2(c-1)}{2c}$ and $\frac{a^2c(c+1)}{2}$, or more specifically, $\frac{(c+1)^2(c-1)}{c}$ and $c(c+1)$.

$$\frac{(c+1)^2(c-1)}{c} = \frac{c^3+c^2-c-1}{c} = c^2+c-1-\frac{1}{c} < c^2+c = c(c+1)$$

Therefore, for any score vectors where $x_{k^*} < y_{k^*} \leq y_{l^*+1} < x_{l^*+1}$ and $k^* < l^*$ we can state the following:

$$\sum_{p,q=k^*}^{l^*+1} x_p x_q < \sum_{p,q=k^*}^{l^*+1} y_p y_q$$

$$\Rightarrow \sum_{i=k^*}^{l^*+1} x_i{}^2 > \sum_{i=k^*}^{l^*+1} y_i{}^2$$

$$\Rightarrow \sum_{i=1}^{n} x_i{}^2 > \sum_{i=1}^{n} y_i{}^2$$

By the corollary to Theorem 12, either case will result in $h_{\bar{x}} > h_{\bar{y}}$.

We now have a connection between the poset relation $\mathcal{T}(n)$ and Landau's hierarchy number. If a score vector $\bar{x}$ is 'closer' to the transitive score vector than $\bar{y}$ is in the poset relation, then that closeness is also represented in their respective hierarchy numbers.

Is the converse true? Interestingly, it is not. Given $\bar{x}$ and $\bar{y}$ such that $h_{\bar{x}} > h_{\bar{y}}$, it does not necessarily imply that $\bar{x} \leq \bar{y}$ in the poset relation. For example, consider $E = (0,1,3,3,3)$ and $D = (1,1,2,2,4)$, score vectors on $n = 5$ vertices. Looking at their respective inner products we see that $\sum_{i=1}^{5} e_i{}^2 = 28 > 26 = \sum_{i=1}^{5} d_i{}^2$. Clearly $h_E > h_D$, but $E \parallel D$ in the poset structure. For this example, E *does* reside in a lower antichain than D, so in some sense it is 'closer' to the transitive score vector in the poset relation; but that is not reflected in a comparison with D. Although the converse is not necessarily true, we can conclude that if two vectors have the same hierarchy number, $h_{\bar{x}} = h_{\bar{y}}$, then they must be incomparable, $\bar{x} \parallel \bar{y}$.

We will conclude this chapter with a final observation and conjecture regarding the inner products of the score vectors and their placement in the poset $\mathcal{T}(n)$ for $n = 3, \dots, 6$. Looking back at the figures, in addition to the score vectors residing in a horizontal antichain having the same inner products, the values of those inner products have a constant difference of 2 between adjacent antichains.

Conjecture 2. There is a difference of 2 between the inner products of score vectors in two consecutive horizontal antichains in the poset $\mathcal{T}(n)$, for $n \geq 3$.

Conjecture 3. Letting $\bar{x}$ be the transitive score vector and $\bar{y}$ being the (nearly) regular score vector, the height of $\mathcal{T}(n)$ can be found by the formula $\frac{\sum_{i=1}^{n} x_i{}^2 - \sum_{i=1}^{n} y_i{}^2}{2} + 1$ for $n \geq 3$.

CHAPTER 7

FURTHER TOPICS ON THE STRUCTURE OF $\mathcal{T}(n)$

Recall that once a poset relation $\mathcal{P} = (X, P)$ is established extensions of the poset can be explored. Again, an extension of $\mathcal{P}$ is a poset $\mathcal{Q} = (X, Q)$ on the same elements X where $P \subseteq Q$. A larger poset structure can then be studied, consisting of all the extensions of $\mathcal{P}$ with the poset relation of set inclusion on the extensions. The minimum element of this poset is $\mathcal{P}$ and the maximal elements are the famous linear extensions of the poset $\mathcal{E}(P)$. Let's call this larger poset structure the *extension poset* of $\mathcal{P}$. Consider now the extension poset of $\mathcal{T}(n) = (X, P)$, for which it is the minimum element. We will explore a specific class of extensions of $\mathcal{T}(n)$, and a possible route of constructing linear extensions of $\mathcal{T}(n)$.

Let X be the set of all score vectors on n vertices. Given that $\bar{x} = (x_1, \dots, x_n)$ and $\bar{y} = (y_1, \dots, y_n)$ are two elements of X, let $\bar{x} \leq \bar{y}$ if $\sum_{i=1}^{n} x_i{}^m > \sum_{i=1}^{n} y_i{}^m$ for $m \geq 1$. Let $H^m = \{(\bar{x}, \bar{y}) \in X \times X \; s.t. \sum_{i=1}^{n} x_i{}^m > \sum_{i=1}^{n} y_i{}^m\}$. Therefore, $\mathcal{H}^m = (X, H^m)$ is a poset with an irreflexive ($\bar{x} \nleq \bar{x}$), anti-symmetric ($\bar{x} \leq \bar{y} \Rightarrow \bar{y} \nleq \bar{x}$), and transitive ($\bar{x} \leq \bar{y}, \bar{y} \leq \bar{z} \Rightarrow \bar{x} \leq \bar{z}$) binary relation. When $m = 1$, all of the elements of X are pair-wise incomparable since $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$. Hence, $\mathcal{H}^1$ is an antichain. $\mathcal{H}^2$ results in a poset reflecting Landau's hierarchy relation since $\sum_{i=1}^{n} x_i{}^2 > \sum_{i=1}^{n} y_i{}^2 \Leftrightarrow h_{\bar{x}} > h_{\bar{y}}$ by the Corollary to Theorem 12.

Theorem 17. $\mathcal{H}^2 = (X, H^2)$ is an extension of $\mathcal{T}(n) = (X, P)$.

This is a direct result of Theorem 16 in Chapter 6. Given that $\bar{x}$ and $\bar{y}$ are two score vectors on n vertices, if $\bar{x} \leq \bar{y}$ in P, then $\bar{x} \leq \bar{y}$ in H^2. In the case where $n = 3$ and 4, we can visually see from Figures 13 and 14 that $\mathcal{H}^2$ is an extension such that $\mathcal{T}(n) = \mathcal{H}^2$. Whereas, $\mathcal{H}^2$ represents distinct elements in the extension posets for $n = 5$ and 6. Below is a visual representation of $\mathcal{H}^2$, corresponding to the poset $\mathcal{T}(5)$.

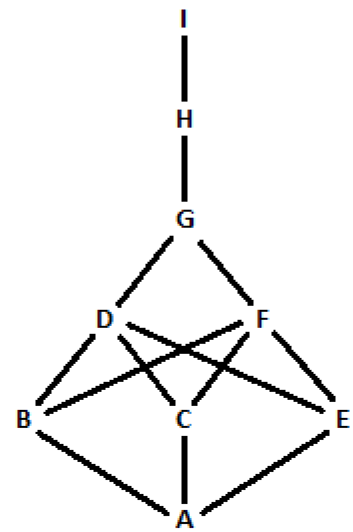


Figure 17. The poset extension $\mathcal{H}^2$ for $n = 5$

Now that one extension has been identified, an obvious question is whether or not other extensions of $\mathcal{T}(n)$ can be found similarly, specifically its linear extensions. Finding linear extensions is the first step in determining the dimension of a poset. In general, a poset's dimension can be very difficult to find, especially for large values of n. If linear extensions can be found using a fairly small amount of computation, the total amount of work necessary to find the dimension can be greatly reduced. The extension $\mathcal{H}^2$ was found by summing the squares of the entries in the score vectors. Do the posets $\mathcal{H}^m$, constructed by summing higher powers of the score vector's entries, result in extensions for values of m greater than 2? For $n = 3, \ldots, 6$ and $m = 3, \ldots, 12$, the answer is yes!

Using the same labeling of the score vectors as in Chapter 5 for $n = 3$ through 6, tables are listed below containing the score vectors' respective values of $\sum_{i=1}^{n} x_i{}^m$ for $m = 1, \ldots, 12$. It is left to the reader to verify that if $\bar{x} \leq \bar{y}$ in $\mathcal{T}(n)$, then $\bar{x} \leq \bar{y}$ in $\mathcal{H}^m$ for $n = 3, \ldots, 6$ and $m = 1, \ldots, 12$.

vectors	x^1	x^2	x^3	x^4	x^5	x^6
A	3	5	9	17	33	65
B	3	3	3	3	3	3

contd.	x^7	x^8	x^9	x^10	x^11	x^12
A	129	257	513	1025	2049	4097
B	3	3	3	3	3	3

Table 2. Score vectors for $n = 3$ and values of $\sum_{i=1}^{n} x_i{}^m$ for $m = 1, \ldots, 12$

vectors	x^1	x^2	x^3	x^4	x^5	x^6
A	6	14	36	106	276	794
B	6	12	30	97	246	732
C	6	12	24	56	96	192
D	6	10	18	42	66	130

contd.	x^7	x^8	x^9	x^10	x^11	x^12
A	2316	6818	20196	60074	179196	535538
B	2190	6564	19686	59052	177150	531444
C	384	768	1536	3072	6144	12288
D	258	514	1026	2050	4098	8194

Table 3. Score vectors for $n = 4$ and values of $\sum_{i=1}^{n} x_i{}^m$ for $m = 1, \ldots, 12$

For $n = 3$, it can be seen from the values above that $\mathcal{T}(3) = \mathcal{H}^2 = \cdots = \mathcal{H}^{12}$, the only element in the extension poset of $\mathcal{T}(3)$. Hence, $L_1 = A, B$ is its only linear extension. Based on the score vectors, it is not too difficult to conclude that $\mathcal{T}(3) = \mathcal{H}^m$ for $m \geq 2$.

For $n = 4$, a linearization occurs when $m = 3$, and it can be seen that $\mathcal{H}^m = \mathcal{H}^3$ for $m = 3, \ldots, 12$. This linear extension can be written as $L_1 = A, B, C, D$.

vectors	x^1	x^2	x^3	x^4	x^5	x^6
A	10	30	100	354	1300	4890
B	10	28	94	340	1270	4828
C	10	28	88	304	1120	4288
D	10	26	82	290	1090	4226
E	10	28	82	244	730	2188
F	10	26	70	194	550	1586
G	10	24	64	180	520	1524
H	10	22	52	130	340	922
I	10	20	40	80	160	320

contd.	x^7	x^8	x^9	x^10	x^11	x^12
A	18700	72354	282340	1108650	4373500	17312754
B	18574	72100	281830	1107628	4371454	17308660
C	16768	66304	263680	1051648	4200448	16789504
D	16642	66050	263170	1050626	4198402	16785410
E	6562	19684	59050	177148	531442	1594324
F	4630	13634	40390	120146	358390	1071074
G	4504	13380	39880	119124	356344	1066980
H	2572	7330	21220	62122	183292	543730
I	640	1280	2560	5120	10240	20480

Table 4. Score vectors for $n = 5$ and values of $\sum_{i=1}^{n} x_i{}^m$ for $m = 1, \ldots, 12$

In the case of $n = 5$, a linear extension occurs when $m = 4$. It can be written as $L_1 = A, B, C, D, E, F, G, H, I$. Again, all extensions after this one result in the same linear extension. That is, $\mathcal{H}^m = \mathcal{H}^4$ for $m = 4, \ldots, 12$.

For $n = 6$, there are actually *two* linear extensions that appear. The first one occurs when $m = 4$. This linearization is used to list the score vectors in the table below. The second one occurs in $\mathcal{H}^5$. They are $L_1 = A, E, B, F, C, D, G, J, O, H, I, K, L, P, Q, M, R, S, N, T, U, V$ and $L_2 = A, E, B, F, C, D, G, H, I, J, O, K, L, P, Q, M, R, S, T, N, U, V$, respectively. This second linearization also repeats for the remaining values of m. That is $\mathcal{H}^m = \mathcal{H}^5$ for $m = 5, \ldots, 12$.

vectors	x^1	x^2	x^3	x^4	x^5	x^6
A	15	55	225	979	4425	20515
E	15	53	219	965	4395	20453
B	15	53	213	929	4245	19913
F	15	51	207	915	4215	19851
C	15	53	207	869	3855	17813
D	15	51	195	819	3675	17211
G	15	49	189	805	3645	17149
J	15	53	201	785	3105	12353
O	15	51	195	771	3075	12291
H	15	47	177	755	**3465**	**16547**
I	15	45	165	705	**3285**	**15945**
K	15	51	183	675	2535	9651
L	15	49	171	625	2355	9049
P	15	47	165	611	2325	8987
Q	15	45	153	561	2145	8385
M	15	47	153	515	1785	6347
R	15	45	147	501	1755	6285
S	15	43	135	451	1575	5683
N	15	45	135	405	1215	3645
T	15	41	123	401	**1395**	**5081**
U	15	41	117	341	1005	2981
V	15	39	105	291	825	2379

contd.	x^7	x^8	x^9	x^10	x^11	x^12
A	96825	462979	2235465	10874275	53201625	261453379
E	96699	462725	2234955	10873253	53199579	261449285
B	94893	456929	2216805	10817273	53028573	260930129
F	94767	456675	2216295	10816251	53026527	260926035
C	84687	410309	2012175	9942773	49359567	245734949
D	82755	404259	1993515	9885771	49186515	245211699
G	82629	404005	1993005	9884749	49184469	245207605
J	49281	196865	786945	3146753	12584961	50335745
O	49155	196611	786435	3145731	12582915	50331651
H	**80697**	**397955**	**1974345**	**9827747**	**49011417**	**244684355**
I	**78765**	**391905**	**1955685**	**9770745**	**48838365**	**244161105**
K	37143	144195	563655	2215251	8742903	34617315
L	35211	138145	544995	2158249	8569851	34094065
P	35085	137891	544485	2157227	8567805	34089971
Q	33153	131841	525825	2100225	8394753	33566721
M	23073	85475	321705	1226747	4727793	18375635
R	22947	85221	321195	1225725	4725747	18371541

S	21015	79171	302535	1168723	4552695	17848291
N	10935	32805	98415	295245	885735	2657205
T	**19083**	**73121**	**283875**	**1111721**	**4379643**	**17325041**
U	8877	26501	79245	237221	710637	2129861
V	6945	20451	60585	180219	537585	1606611

Table 5. Score vectors for $n = 6$ and values of $\sum_{i=1}^{n} {x_i}^m$ for $m = 1, \ldots, 12$

The values bolded for $m = 5, \ldots, 12$ are listed out of order. Visually, there is a pattern in the placement of this final linearization in $\mathcal{T}(n)$ for $n = 3, \ldots, 6$. Starting with the transitive score vector, the linear ordering 'climbs up' diagonal chains. See the diagram for $\mathcal{T}(6)$ below.

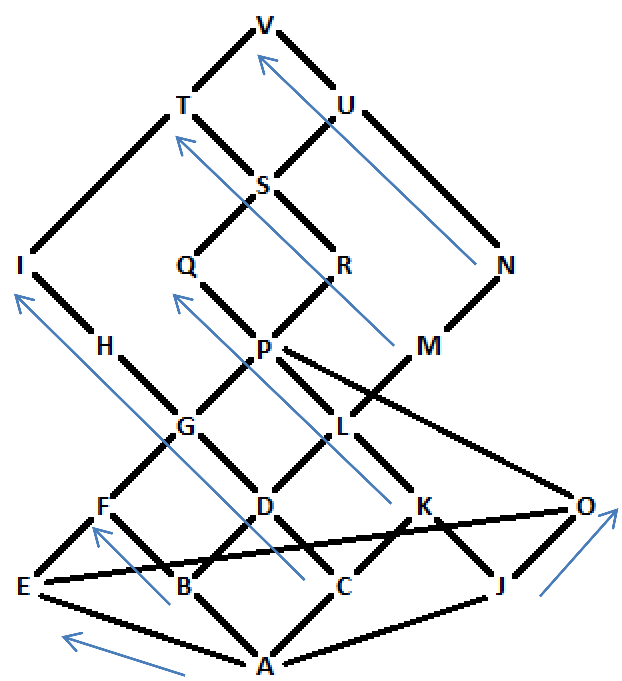


Figure 18. The linearization $L_2 = A, E, B, F, C, D, G, H, I, J, O, K, L, P, Q, M, R, S, T, N, U, V$ represented in the poset $\mathcal{T}(6)$

Numerically, there also seems to be a pattern in the ordering of the score vectors themselves. Consider the first few score vectors for $n = 6$ in the linear order given above.

$A = (0,1,2,3,4,5)$

$E = (1,1,1,3,4,5)$

$B = (0,2,2,2,4,5)$

$F = (1,1,2,2,4,5)$

$C = (0,1,3,3,3,5)$

$D = (0,2,2,3,3,5)$

$G = (1,1,2,3,3,5)$

Starting with the transitive score vector, the scores begin to be 'filtered down' the vector, beginning with the most accessible scores. The third entry in A, 2, can have one of its scores given to a lower valued entry while still maintaining a score vector. Sharing a value of 1 with the first entry, the next score vector is attained, $E = (1,1,1,3,4,5)$. As that is the most 'filtering down' that can be done from the third entry, we will return to the transitive score vector and reduce the score value of the fourth entry, 3, resulting in $B = (0,2,2,2,4,5)$ and $F = (1,1,2,2,4,5)$, consecutively. This pattern continues on until all the scores are as equally shared as possible. Note, this ordering places all the score vectors with sources first, then all the score vectors with a maximal score of $n - 1$ next, etc.

Using this ordering on the score vectors for $n = 7$, let's determine if the sums of powers of their entries similarly result in a linearization. Considering the values in the table below, we are not yet determining if $\mathcal{H}^m$ for $m = 3, \ldots, 10$ result in extensions, but just if a linearization of the elements is attained; and if so, for what values of m.

Vectors	n=7							x^1	x^2	x^3
A=	0	1	2	3	4	5	6	21	91	441
B=	1	1	1	3	4	5	6	21	89	435
C=	0	2	2	2	4	5	6	21	89	429
D=	1	1	2	2	4	5	6	21	87	423
E=	0	1	3	3	3	5	6	21	89	423
F=	0	2	2	3	3	5	6	21	87	411
G=	1	1	2	3	3	5	6	21	85	405
H=	1	2	2	2	3	5	6	21	83	393
I=	2	2	2	2	2	5	6	21	81	381
J=	0	1	2	4	4	4	6	21	89	417
K=	1	1	1	4	4	4	6	21	87	411
L=	0	1	3	3	4	4	6	21	87	399
M=	0	2	2	3	4	4	6	21	85	387
N=	1	1	2	3	4	4	6	21	83	381
O=	1	2	2	2	4	4	6	21	81	369
P=	0	2	3	3	3	4	6	21	83	369
Q=	1	1	3	3	3	4	6	21	81	363
R=	1	2	2	3	3	4	6	21	79	351
S=	2	2	2	2	3	4	6	21	77	339
T=	0	3	3	3	3	3	6	21	81	351
U=	1	2	3	3	3	3	6	21	77	333
V=	2	2	2	3	3	3	6	21	75	321
W=	0	1	2	3	5	5	5	21	89	411
X=	1	1	1	3	5	5	5	21	87	405
Y=	0	2	2	2	5	5	5	21	87	399
Z=	1	1	2	2	5	5	5	21	85	393
AA=	0	1	2	4	4	5	5	21	87	387
BB=	1	1	1	4	4	5	5	21	85	381
CC=	0	1	3	3	4	5	5	21	85	369
DD=	0	2	2	3	4	5	5	21	83	357
EE=	1	1	2	3	4	5	5	21	81	351
FF=	1	2	2	2	4	5	5	21	79	339
GG=	0	2	3	3	3	5	5	21	81	339
HH=	1	1	3	3	3	5	5	21	79	333
II=	1	2	2	3	3	5	5	21	77	321
JJ=	2	2	2	2	3	5	5	21	75	309
KK=	0	1	3	4	4	4	5	21	83	345
LL=	0	2	2	4	4	4	5	21	81	333
MM=	1	1	2	4	4	4	5	21	79	327
NN=	0	2	3	3	4	4	5	21	79	315
OO=	1	1	3	3	4	4	5	21	77	309
PP=	1	2	2	3	4	4	5	21	75	297

QQ=	2	2	2	2	4	4	5	21	73	285
RR=	0	3	3	3	3	4	5	21	77	297
SS=	1	2	3	3	3	4	5	21	73	279
TT=	2	2	2	3	3	4	5	21	71	267
UU=	1	3	3	3	3	3	5	21	71	261
VV=	2	2	3	3	3	3	5	21	69	249
WW=	0	1	4	4	4	4	4	21	81	321
XX=	0	2	3	4	4	4	4	21	77	291
YY=	1	1	3	4	4	4	4	21	75	285
ZZ=	1	2	2	4	4	4	4	21	73	273
AAA=	0	3	3	3	4	4	4	21	75	273
BBB=	1	2	3	3	4	4	4	21	71	255
CCC=	2	2	2	3	4	4	4	21	69	243
DDD=	1	3	3	3	3	4	4	21	69	237
EEE=	2	2	3	3	3	4	4	21	67	225
FFF=	2	3	3	3	3	3	4	21	65	207
GGG=	3	3	3	3	3	3	3	21	63	189

contd.	x^4	x^5	x^6	x^7	x^8	x^9	x^10
A	2275	12201	67171	376761	2142595	12313161	71340451
B	2261	12171	67109	376635	2142341	12312651	71339429
C	2225	12021	66569	374829	2136545	12294501	71283449
D	2211	11991	66507	374703	2136291	12293991	71282427
E	2165	11631	64469	364623	2089925	12089871	70408949
F	2115	11451	63867	362691	2083875	12071211	70351947
G	2101	11421	63805	362565	2083621	12070701	70350925
H	2051	11241	63203	360633	2077571	12052041	70293923
I	2001	11061	62601	358701	2071521	12033381	70236921
J	**2081**	10881	59009	329217	1876481	10864641	63612929
K	**2067**	10851	58947	329091	1876227	10864131	63611907
L	1971	10311	56307	317079	1823811	10641351	62681427
M	1921	10131	55705	315147	1817761	10622691	62624425
N	1907	10101	55643	315021	1817507	10622181	62623403
O	1857	9921	55041	313089	1811457	10603521	62566401
P	1811	9561	53003	303009	1765091	10399401	61692923
Q	1797	9531	52941	302883	1764837	10398891	61691901
R	1747	9351	52339	300951	1758787	10380231	61634899
S	1697	9171	51737	299019	1752737	10361571	61577897
T	**1701**	8991	50301	290871	1712421	10176111	60761421
U	1637	8781	49637	288813	1706117	10156941	60703397
V	1587	8601	49035	286881	1700067	10138281	60646395
W	**1973**	9651	47669	236691	1178693	5879571	29356949

X	**1959**	9621	*47607	236565	1178439	5879061	29355927
Y	**1923**	9471	*47067	234759	1172643	5860911	29299947
Z	**1909**	9441	47005	234633	1172389	5860401	29298925
AA	**1779**	8331	39507	189147	912579	4431051	21629427
BB	**1765**	8301	39445	189021	912325	4430541	21628405
CC	**1669**	7761	36805	177009	859909	4207761	20697925
DD	1619	7581	36203	175077	853859	4189101	20640923
EE	1605	7551	36141	174951	853605	4188591	20639901
FF	1555	7371	35539	173019	847555	4169931	20582899
GG	1509	7011	33501	162939	801189	3965811	19709421
HH	1495	6981	33439	162813	800935	3965301	19708399
II	1445	6801	32837	160881	794885	3946641	19651397
JJ	1395	6621	32235	158949	788835	3927981	19594395
KK	**1475**	6441	28643	129465	593795	2759241	12970403
LL	**1425**	6261	28041	127533	587745	2740581	12913401
MM	**1411**	6231	27979	127407	587491	2740071	12912379
NN	1315	5691	25339	115395	535075	2517291	11981899
OO	1301	5661	25277	115269	534821	2516781	11980877
PP	1251	5481	24675	113337	528771	2498121	11923875
QQ	1201	5301	24073	111405	522721	2479461	11866873
RR	**1205**	*5121	22637	103257	482405	2294001	11050397
SS	1141	*4911	21973	101199	476101	2274831	10992373
TT	1091	*4731	21371	99267	470051	2256171	10935371
UU	1031	*4341	19271	89061	423431	2051541	10060871
VV	981	*4161	18669	87129	417381	2032881	10003869
WW	**1281**	*5121	20481	81921	327681	1310721	5242881
XX	**1121**	*4371	17177	67851	268961	1068771	4254377
YY	**1107**	*4341	17115	67725	268707	1068261	4253355
ZZ	**1057**	*4161	16513	65793	262657	1049601	4196353
AAA	**1011**	3801	14475	55713	216291	845481	3322875
BBB	947	3591	13811	53655	209987	826311	3264851
CCC	897	3411	13209	51723	203937	807651	3207849
DDD	837	3021	11109	41517	157317	603021	2333349
EEE	787	2841	10507	39585	151267	584361	2276347
FFF	677	2271	7805	27447	98597	361071	1344845
GGG	567	1701	5103	15309	45927	137781	413343

Table 6. Score vectors for $n = 7$ and values of $\sum_{i=1}^{n} {x_i}^m$ for $m = 1, \ldots, 10$

There are repeats in values of $\sum_{i=1}^{n} x_i{}^m$ for $m = 1,2,3,5$, and 6, hence none of their corresponding posets represent a linearization of the elements in $\mathcal{T}(7)$. The repeats for $m = 5$ and 6 are starred for easy identification. It is not too surprising that for $m = 1$ through 3, $\mathcal{H}^m$ is not a linear ordering; but it is interesting that $\mathcal{H}^4$ is a linear ordering while $\mathcal{H}^5$ and $\mathcal{H}^6$ are not. The ordering in $\mathcal{H}^4$ is *not* the predicted ordering for the poset $\mathcal{T}(7)$. Their values which are listed out of order are bolded. However, the linear ordering manifest in $\mathcal{H}^7$ *is* the predicted linear ordering. In fact, this linearization continues to be seen in $\mathcal{H}^8$ up through $\mathcal{H}^{12}$, though not all the values are shown in the table above. Whether or not $\mathcal{H}^m$ results in actual extensions of the poset $\mathcal{T}(7)$, the pattern of a linearization being attained for some value of m and maintained thereafter remains consistent for $n = 3$ up through 7.

Conjecture 4. $\mathcal{H}^m = (X, H^m)$ will always result in extensions of $\mathcal{T}(n)$, for $m \geq 2$. In other words, if $\bar{x} \leq \bar{y}$ in $\mathcal{T}(n)$, then $\sum_{i=1}^{n} x_i{}^m > \sum_{i=1}^{n} y_i{}^m$ for $m \geq 2$.

The proof of this for $m = 2$ was detailed in Chapter 6. Essentially, given $\bar{x} \leq \bar{y}$ in $\mathcal{T}(n)$, the terms in the expansion of both sides of the equation $\left(\sum_{i=k^*}^{l^*+1} x_i\right)^2 = \left(\sum_{i=k^*}^{l^*+1} y_i\right)^2$were compared. All values which did *not* contribute to score vector's respective inner products had a strict inequality such that $2\sum_{p,q=k^*}^{l^*+1} x_p x_q < 2\sum_{p,q=k^*}^{l^*+1} y_p y_q$. This implied $\sum_{i=1}^{n} x_i{}^2 > \sum_{i=1}^{n} y_i{}^2$.

In proving this for values of $m > 2$, using the same technique will provide an interesting challenge. Considering the expansion of the necessary equation $\left(\sum_{i=k^*}^{l^*+1} x_i\right)^m = \left(\sum_{i=k^*}^{l^*+1} y_i\right)^m$, the number of values *not* contributing to the vector's respective inner products is much larger than those that are. A generalization of these values may exist so that it still can be proven that $\sum_{i=1}^{n} x_i{}^m > \sum_{i=1}^{n} y_i{}^m$.

Conjecture 5. There exists some $m \in \mathbb{N}$ such that $\mathcal{H}^m$ is a linear extension.

Conjecture 6. There exists some $m^* \in \mathbb{N}$ such that $\mathcal{H}^{m^*}$is a linear extension, and $\mathcal{H}^m = \mathcal{H}^{m^*}$ for all $m \geq m^*$.

Given two score vectors $\bar{x} = (x_1, \dots, x_n)$ and $\bar{y} = (y_1, \dots, y_n)$ where $\bar{x} \neq \bar{y}$, it seems logical that there must exist some value $m^* \in \mathbb{N}$ such $\sum_{i=1}^{n} x_i^{m^*}$ would be unequal to $\sum_{i=1}^{n} y_i^{m^*}$. If not, that would be a contradiction to the fact that $\bar{x} \neq \bar{y}$. Without loss of generality, assume $\sum_{i=1}^{n} x_i^{m^*} > \sum_{i=1}^{n} y_i^{m^*}$. That being the case, there must be at least one value $j \in \{1, \dots, n\}$ such that $x_j^{m^*} > y_j^{m^*}$. This clearly implies that $x_j \neq y_j \Rightarrow x_j^{m} \neq y_j^{m}$, and more specifically, $x_j^{m} > y_j^{m}$ for all values of m. The difference $x_j^{m} - y_j^{m}$ will only increase as m does. It seems this would lead to an increase in the difference between $\sum_{i=1}^{n} x_i^{m}$ and $\sum_{i=1}^{n} y_i^{m}$ for $m \geq m^*$ when m^* is sufficiently large.

In trying to find possible routes for the proofs to the above conjectures, induction strictly on m will not be an option. Given two score vectors $\bar{x}$ and $\bar{y}$, if $\sum_{i=1}^{n} x_i^{m} > \sum_{i=1}^{n} y_i^{m}$ for some m, that does not necessarily imply that $\sum_{i=1}^{n} x_i^{m+1} > \sum_{i=1}^{n} y_i^{m+1}$. That is to say, $\mathcal{H}^{m+1}$ is not necessarily an extension of $\mathcal{H}^{m}$. For example, look at the two score vectors $E = (0,1,3,3,3)$ and $D = (1,1,2,2,4)$ in the poset $\mathcal{T}(5)$. Below are the sums of the entries of E and D raised to different powers of m.

$$\sum_{i=1}^{5} e_i^{2} = 28 > 26 = \sum_{i=1}^{5} d_i^{2}$$

$$\sum_{i=1}^{5} e_i^{3} = 82 = \sum_{i=1}^{5} d_i^{3}$$

$$\sum_{i=1}^{5} e_i^{4} = 244 < 290 = \sum_{i=1}^{5} d_i^{4}$$

This implies that $E \leq D$ in $\mathcal{H}^2$, $E \parallel D$ in $\mathcal{H}^3$, and $D \leq E$ in $\mathcal{H}^4$. In this case $E \parallel D$ in the original poset, so there are no contradictions with $\mathcal{H}^2$, $\mathcal{H}^3$, and $\mathcal{H}^4$ being extensions of $\mathcal{T}(5)$. This just confirms the suspicion that the crux of proving $\mathcal{H}^m$ is an extension of $\mathcal{T}(n)$ will be found in the poset relation on $\mathcal{T}(n)$ itself.

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