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Algebraic Properties of Killing Vectors For Lorentz Metrics In Four Dimensions

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ALGEBRAIC PROPERTIES OF KILLING VECTORS FOR
LORENTZ METRICS IN FOUR DIMENSIONS

by

Jesse W. Hicks

A report submitted in partial fulfillment
of the requirements for the degree

of

MASTER OF SCIENCE

in

Mathematics

Approved:

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2011

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ABSTRACT

Algebraic Properties of Killing Vectors For

Lorentz Metrics In Four Dimensions

by

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Utah State University, 2011

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Department: Mathematics and Statistics

Four-dimensional space-times with symmetry play a central role in the theory of general relativity. In 1961, in the book *Einstein Spaces*, A.Z. Petrov gave a complete local classification of four-dimensional space-times based upon their local isometry group, that is, their Lie algebra of Killing vector fields. In this report we discuss algebraic and geometric properties of these Lie algebras. A database of these properties has been computed for the five-dimensional Lie algebras of Killing vectors found in Petrov. As an application of our work, we present diffeomorphisms between a few pairs of these Lie algebras of Killing vectors.

(95)

DEDICATION

To my beautiful wife Jodi.

ACKNOWLEDGMENTS

I would like to express my deep gratitude to Dr. Ian Anderson for mentoring and advising me these last two years. I know I didn't make it easy on him, but without Dr. Anderson's enduring patience and willingness to take a chance on me I wouldn't have made it even this far.

Also, I would like thank Dr. Mark Fels for his help in my grasping concepts in differential geometry. His lectures, notes, and style proved invaluable. I would also like to thank Dr. Charles Torre for all his input throughout, even if somewhat indirectly, I know it was considerable and I am very grateful. I want to thank Dr. Pawel Nurowski for helping me resolve some mathematical issues and being so willing to help. I thank Dr. Zhaohu Nie for his willingness to be on the committee on somewhat short notice and for his help with some problems.

I would like to express my love and thanks to my beautiful wife Jodi. Without her I wouldn't even bother with such a difficult subject, but with her love and support it somehow seems easier.

Jesse W. Hicks

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CHAPTER 1

INTRODUCTION

In Chapter 5 of the book *Einstein Spaces* [10], A.Z. Petrov gives a complete local classification of four-dimensional space-times with symmetry. Each entry in his classification consists of a list of vector fields and a family of four-dimensional Lorentz metrics g for which the vector fields are a basis for all infinitesimal isometries of g .

In this report I will discuss algebraic and geometric properties of Lie algebras of Killing vectors. In particular, I create a database of these properties which I've computed for the five-dimensional Lie algebras of Killing vectors found in Petrov's list.

Motivation for creating a database such as this is found in the following questions:

- Exactly what pairs of Petrov's Lie algebras of Killing vectors define isomorphic abstract Lie algebras?
- Among those with isomorphic abstract Lie algebras, which pairs of Killing vectors have diffeomorphisms between them?
- Given a pair of Lie algebras of Killing vectors from Petrov's classification for which there exists a diffeomorphism ϕ such that ϕ_* maps one space of Killing vectors to the other, does there exist a diffeomorphism ψ such that ψ^* maps one corresponding metric to the other?

Let Γ be a Lie algebra of Killing vectors for a Lorentz metric and \mathfrak{g} denote the abstract Lie algebra of Γ . The properties computed, from which the database was created, are the following:

- Algebra Type: Abelian, Nilpotent, Solvable, Semi-simple, or Generic
- Derived Series Dimensions
- Decomposability of \mathfrak{g}
- Decomposability of the nilradical $\text{Nil}(\mathfrak{g})$

- Dimensions of the Decomposition of \mathfrak{g}
- Orbit Dimension
- Dimension of the Nilradical
- Dimension of the Lie Algebra of Derivations \mathcal{D} of \mathfrak{g}
- Dimension of the Lie Algebra of Derivations constructed from the abstract Lie algebra of \mathcal{D}
- Dimensions of the Decomposition of the Radical of the Derivations $\text{Rad}(\mathcal{D})$
- Isotropy Type of Γ_p
- *True* or *False*, according to whether or not there is a subspace $C \subset \Gamma$ such that $\{\Gamma_p, C\}$ is a reductive pair.
- *True* or *False*, according to whether or not $\{\Gamma_p, C\}$ is a symmetric pair, whenever $\{\Gamma_p, C\}$ is a reductive pair.
- Subspace Type of Γ

All the above properties could be computed for a Lie algebra of Killing vectors in any dimension except isotropy type. The isotropy type is defined for the Lie algebras of Killing vectors of four-dimensional Lorentz metrics.

Here is a sample entry from the database:

```
#####
# [33, 7, 1]
#####
JTable[[33, 7, 1]]:=table();
JTable[[33, 7, 1]]["AlgebraType"] := "Solvable";
JTable[[33, 7, 1]]["Indecomposable"] := true;
JTable[[33, 7, 1]]["DecompositionDimensions"] := [5];
JTable[[33, 7, 1]]["IsotropySubalgebraReductivePair"] := false;
JTable[[33, 7, 1]]["IsotropySubalgebraSymmetricPair"] := NA;
JTable[[33, 7, 1]]["DerivedSeriesDimensions"] := [5,3,1,0];
```

```

JTable[[33, 7, 1]]["OrbitDimension"] := 4;
JTable[[33, 7, 1]]["IsotropyType"] := "F14";
JTable[[33, 7, 1]]["NilradicalDimension"] := 3;
JTable[[33, 7, 1]]["NilradicalIndecomposable"] := true;
JTable[[33, 7, 1]]["DerivationsDimension"]:= 5;
JTable[[33, 7, 1]]["DecompositionOfRadicalOfDerivationsDimensions"]:= [5];
JTable[[33, 7, 1]]["DerivationsOfDerivationsDimension"]:= 5;
JTable[[33, 7, 1]]["SubspaceType"]:= nonnull;
JTable[[33, 7, 1]]["VectorFields"]:=
  [D_x2, D_x3, x3*D_x2-(1/2)*sqrt(4-q^2)*tan((1/2)*sqrt(4-q^2)*x1)*D_x3,
  -D_x1+(q*x2+(1/2)*x3^2)*D_x2-(1/2)*x3*(-q+sqrt(4-q^2)*tan((1/2)
  *sqrt(4-q^2)*x1))*D_x3, (exp(-q*x1)+2*x2)*D_x2+x3*D_x3+D_x4];

```

The notation $[33, 7, 1]$ is used in the database and denotes $(33, 7)$ in Petrov [10], with $\varepsilon = 1$. There are many entries in the database that follow this same obvious convention. The parameters in each case are denoted ε more often than not, though on occasion it is denoted e_4 , or something very similar, with $\varepsilon, e_4 \in \{0, \pm 1\}$. In some instances, different values of these parameters yield distinct Lie algebras.

In the next chapter we give a review of space-time isometries. We'll proceed to define the above properties and demonstrate how to compute them by hand and by using Maple.

In Chapter 3, we will demonstrate the calculation of all these properties for a particular example from the database. The database itself is presented in an appendix.

In Chapter 4, we use the database to identify by diffeomorphism some Lie algebras of Killing vectors in Petrov's classification.

In Chapter 5 we will present ideas for future research.

CHAPTER 2
REVIEW OF SPACE-TIME ISOMETRIES

We review basic background material and properties of space-times used to create a database for Petrov's Lie algebras of infinitesimal generators of local group actions on 4-dimensional space-times, [10]. The database is a table that can be read and explored in Maple. We'll look at properties of abstract Lie algebras which are preserved under Lie algebra homomorphisms (to be defined). Also, we define certain properties of Lie subalgebras specific to any Lie algebra of vector fields Γ that are preserved under push-forward by diffeomorphisms that may not, however, be preserved under a Lie algebra isomorphism when Γ is viewed as an abstract Lie algebra.

2.1 Group Actions and Isometries

Let M be a differentiable manifold and $p \in M$. We will denote the set of all smooth real valued functions defined on some open neighborhood U of p by $C^\infty(p)$. We denote the space of linear functions $X_p : C^\infty(p) \rightarrow \mathbb{R}$, called *derivations*, which satisfy the Leibniz rule $X_p(fg) = f(p)X_p(g) - g(p)X_p(f)$, by T_pM . Let $T(M)$ denote the tangent bundle, that is, $T(M) = \cup_{p \in M} T_pM$, and π be the natural projection $\pi : T(M) \rightarrow M$, given by $X_p \mapsto p$. The set of all smooth vector fields on an n -dimensional differentiable manifold M will be denoted $\mathfrak{X}(M)$. Note that $\mathfrak{X}(M)$ can be viewed as a vector space *over the reals*, as any linear combination of smooth vectors fields is again a smooth vector field.

A *metric tensor* g is a mapping on M given by $p \mapsto g_p$, where $g_p : T_pM \times T_pM \rightarrow \mathbb{R}$ is a non-degenerate symmetric bilinear form, and where g varies smoothly with $p \in M$. As a symmetric bilinear form, we may find the (symmetric) matrix representation $G = (G_{ij})$ of $g(p)$ by computing $G_{ij} = g(p)(X_p^i, X_p^j)$, for a basis $\{X_p^i\}$ of T_pM . Define the *quadratic form* Q_p on T_pM associated to g at p by $Q(X_p) = g_p(X_p, X_p)$. Sylvester's Law of Inertia states that we may diagonalize G , where the new diagonal values a_{ii} are in the set $\{0, -1, 1\}$. But $a_{ii} \neq 0$ in our case, since G is non-degenerate. If r is the number of positive 1's in the diagonal and if s is the number of negative 1's in the diagonal, the *signature* of Q_p is defined

to be (r, s) . Also by Sylvester's Law of Inertia, however, the signature of Q_p is independent of how G was diagonalized (see [6], page 252). If the signature of Q_p is constant with respect to p , then the *signature of g* is (r, s) . A *Lorentzian manifold* is a four-dimensional manifold with a metric tensor having signature $(3,1)$.

One of the fundamental structures underlying our discussion is the following:

Definition 2.1.1. *A group G endowed with an m -dimensional C^∞ manifold structure with the additional properties*

- i. the group operation $* : G \times G \rightarrow G, (g, g') \mapsto g * g'$ is a C^∞ operation,*
- ii. the map $i : G \rightarrow G$ given by $i(g) = g^{-1}$ is C^∞ ,*

is called an m -dimensional Lie group.

We will drop the symbol $*$ when denoting a group operation. Case in point, a Lie group *homomorphism* between two Lie groups G and G' is any smooth map $\psi : G \rightarrow G'$ such that $\psi(gg') = \psi(g)\psi(g')$, noting that on the left the group operation is that of G and on the right the group operation is that of G' . A bijective Lie group homomorphism is a Lie group *isomorphism*.

Definition 2.1.2. *Let G be any group and E any set. A group action of G on E is a mapping $\mu : G \times E \rightarrow E$ satisfying*

- i. $\mu(e, x) = x$, for all $x \in E$ and e the identity in G , and*
- ii. $\mu(g, \mu(g', x)) = \mu(gg', x)$, for all $g, g' \in G, x \in E$.*

We will say “ G acts on E by μ .”

We will be concerned with the case where $E = M$ is a differentiable manifold, G a Lie group, and μ is smooth and refer to μ as a *smooth action* or *C^∞ action*.

Fix $g \in G$. Then define $\mu_g : M \rightarrow M$ to be given by $\mu_g(x) = \mu(g, x)$. Observe that $\mu_g \circ \mu_h(x) = \mu_g(\mu_h(x)) = \mu_g(\mu(h, x)) = \mu(g, \mu(h, x)) = \mu(gh, x) = \mu_{gh}$. Then for $h = g^{-1}$,

we have $\mu_g \circ \mu_{g^{-1}} = \mu_e$ and therefore $\mu_g \circ \mu_{g^{-1}} = \mathbb{1}_M$, the identity on M . Then $\mu_{g^{-1}} = \mu_g^{-1}$ and since μ is smooth, μ_g is smooth for all $g \in G$, making μ_g a diffeomorphism from M to M .

Definition 2.1.3. Let G be a Lie group acting on a manifold M by a smooth action μ . The orbit of $x \in M$ under the action μ is the set $Gx = \{\mu(g, x) \mid g \in G\}$.

We will simply write gx for $\mu(g, x)$ when it's convenient and speak of the orbit of x without explicitly stating which action it is under as it is always clear in context.

Example 2.1.4. Let $\mu : G \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the smooth action given by $\mu((a, b, c), (x, y)) = (x + ay + bz, y + cz, z)$, with multiplication in $G = \mathbb{R}^3$ given by $(a, b, c)(d, e, f) = (a + d, e + af + b, c + f)$. Note that $(a, b, c)^{-1} = (-a, ac - b, -c)$ and the identity is $(0, 0, 0)$. Let $x = (1, 2, 3)$ and observe that

$$\begin{aligned} Gx &= \{gx \mid g \in G\} \\ &= \{\mu((a, b, c), (1, 2, 3)) \mid (a, b, c) \in G\} \\ &= \{(1 + 2a + 3b, 2 + 3c, 3) \mid (a, b, c) \in G\}. \end{aligned}$$

Note that orbits are submanifolds of M with varying dimension but not necessarily regular submanifolds. See Olver [8].

Definition 2.1.5. The orbit dimension of Gx is its dimension as a submanifold of M .

We now look at a special case of a group action.

Definition 2.1.6. A smooth group action $\psi : \mathbb{R} \times M \rightarrow M$ of the Lie group \mathbb{R} on a differentiable manifold M is called a flow.

Observe that if we fix $t \in \mathbb{R}$, then ψ_t is a diffeomorphism and $\psi_t \circ \psi_s = \psi_{t+s}$.

Definition 2.1.7. The infinitesimal generator for a flow ψ is the smooth vector field X on M given by $p \mapsto X_p$, where $X_p : C^\infty(p) \rightarrow \mathbb{R}$ is given by

$$X_p(f) = \lim_{t \rightarrow 0} \frac{f(\psi_t(p)) - f(p)}{t}.$$

Observe that X_p is a derivation since for any $a, b \in \mathbb{R}$ and $f, g \in C^\infty(p)$,

$$\begin{aligned} X_p(af + bg) &= \lim_{t \rightarrow 0} \frac{(af + bg)(\psi_t(p)) - (af + bg)(p)}{t} \\ &= \lim_{t \rightarrow 0} \frac{af(\psi_t(p)) + bg(\psi_t(p)) - af(p) - bg(p)}{t} \\ &= a \lim_{t \rightarrow 0} \frac{f(\psi_t(p)) - f(p)}{t} + b \lim_{t \rightarrow 0} \frac{g(\psi_t(p)) - g(p)}{t} \\ &= aX_p(f) + bX_p(g), \end{aligned}$$

which gives linearity. The Leibniz rule is shown by

$$\begin{aligned} X_p(fg) &= \lim_{t \rightarrow 0} \frac{(fg)(\psi_t(p)) - (fg)(p)}{t} \\ &= \lim_{t \rightarrow 0} \frac{f(\psi_t(p))g(\psi_t(p)) - f(p)g(p)}{t} \\ &= \lim_{t \rightarrow 0} \frac{f(\psi_t(p))g(\psi_t(p)) + f(\psi_t(p))g(p) - f(\psi_t(p))g(p) - f(p)g(p)}{t} \\ &= \lim_{t \rightarrow 0} \frac{f(\psi_t(p))[g(\psi_t(p)) - g(p)] + g(p)[f(\psi_t(p)) - f(p)]}{t} \\ &= \lim_{t \rightarrow 0} \frac{f(\psi_t(p))[g(\psi_t(p)) - g(p)]}{t} + \lim_{t \rightarrow 0} \frac{g(p)[f(\psi_t(p)) - f(p)]}{t} \\ &= f(p)X_p(g) - g(p)X_p(f), \end{aligned}$$

noting that f and ψ are assumed smooth and due to the fact that ψ is a smooth group action, we have $f(\psi_t(p)) \rightarrow f(\psi_0(p)) = f(\psi(0, p)) = f(p)$ as $t \rightarrow 0$. Therefore the infinitesimal generator of a flow on M is a smooth vector field on M . See Boothby [1], pages 113-119.

Note that in local coordinates for a flow ψ , if $\bar{x}^i = \psi^i(t, x^i)$, then

$$X_p = \sum_{i=1}^n \frac{\partial \psi^i}{\partial t} \Big|_{t=0} \frac{\partial}{\partial x^i},$$

where n is the dimension of the manifold. There is an interesting connection between smooth vector fields and flows, with proof in Boothby [1]:

Theorem 2.1.8. *Let X be a smooth vector field on a differentiable manifold M . Then X is the infinitesimal generator of a unique local flow ψ on M .*

Note that by *local* we mean, in essence, that for any given point $p \in \mathbb{R} \times \mathbb{M}$, the action of the flow is defined in a neighborhood $W \subset \mathbb{R} \times \mathbb{M}$ such that $p \in W$. For more information

on this theorem and local flows, see Boothby [1], pages 127-134, specifically Theorems 4.5 and 4.6 and their proofs.

Definition 2.1.9. *Let G be an n -dimensional Lie group, with identity e , acting on a differentiable manifold M by the C^∞ action μ . Let $\sigma : \mathbb{R} \rightarrow M$ be a 1-parameter subgroup of G . That is, $\sigma(0) = e$ and $\sigma(t+s) = \sigma(t)\sigma(s)$. Then the associated infinitesimal generator on M is the vector field defined by the flow $\psi_t(p) := \mu(\sigma(t), p)$.*

Observe that $\psi_{t+s}(p) = \mu(\sigma(t+s), p) = \mu(\sigma(t)\sigma(s), p) = \mu(\sigma(t), \mu(\sigma(s), p)) = (\psi_t \circ \psi_s)(p)$ and since μ is smooth, ψ_t is smooth. We now introduce a product on $\mathfrak{X}(M)$.

Proposition 2.1.10. *The mapping $[\cdot, \cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$, given by $[X, Y](f) = X(Y(f)) - Y(X(f))$, defines a vector field called the commutator of vector fields.*

Proof: We demonstrate that $[X, Y]$ is a smooth vector field in $\mathfrak{X}(M)$. If $X = \sum_{i=1}^n X^i(\mathbf{x}) \frac{\partial}{\partial x^i}$ and $Y = \sum_{i=1}^n Y^i(\mathbf{x}) \frac{\partial}{\partial x^i}$ in local coordinates, then

$$\begin{aligned} [X, Y](f) &= \sum_{i=1}^n X^i(\mathbf{x}) \frac{\partial}{\partial x^i} \left(\sum_{j=1}^n Y^j(\mathbf{x}) \frac{\partial f}{\partial x^j} \right) \\ &\quad - \sum_{j=1}^n Y^j(\mathbf{x}) \frac{\partial}{\partial x^j} \left(\sum_{i=1}^n X^i(\mathbf{x}) \frac{\partial f}{\partial x^i} \right) \\ &= \sum_{i=1}^n \sum_{j=1}^n X^i(\mathbf{x}) \frac{\partial Y^j(\mathbf{x})}{\partial x^i} \frac{\partial f}{\partial x^j} - \sum_{i=1}^n \sum_{j=1}^n Y^j(\mathbf{x}) \frac{\partial X^i(\mathbf{x})}{\partial x^j} \frac{\partial f}{\partial x^i} \\ &= \sum_{i=1}^n \left[\sum_{j=1}^n \left(X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \right) \right] \frac{\partial f}{\partial x^i}. \end{aligned}$$

Since X and Y are C^∞ vector fields, X^i and Y^i are C^∞ functions and thus

$$\sum_{j=1}^n \left(X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \right)$$

is a C^∞ function. Note that $[X, Y]$ is linear since

$$\begin{aligned} [X, Y](af + bg) &= X(Y(af + bg)) - Y(X(af + bg)) \\ &= X(aY(f) + bY(g)) - Y(aX(f) + bY(g)) \end{aligned}$$

$$\begin{aligned}
&= aX(Y(f)) + bX(Y(g)) - aY(X(f)) - bX(Y(g)) \\
&= aX(Y(f)) - aY(X(f)) + bX(Y(g)) - bX(Y(g)) \\
&= a[X, Y](f) + b[X, Y](g)
\end{aligned}$$

Note that the Leibniz rule is satisfied as

$$\begin{aligned}
[X, Y](fg) &= X(Y(fg)) - Y(X(fg)) \\
&= X(gY(f) + fY(g)) - Y(gX(f) + fX(g)) \\
&= X(gY(f)) + X(fY(g)) - Y(gX(f)) - Y(fX(g)) \\
&= X(gY(f)) - Y(gX(f)) + X(fY(g)) - Y(fX(g)) \\
&= gX(Y(f)) + Y(f)X(g) - gY(X(f)) - X(f)Y(g) \\
&\quad + fX(Y(g)) + Y(g)X(f) - fY(X(g)) - X(g)Y(f) \\
&= gX(Y(f)) - gY(X(f)) + fX(Y(g)) - fY(X(g)) \\
&= g[X, Y](f) + f[X, Y](g)
\end{aligned}$$

Therefore $[X, Y]$ defines a C^∞ vector field. \square

Note that this shows $\mathfrak{X}(M)$ to be closed under the commutator of vector fields. However, the following result shows that the set of infinitesimal generators of a smooth action is also closed under the commutator of vector fields. See Olver [8], page 51.

Theorem 2.1.11. *If X and Y are infinitesimal generators for a smooth action μ on a manifold M , then $[X, Y]$ is also an infinitesimal generator for μ .*

We now note a property used in the description of the subspace of infinitesimal generators that the database needs. We use the same language as definition 2.1.5.

Definition 2.1.12. *Let Γ be the set of infinitesimal generators of the smooth action μ of a Lie group G on a manifold M . If $\Lambda = \text{span}(\Gamma)$, then the orbit dimension of Γ is $\dim(\Lambda)$.*

Example 2.1.13. As an example, let

$$\Gamma = \{ \partial_y, \partial_z, -\partial_x + y\partial_y + z\partial_z, -z\partial_y + y\partial_z, \alpha\partial_x + \partial_w \},$$

for α a constant. Then let $\beta = \{\partial_y, \partial_z, -\partial_x + y\partial_y + z\partial_z, \alpha\partial_x + \partial_w\}$ and observe that $\text{span}(\beta) = \text{span}(\Gamma)$ and thus the orbit dimension for Γ is 4. This is easily calculated in Maple:

```
> beta := DGbasis(Gamma);
      beta := [partial_y, partial_z, -partial_x + y partial_y + z partial_z, alpha partial_x + partial_w]
> nops(beta);
```

4

This is assuming one has set up Γ first.

Now, let N and M be differentiable manifolds. Given a smooth function $\phi : M \rightarrow N$, then for any $f \in C^\infty(\phi(p))$ for $p \in M$, the *push-forward* of $X_p \in T_p M$ by ϕ is defined to be the mapping $\phi_* : T_p M \rightarrow T_{\phi(p)} N$ given by

$$\phi_*(X_p)(f) = X_p(f \circ \phi).$$

In local coordinates $\{x^i\}_{i=1}^m$ for M and $\{y^j\}_{j=1}^n$ for N we have

$$\begin{aligned} \phi_*(X_p)(f) &= \sum_{i=1}^n X^i \frac{\partial f \circ \phi}{\partial x^i} \Big|_p \\ &= \sum_{i=1}^n X^i \left(\sum_{j=1}^m \frac{\partial f}{\partial y^j} \Big|_p \frac{\partial \phi^j}{\partial x^i} \Big|_p \right) \\ &= \sum_{j=1}^m \left(\sum_{i=1}^n X^i \frac{\partial \phi^j}{\partial x^i} \Big|_p \right) \frac{\partial f}{\partial y^j} \Big|_p \end{aligned}$$

Let $p \in M$ and g be a metric tensor field on N . Then the *pullback of g along ϕ* is defined to be

$$(\phi^*g)_p(X_p, Y_p) = g_{\phi(p)}(\phi_*(X_p), \phi_*(Y_p)).$$

Now, if $\phi : M \rightarrow M$ is a diffeomorphism, then we say g is *invariant under ϕ* if $(\phi^*g)_p = g_p$, and call ϕ an *isometry* of g . The set of isometries of a metric tensor form a group with composition as group operation, simply called the *group of isometries* of g . However,

the group of isometries, critical to our discussion, can be given even more structure under certain conditions. This is shown in the following very important theorem which can be found with proof in Kobayashi [7].

Theorem 2.1.14. *Let M be an n -dimensional Lorentzian manifold with metric g . The group of isometries of g form a Lie group. Furthermore, the maximal dimension for the group of isometries is at most $\frac{n(n+1)}{2}$.*

Definition 2.1.15. *Let g be a metric on M . Let X be a vector field with unique local flow ψ_t (by Theorem 2.1.8). If ψ_t is an isometry for g , so $\psi_t^*g_{\phi(p)} = g_p$, then we call X an infinitesimal isometry or Killing vector of g .*

Let ψ_t be the unique local flow for a vector field X . The *Lie derivative* of the metric g with respect to a vector field X is defined to be

$$(\mathcal{L}_X g)_p = \lim_{t \rightarrow 0} \frac{\psi_t^* g_{\psi_t(p)} - g_p}{t}.$$

Then if ψ_t is an isometry for g we see $(\mathcal{L}_X g)_p = 0$ for all $p \in M$. In other words, if X is a Killing vector for g , then $(\mathcal{L}_X g)_p = 0$. The converse of this, that is, $(\mathcal{L}_X g)_p = 0$ for all $p \in M$ implies X is a Killing vector of g , is also true. It can be shown that (see Fels [4])

$$\mathcal{L}_{X_p}(g_p) = \sum_{i,j=1}^n \left(\sum_{k=1}^n g_{ik}(\mathbf{x}) \frac{\partial X^k}{\partial x^j} \Big|_p + g_{kj}(\mathbf{x}) \frac{\partial X^k}{\partial x^i} \Big|_p + X^k(\mathbf{x}) \frac{\partial g_{ij}}{\partial x^k} \Big|_p \right) dx^i dx^j,$$

in local coordinates – a form we’ll wish to use for computation later in the chapter. It is the actions of Lie groups of isometries, called *groups of motions*, on Lorentzian manifolds from which Petrov constructed the classification found in [10].

In the next section we discuss in more detail the structure the commutator of vector fields endows $\mathfrak{X}(M)$. We develop properties of this structure needed to create the Maple database.

2.2 Lie Algebraic Properties

We first give the general definition of the most important (for our work) algebraic structure with which we may endow $\mathfrak{X}(M)$. This structure and many of its properties are the concern of this section and critical to this paper.

Definition 2.2.1. A vector space V over a field K with a bilinear mapping $[\cdot, \cdot] : V \times V \rightarrow V$ satisfying, for all $x, y, z \in V$,

$$(i) \text{ the skew-symmetric property: } [x, y] = -[y, x]$$

$$(ii) \text{ the Jacobi identity: } [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

is called a Lie algebra over K .

The mapping $[\cdot, \cdot]$ is called the *Lie bracket*. If a subspace W of a Lie algebra V is closed under the Lie bracket of V , then W is called a *Lie subalgebra*.

Now, the notation of the commutator of vector fields was no accident. That is, it satisfies definition (2.2.1). Thus $\mathfrak{X}(M)$ is a Lie algebra under the commutator of vector fields. This can be shown by the following. The commutator of vector fields is bilinear since

$$\begin{aligned} [aX + bY, Z](f) &= (aX + bY)(Z(f)) - Z((aX + bY)(f)) \\ &= aX(Z(f)) + bY(Z(f)) - aZ(X(f)) - bZ(Y(f)) \\ &= aX(Z(f)) + bY(Z(f)) - aZ(X(f)) - bZ(Y(f)) \\ &= a[X, Z](f) + b[Y, Z](f), \end{aligned}$$

and similarly for $[X, aY + bZ]$. Obviously, $[X, Y] = -[Y, X]$, making it skew-symmetric. Also, the Jacobi identity $([X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]])(f) = 0$, is satisfied since

$$\begin{aligned} ([X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]])(f) &= \\ &= [X, [Y, Z]](f) + [Y, [Z, X]](f) + [Z, [X, Y]](f) \\ &= X([Y, Z](f)) - [Y, Z](X(f)) \\ &\quad + Y([Z, X](f)) - [Z, X](Y(f)) \\ &\quad + Z([X, Y](f)) - [X, Y](Z(f)) \\ &= X(Y(Z(f)) - Z(Y(f))) - Y(Z(X(f))) + Z(Y(X(f))) \\ &\quad + Y(Z(X(f)) - X(Z(f))) - Z(X(Y(f))) + X(Z(Y(f))) \\ &\quad + Z(X(Y(f)) - Y(X(f))) - X(Y(Z(f))) + Y(X(Z(f))) \end{aligned}$$

$$\begin{aligned}
&= X(Y(Z(f))) - X(Z(Y(f))) - Y(Z(X(f))) + Z(Y(X(f))) \\
&\quad + Y(Z(X(f))) - Y(X(Z(f))) - Z(X(Y(f))) + X(Z(Y(f))) \\
&\quad + Z(X(Y(f))) - Z(Y(X(f))) - X(Y(Z(f))) + Y(X(Z(f))) \\
&= 0.
\end{aligned}$$

As a Lie algebra, $\mathfrak{X}(M)$ contains finite-dimensional Lie subalgebras that are of prime importance to us. In fact, we've seen these Lie algebras already, we simply didn't call them by that name. To wit, the span of infinitesimal generators of a group action, and thus the span of infinitesimal isometries of a Lorentzian manifold, are both finite-dimensional Lie subalgebras of $\mathfrak{X}(M)$ by Theorem 2.1.11.

We now turn our attention to investigating properties of Lie algebras that every Lie algebra shares in common (in particular Lie algebras of infinitesimal isometries).

Let \mathfrak{g} be a Lie algebra. An *ideal* of \mathfrak{g} is a subspace \mathfrak{a} of \mathfrak{g} such that $[\mathfrak{a}, \mathfrak{g}] \subseteq \mathfrak{a}$, where $[\mathfrak{a}, \mathfrak{g}] = \text{span}\{[X, Y] \mid X \in \mathfrak{a}, Y \in \mathfrak{g}\}$. Note that an ideal is also a Lie subalgebra since for any ideal \mathfrak{a} in \mathfrak{g} we clearly have $[\mathfrak{a}, \mathfrak{a}] \subseteq \mathfrak{a}$. Also observe that \mathfrak{g} is an ideal of itself since it is closed under the Lie bracket. Due to the skew-symmetry of the Lie bracket there is no need to distinguish between left or right ideals.

Proposition 2.2.2. *For any ideals \mathfrak{h} and \mathfrak{l} of \mathfrak{g} , $[\mathfrak{h}, \mathfrak{l}]$ is an ideal of \mathfrak{g} .*

Proof: These steps follow closely to [3], pages 11-12. Note that by definition $[\mathfrak{h}, \mathfrak{l}]$ is a subspace of \mathfrak{g} . Let $h \in \mathfrak{h}$, $l \in \mathfrak{l}$, and $x \in \mathfrak{g}$. Then observe by the Jacobi identity

$$\begin{aligned}
[[h, l], x] &= [h, [l, x]] + [l, [x, h]] \\
&= -[h, [x, l]] - [[x, h], l]
\end{aligned}$$

But $[x, l] \in \mathfrak{l}$ and $[x, h] \in \mathfrak{h}$ as \mathfrak{l} and \mathfrak{h} are ideals and thus the sum

$$-[h, [x, l]] - [[x, h], l] \in [\mathfrak{h}, \mathfrak{l}]$$

and so

$$(2.1) \quad [[h, l], x] = [h, [x, l]] + [[x, h], l] \in [\mathfrak{h}, \mathfrak{l}]$$

Now let $u \in [\mathfrak{h}, \mathfrak{l}]$. Then $u = \sum c_i [h_i, l_i]$, where the $h_i \in \mathfrak{h}$ and $l_i \in \mathfrak{l}$, since $u \in \text{span}\{[X, Y] \mid X \in \mathfrak{a}, Y \in \mathfrak{g}\}$. Then

$$[u, x] = \left[\sum c_i [h_i, l_i], x \right] = c_i \sum [[h_i, l_i], x] \in [\mathfrak{h}, \mathfrak{l}],$$

for each i by (2.1), noting $x \in \mathfrak{g}$ was arbitrary. \square

Now we may define a very important concept.

Definition 2.2.3. *The derived algebra is the ideal $\mathfrak{g}^{(1)} := [\mathfrak{g}, \mathfrak{g}]$.*

Example 2.2.4. Let \mathfrak{g} be an abstract finite-dimensional Lie algebra with basis $\{e_1, e_2, e_3, e_4\}$ and structure equations $\{[e_1, e_4] = e_1, [e_2, e_3] = e_2\}$. We can view the structure equations best in the form of a multiplication table:

$[\cdot, \cdot]$	e_1	e_2	e_3	e_4
e_1	·	·	·	e_1
e_2	·	·	e_2	·
e_3	·	$-e_2$	·	·
e_4	$-e_1$	·	·	·

Let $x, y \in \mathfrak{g}$. Then $x = \sum c_i e_i$ and $y = \sum d_j e_j$ for some constants c_i and d_j and thus we must have

$$\begin{aligned} [x, y] &= \left[\sum c_i e_i, \sum d_j e_j \right] \\ &= \sum d_j \sum c_i [e_i, e_j] \\ &= \sum d_j \sum c_i e_{ij}, \end{aligned}$$

where $e_{ij} \in \{e_1, e_2, -e_1, -e_2\}$, by the multiplication table. But we've simply shown $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] = \text{span}\{e_1, e_2\}$. We can easily compute the Derived algebra in Maple (we will be assuming from here on out that one has already imported the pertinent packages; more information regarding the different differential geometry packages within Maple can be found in Maple's help pages):

```

> Eq := [[e1, e4] = e1, [e2, e3] = e2], [e1, e2, e3, e4];
> DGsetup(LieAlgebraData(Eq, g));
> DerivedAlgebra(g);

```

$$[e_1, e_2].$$

If we define $\mathfrak{g}^{(0)} := \mathfrak{g}$, we can define recursively the chain of ideals $\mathfrak{g}^{(k)} := [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}]$, for $k \geq 1$. Note the obvious consequence of this definition, which explains why we used the term chain: $\mathfrak{g} = \mathfrak{g}^{(0)} \supseteq \mathfrak{g}^{(1)} \supseteq \mathfrak{g}^{(2)} \supseteq \dots$.

Definition 2.2.5. *The sequence $\mathfrak{g} = \mathfrak{g}^{(0)} \supseteq \mathfrak{g}^{(1)} \supseteq \mathfrak{g}^{(2)} \dots$ will be called the derived series.*

Example 2.2.6. Let \mathfrak{g} be the Lie algebra of Example 2.2.4. Then recall we've already shown $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] = \text{span}\{e_1, e_2\}$. Then observe the multiplication table for $\mathfrak{g}^{(1)}$, which has as an obvious basis $\{e_1, e_2\}$:

$$\begin{array}{c|cc} [\cdot, \cdot] & e_1 & e_2 \\ \hline e_1 & \cdot & \cdot \\ e_2 & \cdot & \cdot \end{array}$$

Clearly then, $\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] = \text{span}\{0\} = \{0\}$, which we'll write just as 0 from here on, it being simply understood given whatever context we happen to be in. Then observe that we must have $\mathfrak{g} = \mathfrak{g}^{(0)} \supseteq \mathfrak{g}^{(1)} \supseteq \mathfrak{g}^{(2)} = 0 = \mathfrak{g}^{(3)} = \mathfrak{g}^{(4)} \dots$. In Maple we simply perform the following:

```
> Eq := [[e1, e4] = e1, [e2, e3] = e2], [e1, e2, e3, e4];
> DGsetup(LieAlgebraData(Eq, g));
> Series(g, "Derived");
```

$$[[e1, e2, e3, e4], [e1, e2], []].$$

Observe from the previous example that the dimensions of the ideals in the derived series were $\dim(\mathfrak{g}^{(0)}) = 4$, $\dim(\mathfrak{g}^{(1)}) = 2$, $\dim(\mathfrak{g}^{(2)}) = 0$, $\dim(\mathfrak{g}^{(3)}) = 0$, and so on. We will denote this as $[4, 2, 0]$, called the *dimensions of the derived series* in this paper. In Maple it is computed, once the above commands are computed, by

```
> map(nops, %);
```

$$[4, 2, 0]$$

This leads us again to a new and very important concept.

Definition 2.2.7. A Lie algebra \mathfrak{g} is said to be solvable if for some k we have $\mathfrak{g}^{(k)} = 0$.

So the Lie algebra in Example 2.2.6 is solvable. Not every Lie algebra is solvable as we shall see later. Now, if we let $\mathfrak{g}_{(0)} := \mathfrak{g}$, we can again define recursively a chain of ideals $\mathfrak{g}_{(k)} := [\mathfrak{g}, \mathfrak{g}_{(k-1)}]$, for $k \geq 1$. Note the similar consequence of this definition: $\mathfrak{g} = \mathfrak{g}_{(0)} \supseteq \mathfrak{g}_{(1)} \supseteq \mathfrak{g}_{(2)} \supseteq \cdots$. To see an example of Maple determining if a Lie algebra is solvable, see Example 2.2.11.

Definition 2.2.8. The sequence $\mathfrak{g} = \mathfrak{g}_{(0)} \supseteq \mathfrak{g}_{(1)} \supseteq \mathfrak{g}_{(2)} \supseteq \cdots$ is called the lower central series.

Example 2.2.9. Let \mathfrak{g} be an abstract Lie algebra with basis $\{e_1, e_2, e_3, e_4\}$ and structure equations $\{[e_2, e_4] = e_1\}$. Here is the multiplication table for this Lie algebra:

$[\cdot, \cdot]$	e_1	e_2	e_3	e_4
e_1	·	·	·	·
e_2	·	·	·	e_1
e_3	·	·	·	·
e_4	·	$-e_1$	·	·

Then observe the derived algebra $\mathfrak{g}^{(1)} = \text{span}\{e_1\}$, using the same technique from Example 2.2.4. But from the definition of derived algebra and lower central series we have $\mathfrak{g}_{(1)} = \mathfrak{g}^{(1)} = \text{span}\{e_1\}$. Now note the multiplication table for $\mathfrak{g}_{(2)} = [\mathfrak{g}, \mathfrak{g}^{(1)}]$ (we are working on the bases):

$[\cdot, \cdot]$	e_1
e_1	0
e_2	0
e_3	0
e_4	0

Then clearly $\mathfrak{g}_{(2)} = \text{span}\{0\} = 0 = \mathfrak{g}_{(3)} = \cdots$. That is, we have for the lower central series of this Lie algebra $\mathfrak{g}_{(1)} = \text{span}\{e_1\} \supseteq \mathfrak{g}_{(2)} = \text{span}\{0\} = 0 = \mathfrak{g}_{(3)} = \cdots$.

The lower central series is not used in the database directly, but it is the idea behind a very important definition we need.

Definition 2.2.10. Let \mathfrak{g} be a Lie algebra. If there is a k such that $\mathfrak{g}_{(k)} = 0$, then \mathfrak{g} is said to be nilpotent.

By definition then, one can easily see that the Lie algebra of Example 2.2.9 is nilpotent. But, we can easily see that it is indeed solvable too. In fact all nilpotent Lie algebras are solvable. Before we prove this, we show by example that the converse is false.

Example 2.2.11. Let \mathfrak{g} be the solvable abstract Lie algebra of Example 2.2.4. Here is the multiplication table reproduced:

$[\cdot, \cdot]$	e_1	e_2	e_3	e_4
e_1	·	·	·	e_1
e_2	·	·	e_2	·
e_3	·	$-e_2$	·	·
e_4	$-e_1$	·	·	·

Recall that the derived algebra is $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] = \text{span}\{e_1, e_2\}$. Then we have $\mathfrak{g}_{(1)} = [\mathfrak{g}, \mathfrak{g}_{(0)}] = \text{span}\{e_1, e_2\}$, by definition. Here is the multiplication table for $\mathfrak{g}_{(2)} = [\mathfrak{g}, \mathfrak{g}_{(1)}]$ (again, working with the bases):

$[\cdot, \cdot]$	e_1	e_2
e_1	·	·
e_2	·	·
e_3	·	$-e_2$
e_4	$-e_1$	·

Thus

$$\mathfrak{g}_{(2)} = [\mathfrak{g}, \mathfrak{g}_{(1)}] = \text{span}\{-e_1, -e_2\} = \text{span}\{e_1, e_2\} = \mathfrak{g}_{(1)}.$$

Therefore have $\mathfrak{g}_{(1)} = \mathfrak{g}_{(2)} = \mathfrak{g}_{(3)} = \dots$. Thus there is no k for which $\mathfrak{g}_{(k)} = 0$ and therefore \mathfrak{g} is not nilpotent. We can easily show this in Maple:

```

> Eq := [[e1, e4] = e1, [e2, e3] = e2], [e1, e2, e3, e4];
> DGsetup(LieAlgebraData(Eq, g));
> Query(g, "Nilpotent");

```


false

Proposition 2.2.12. *Every nilpotent Lie algebra is solvable.*

Proof: Let \mathfrak{g} be a nilpotent Lie algebra. We'll proceed by way of induction. By definition of lower central series and derived series, we have $\mathfrak{g}^{(1)} = \mathfrak{g}_{(1)}$. Therefore $\mathfrak{g}^{(1)} \subseteq \mathfrak{g}_{(1)}$. Suppose $\mathfrak{g}^{(n)} \subseteq \mathfrak{g}_{(n)}$, for some $n \geq 1$, and let $x \in \mathfrak{g}^{(n+1)} = [\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}]$. Then $x = \sum c_i [x_i, y_i]$, for $x_i, y_i \in \mathfrak{g}$. Note that $x_i \in \mathfrak{g}$ and by the induction hypothesis $y_i \in \mathfrak{g}_{(n)}$. Thus $x = \sum c_i [x_i, y_i] \in [\mathfrak{g}, \mathfrak{g}_{(n)}] = \mathfrak{g}_{(n+1)}$, giving $\mathfrak{g}^{(n+1)} \subseteq \mathfrak{g}_{(n+1)}$. \square

We note in passing that a Lie algebra is said to be *abelian* if the Lie bracket is always zero. For the purposes of the database, however, if a Lie algebra is non-abelian and nilpotent, it will be classified as nilpotent and the algebras that are not nilpotent but solvable will be classified as solvable. Of course, not all Lie algebras are abelian, nilpotent, or solvable. Namely the generic and semi-simple Lie algebras to be introduced shortly. But before we introduce these new algebraic properties we need more terminology.

Suppose I is an ideal in a Lie algebra \mathfrak{g} . If for any other ideal J in \mathfrak{g} such that $I \subseteq J$, then either $J = I$ or $J = \mathfrak{g}$ and we'll say I is the *maximal* ideal in \mathfrak{g} . It turns out that every finite-dimensional Lie algebra is guaranteed a unique maximal solvable ideal and a unique maximal nilpotent ideal. This can be seen by showing that if I and J are solvable (respectively nilpotent) ideals in \mathfrak{g} , then $I + J$ is a solvable (resp. nilpotent) ideal in \mathfrak{g} . Then, we suppose we have two largest solvable (resp. nilpotent) ideals I and J in \mathfrak{g} and observe that $I + J$ is thus a solvable (resp. nilpotent) ideal containing both I and J . But then $I + J = I$ and $I + J = J$ and thus $I = J$. For full proof of the solvable case, see [12] page 122, [2] page 45, [3] page 30; and for the nilradical case see [2] page 42.

Definition 2.2.13. *Let \mathfrak{g} be a Lie algebra. The radical of \mathfrak{g} is the maximal solvable ideal in \mathfrak{g} . We will denote this by $\text{Rad}(\mathfrak{g})$.*

Definition 2.2.14. *Let \mathfrak{g} be a Lie algebra. The nilradical of \mathfrak{g} is the maximal nilpotent ideal in \mathfrak{g} . We will denote this by $\text{Nil}(\mathfrak{g})$.*

An immediate consequence of these definitions is the fact that the radical of any abelian, nilpotent, or solvable Lie algebra \mathfrak{g} is \mathfrak{g} itself, namely $\text{Rad}(\mathfrak{g}) = \mathfrak{g}$ in these cases and in any abelian or nilpotent Lie algebra \mathfrak{g} , we have $\text{Nil}(\mathfrak{g}) = \mathfrak{g}$.

Example 2.2.15. Let \mathfrak{g} be the abstract Lie algebra with basis $\{e_1, e_2, e_3, e_4, e_5\}$ and structure equations $\{[e_2, e_3] = e_1, [e_2, e_4] = e_3, [e_3, e_4] = -e_2\}$. Here is the multiplication table:

$[\cdot, \cdot]$	e_1	e_2	e_3	e_4	e_5
e_1	·	·	·	·	·
e_2	·	·	e_1	e_3	·
e_3	·	$-e_1$	·	$-e_2$	·
e_4	·	$-e_3$	e_2	·	·
e_5	·	·	·	·	·

Observe that this algebra is not nilpotent since $\mathfrak{g}_{(1)} = \text{span}\{e_1, e_2, e_3\}$ and $\mathfrak{g}_{(2)} = [\mathfrak{g}, \mathfrak{g}_{(1)}] = \text{span}\{e_1, e_2, e_3\} = \mathfrak{g}_{(1)}$, which can easily be seen by ignoring columns 4 and 5 of the table. However, look what happens if vector e_4 isn't there: we would have a terminating sequence. That is to say, look at the subalgebra with basis $\{e_1, e_2, e_3, e_5\}$ and multiplication table

$[\cdot, \cdot]$	e_1	e_2	e_3	e_5
e_1	·	·	·	·
e_2	·	·	e_1	·
e_3	·	$-e_1$	·	·
e_5	·	·	·	·

Then, calling this subalgebra \mathfrak{h} , we have $\mathfrak{h}_{(1)} = \text{span}\{e_1\}$ and the multiplication table for \mathfrak{h} gives $\mathfrak{h}_{(2)} = \text{span}\{0\}$ (ignore columns 2 through 5), making \mathfrak{h} nilpotent. Since $\dim(\mathfrak{h}) = 4$, and \mathfrak{g} is not nilpotent, we see that $\text{Nil}(\mathfrak{g}) = \mathfrak{h}$. However, note that $\mathfrak{h}^{(1)} = \text{span}\{e_1\}$ and $\mathfrak{h}^{(2)} = \text{span}\{0\}$. Therefore $\text{Rad}(\mathfrak{g}) = \mathfrak{h}$ too. Now let's see what Maple can do. Let \mathfrak{l} be the finite-dimensional Lie algebra of Killing vector fields with basis

$$L = \{ \partial_y, \partial_z, \partial_x + z \partial_z, \partial_x + y \partial_y, \alpha \partial_x + \partial_w \},$$

where α is a constant. Here are the Maple commands:

```

> DGsetup(LieAlgebraData(L, l));
> Radical();
```

$$[e5, e4, e3, e2, e1]$$

$$> \text{Niladical()};$$

$$[e1, e2, e5],$$

from which we see \mathfrak{l} is solvable with $\text{Rad}(\mathfrak{l}) = \mathfrak{l}$ and a 3-dimensional nilradical with $\text{Nil}(\mathfrak{l}) = \text{span}\{\partial_y, \partial_z, \partial_x + \partial_w\}$, the first, second and fifth vector fields in the basis list L .

Definition 2.2.16. *Let \mathfrak{g} be a Lie algebra. If $\text{Rad}(\mathfrak{g}) = 0$, then \mathfrak{g} is said to be semi-simple.*

Another way to put this is, a Lie algebra \mathfrak{g} is semi-simple if whenever \mathfrak{g} contains a solvable ideal, it must be the zero ideal. From this definition we see a nilpotent or solvable Lie algebra cannot be semi-simple. To detect if a Lie algebra is semi-simple, we would like some criterion to help us minimize our efforts computationally. We will introduce *Cartan's Criterion for semi-simplicity*, which can be used to determine semi-simplicity. This is the method Maple uses to query a Lie algebra's semi-simplicity. But before we introduce the criterion we need more machinery.

By a *Lie algebra homomorphism* between two Lie algebras \mathfrak{g}_1 and \mathfrak{g}_2 , we mean a linear map $\phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ with the additional property $\phi([x, y]_{\mathfrak{g}_1}) = [\phi(x), \phi(y)]_{\mathfrak{g}_2}$, where $[\cdot, \cdot]_{\mathfrak{g}_i}$ denotes the Lie bracket of \mathfrak{g}_i , $i = 1, 2$, respectively. By a *Lie algebra representation* of a vector space V we mean any Lie algebra homomorphism $\rho := \mathfrak{g} \rightarrow \text{gl}(V)$, where $\text{gl}(V)$ denotes the vector space of all linear mappings from V to V (over the same field which \mathfrak{g} is over and with composition as the "addition" of vectors). It can be shown that $\text{gl}(V)$ is indeed a Lie algebra with Lie bracket given by $[f, g] = f \circ g - g \circ f$.

Definition 2.2.17. *Define $\text{ad} : \mathfrak{g} \rightarrow \text{gl}(\mathfrak{g})$ by $x \mapsto \text{adx}$ where $(\text{adx})(y) := [x, y]$. The map ad is called the adjoint representation.*

From the bilinearity of the Lie bracket, the map adx is linear on \mathfrak{g} . Let $x, y \in \mathfrak{g}$ and

$c \in \mathbb{R}$ and define $w = cx + y$ and observe

$$\begin{aligned}
 \text{ad}(w)(z) &= (\text{ad}w)(z) \\
 &= [cx + y, z] \\
 &= c[x, z] + [y, z], \text{ by the Jacobi identity,} \\
 &= c(\text{ad}x)(z) + (\text{ad}y)(z) \\
 &= c\text{ad}(x)(z) + \text{ad}(y)(z),
 \end{aligned}$$

showing ad to be linear. Now let $x, y, z \in \mathfrak{g}$ and observe

$$\begin{aligned}
 \text{ad}([x, y])(z) &= [[x, y], z] \\
 &= [x, [y, z]] + [y, [z, x]], \text{ by the Jacobi identity,} \\
 &= [x, [y, z]] - [y, [x, z]] \\
 &= (\text{ad}x)(\text{ad}y)(z) - (\text{ad}y)(\text{ad}x)(z) \\
 &= [\text{ad}(x), \text{ad}(y)](z).
 \end{aligned}$$

This shows ad to be a Lie algebra homomorphism.

Definition 2.2.18. *Let tr be the usual trace of a linear mapping on a vector space (or Lie algebra \mathfrak{g}) to itself. We define the Killing form of \mathfrak{g} to be $K(x, y) := \text{tr}(\text{ad}x \circ \text{ad}y)$, on $\mathfrak{g} \times \mathfrak{g}$. Note that if A and B are the matrix representations of $\text{ad}x$ and $\text{ad}y$ respectively, then $K(x, y) = \text{Tr}(AB)$, where Tr is the usual trace of a matrix.*

The Killing form is symmetric since $\text{Tr}(AB) = \text{Tr}(BA)$. Also, it is a bilinear form as the composition $\text{ad}x \circ \text{ad}y$ is bilinear and Tr is linear. That is, K is a symmetric bilinear form. A symmetric bilinear form B on $V \times V$, V a vector space, is said to be *non-degenerate* if there is no $v \in V$ with $v \neq 0$ such that $B(v, u) = 0$, for all $u \in V$. Now we are ready to state Cartan's Criterion for semi-simplicity, without proof. The proof can be found in Sattinger and Weaver [12] on page 128, or in de Graaf [2] on page 91.

Theorem 2.2.19. (Cartan's Criterion for semi-simplicity) *Let \mathfrak{g} be a Lie algebra. Then \mathfrak{g} is semi-simple iff the Killing form K of \mathfrak{g} is non-degenerate.*

We give an example to show how one might use this.

Example 2.2.20. Let \mathfrak{g} be an abstract Lie algebra with basis $\{e_1, e_2, e_3\}$ and structure equations $\{[e_1, e_2] = e_3, [e_1, e_3] = -e_2, [e_2, e_3] = e_1\}$. Here is the multiplication table on the basis:

$[\cdot, \cdot]$	e_1	e_2	e_3
e_1	\cdot	e_3	$-e_2$
e_2	$-e_3$	\cdot	e_1
e_3	e_2	$-e_1$	\cdot

Note from the table that this Lie algebra is nonabelian and neither nilpotent nor solvable since the derived algebra $\mathfrak{g}^{(1)} = \text{span}\{e_1, e_2, e_3\} = \mathfrak{g}$. Since the Killing form is a symmetric bilinear form, we may compute its matrix representation. First we'll compute the matrix representations of the linear maps ade_i . Then we'll use those to compute the Killing form on the basis elements, giving the columns of its matrix representation. Observe the following:

$$(\text{ade}_1)(e_2) = [e_1, e_2] = e_3$$

and

$$(\text{ade}_1)(e_3) = [e_1, e_3] = -e_2$$

and thus in matrix form

$$\text{ade}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Similarly, we find

$$\text{ade}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

and

$$\text{ade}_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then we have

$$K(e_1, e_1) = \text{Tr}(\text{ade}_1 \text{ade}_1) = -2$$

$$K(e_1, e_2) = \text{Tr}(\text{ade}_1 \text{ade}_2) = 0$$

$$K(e_1, e_3) = \text{Tr}(\text{ade}_1 \text{ade}_1) = 0$$

$$K(e_2, e_2) = \text{Tr}(\text{ade}_2 \text{ade}_2) = -2$$

$$K(e_2, e_3) = \text{Tr}(\text{ade}_2 \text{ade}_3) = 0,$$

and

$$K(e_3, e_3) = \text{Tr}(\text{ade}_3 \text{ade}_3) = -2.$$

Since the Killing form is symmetric we didn't have to calculate other combinations of the bases. Then the matrix representation for the Killing form is

$$K = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Let $\mathbf{v} = [v_1 \ v_2 \ v_3]$ and $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, written as a column vector. Then

$$(2.2) \quad K(v, u) = vKu = -2u_1v_1 - 2u_2v_2 - 2u_3v_3 = 0$$

if and only if $u_1v_1 + u_2v_2 + u_3v_3 = 0$. If we fix $\mathbf{u} = \mathbf{u}_0$ and let \mathbf{v} vary, then clearly (2.2) would imply $\mathbf{u} = \mathbf{0}$. That is to say, K is non-degenerate. Therefore, by Cartan's Criterion for semi-simplicity, \mathfrak{g} is semi-simple. Note that Maple gives the same result. We can initialize \mathfrak{g} and use the Query command,

```
> Eq := [[e1, e2] = e3, [e1, e3] = -e2, [e2, e3] = e1], [e1, e2, e3] :
> DGsetup(LieAlgebraData(Eq, g)) :
> Query(g, "Semisimple");
```

true

Not every Lie algebra can be classified as one of the following: abelian, nilpotent, solvable, or semi-simple. For those that cannot fit one of these descriptions and for the purposes of the database we'll call them *generic*.

This next definition provides a great way to classify Lie algebras and their subalgebras into two broad categories: the indecomposable kind and the decomposable kind.

Definition 2.2.21. Let \mathfrak{g} be a Lie algebra and $\mathfrak{h}_i \subset \mathfrak{g}$ be Lie subalgebras, for $i = 1, \dots, n$.

If

$$i) \mathfrak{g} = \{h_1 + h_2 + \dots + h_n \mid h_i \in \mathfrak{h}_i\} = \sum \mathfrak{h}_i,$$

$$ii) \mathfrak{h}_i \cap \mathfrak{h}_j = 0, \text{ whenever } i \neq j,$$

$$iii) [\mathfrak{h}_i, \mathfrak{h}_j] = 0, \text{ whenever } i \neq j,$$

then \mathfrak{g} is said to be the (internal) direct sum of the Lie algebras \mathfrak{h}_i , written $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \dots \oplus \mathfrak{h}_n$. A Lie algebra \mathfrak{g} is said to be indecomposable if it cannot be written as a direct sum of Lie algebras and decomposable if it can.

Example 2.2.22. Let \mathfrak{g} be an abstract Lie algebra with basis $\{e_1, e_2, e_3, e_4, e_5\}$ and structure equations $\{[e_1, e_3] = e_1, [e_1, e_4] = e_2, [e_2, e_3] = e_2, [e_2, e_4] = -e_1\}$. Here is the multiplication table for these equations:

$[\cdot, \cdot]$	e_1	e_2	e_3	e_4	e_5
e_1	\cdot	\cdot	e_1	e_2	\cdot
e_2	\cdot	\cdot	e_2	$-e_1$	\cdot
e_3	$-e_1$	$-e_2$	\cdot	\cdot	\cdot
e_4	$-e_2$	e_1	\cdot	\cdot	\cdot
e_5	\cdot	\cdot	\cdot	\cdot	\cdot

From this table we can see that if we define the following subalgebras $\mathfrak{h} = \text{span}\{e_1, e_2, e_3, e_4\}$ and $\mathfrak{k} = \text{span}\{e_5\}$ then clearly $\mathfrak{g} = \mathfrak{h} + \mathfrak{k}$, $\mathfrak{h} \cap \mathfrak{k} = 0$, and $[\mathfrak{h}, \mathfrak{k}] = 0$. Therefore $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$, making \mathfrak{g} decomposable.

Now, let \mathfrak{l} be the abstract Lie algebra given by the structure equations $\{[e_1, e_2] = -e_4, [e_1, e_3] = e_1, [e_2, e_4] = -e_1, [e_3, e_4] = -e_4, [e_3, e_5] = -e_5\}$ (noting that these are not the same e_i 's as for \mathfrak{g}) with multiplication table

$[\cdot, \cdot]$	e_1	e_2	e_3	e_4	e_5
e_1	\cdot	$-e_4$	e_1	\cdot	\cdot
e_2	e_4	\cdot	\cdot	$-e_1$	\cdot
e_3	$-e_1$	\cdot	\cdot	$-e_4$	$-e_5$
e_4	\cdot	e_1	e_4	\cdot	\cdot
e_5	\cdot	\cdot	e_5	\cdot	\cdot

This is less obvious as to whether it is decomposable or not. However, it can be shown that \mathfrak{l} is in fact indecomposable using an algorithm developed in Winternitz et al, [11]. Here are Maple's results:

```
> Eq1 := [[e1, e3] = e1, [e1, e4] = e2, [e2, e3] = e2, [e2, e4] = -e1],
```

```
    [e1, e2, e3, e4, e5] :
```

```
> DGsetup(LieAlgebraData(Eq1, g)) :
```

```
> Query(g, "Indecomposable");
```

false

```
> Eq2 := [[e1, e2] = -e4, [e1, e3] = e1, [e2, e4] = -e1, [e3, e4] = -e4,
```

```
    [e3, e5] = -e5], [e1, e2, e3, e4, e5] :
```

```
> DGsetup(LieAlgebraData(Eq2, l)) :
```

```
> Query(l, "Indecomposable");
```

true

We make a final note that $e_1 \in \mathfrak{g}$ does not equal $e_1 \in \mathfrak{l}$.

For classifying, the concept of indecomposable is used in two ways. First, a Lie algebra will have "Indecomposable" values *true* or *false*, referring to if the given Lie algebra is indecomposable (true) or not (false). Second, if a Lie algebra is decomposable, then we calculate the dimensions of the bases of each Lie algebra constituting the direct sum. For instance, for \mathfrak{g} in Example 2.2.22, the *decomposition dimensions* would be $[4, 1]$. And if the Lie algebra is indecomposable, the decomposition dimensions would be the same dimension of the Lie algebra itself. So in Example 2.2.22, the decomposition dimensions of \mathfrak{l} would be $[5]$.

We introduce another way to decompose a Lie algebra that is of interest. Let \mathfrak{g} be a Lie algebra. If \mathfrak{h} and \mathfrak{k} are Lie subalgebras of \mathfrak{g} such that $\mathfrak{g} = \mathfrak{h} + \mathfrak{k}$ and $[\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{h}$, then \mathfrak{g}

is said to be the *semi-direct sum* of \mathfrak{h} and \mathfrak{k} . This will be denoted by $\mathfrak{g} = \mathfrak{h} \oplus_s \mathfrak{k}$. Here is a powerful theorem with proof in Fulton and Harris [5], page 499.

Theorem 2.2.23. (Levi's Decomposition Theorem) *Let \mathfrak{g} be a finite-dimensional Lie algebra. Then $\mathfrak{g} = \text{Rad}(\mathfrak{g}) \oplus_s \mathfrak{s}$, where \mathfrak{s} is a semi-simple Lie subalgebra of \mathfrak{g} .*

Definition 2.2.24. *The decomposition of Theorem 2.2.23 will be called the Levi decomposition.*

It's obvious that if \mathfrak{g} is abelian, nilpotent, or solvable, we have $\mathfrak{s} = 0$, since $\text{Rad}(\mathfrak{g}) = \mathfrak{g}$. On the other hand, if \mathfrak{g} is semi-simple, then $\text{Rad}(\mathfrak{g}) = 0$. In either case, the Levi decomposition doesn't provide new insight into the structure of the Lie algebra. However, for generic Lie algebras it may be useful to compute the Levi decomposition and take notice of the semi-simple part. Let's see an example.

Example 2.2.25. Let \mathfrak{g} be the abstract Lie algebra with basis $\{e_1, e_2, e_3, e_4\}$ and structure equations $\{[e_1, e_2] = e_1, [e_1, e_3] = -2e_2, [e_2, e_3] = e_3\}$. Here is the multiplication table:

$[\cdot, \cdot]$	e_1	e_2	e_3	e_4
e_1	\cdot	e_1	$-2e_2$	\cdot
e_2	$-e_1$	\cdot	e_3	\cdot
e_3	$2e_2$	$-e_3$	\cdot	\cdot
e_4	\cdot	\cdot	\cdot	\cdot

Look at the block in the table given by e_1, e_2, e_3 . If we let $\mathfrak{s} = \text{span}\{e_1, e_2, e_3\}$, then $\mathfrak{s}^{(1)} = \mathfrak{s}$ and so \mathfrak{s} is not solvable. Note the Killing form for \mathfrak{s} :

$$K = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 2 & 0 \\ 4 & 0 & 0 \end{bmatrix}$$

Let $v = [x \ y \ z] \in \mathfrak{s}$ and fix $u_0 = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in \mathfrak{s}$, written as a column vector. Then

$$K(v, u) = vKu = 4u_1z + 2u_2y + 4u_3x = 0,$$

if and only if $2u_1z + u_2y + 2u_3x = 0$. From this we can determine that K is non-degenerate and by Cartan's Criterion for semi-simplicity, \mathfrak{s} is semi-simple. Then observe that if we let

$\mathfrak{r} = \text{span}\{e_4\}$ then $\mathfrak{r}^{(1)} = 0$ and so \mathfrak{r} is solvable. Now, any 3-dimensional subalgebras of \mathfrak{g} must then be semi-simple and any 1-dimensional subalgebras must be solvable. An exhaustive search in the 2-dimensional subalgebras will convince us that $\text{Rad}(\mathfrak{g}) = \mathfrak{r}$. Observe that $\mathfrak{g} = \mathfrak{r} + \mathfrak{s}$ and $\mathfrak{r} \cap \mathfrak{s} = 0$. Then we have $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$. But by definition, the radical is an ideal and therefore $[\mathfrak{r}, \mathfrak{s}] \subset \mathfrak{r}$, which can be seen from the multiplication table for \mathfrak{g} . Thus $\mathfrak{g} = \mathfrak{r} \oplus_{\mathfrak{s}} \mathfrak{s}$, and we have found the Levi decomposition of \mathfrak{g} . Maple returns precisely the same results:

```
> Eq := [[e1, e2] = e1, [e1, e3] = -2e2, [e2, e3] = e3], [e1, e2, e3, e4] :
> DGsetup(LieAlgebraData(Eq, g)) :
> LeviDecomposition(g);
```

$$[[e4], [e1, e2, e3]],$$

where the list returned from the command is always of the form $[radical, semi-simple\ part]$.

Now we move to another property used in the database.

Definition 2.2.26. *Let $\psi \in \text{gl}(\mathfrak{g})$, for a Lie algebra \mathfrak{g} . Then ψ is a derivation of \mathfrak{g} if $\psi([x, y]) = [\psi(x), y] + [x, \psi(y)]$, for all $x, y \in \mathfrak{g}$.*

It can be easily shown that the subset of all derivations in $\text{gl}(\mathfrak{g})$ is a Lie subalgebra, recalling that $\text{gl}(\mathfrak{g})$ is itself a Lie algebra with the commutator serving as the Lie bracket. Just observe that if ψ and ϕ are two derivations then $[\psi, \phi] = \psi\phi - \phi\psi$ gives

$$\begin{aligned} [\psi, \phi]([x, y]) &= (\psi\phi - \phi\psi)([x, y]) \\ &= \psi(\phi([x, y])) - \phi(\psi([x, y])) \\ &= \psi([\phi(x), y] + [x, \phi(y)]) - \phi([\psi(x), y] + [x, \psi(y)]) \\ &= \psi([\phi(x), y]) + \psi([x, \phi(y)]) - \phi([\psi(x), y]) - \phi([x, \psi(y)]) \\ &= [\psi(\phi(x)), y] + [\phi(x), \psi(y)] + [\psi(x), \phi(y)] \\ &\quad + [x, \psi(\phi(y))] - [\phi(\psi(x)), y] - [\psi(x), \phi(y)] \end{aligned}$$

$$\begin{aligned}
& -[\phi(x), \psi(y)] - [x, \phi(\psi(y))] \\
& = [(\psi\phi - \phi\psi)(x), y] + [x, (\psi\phi - \phi\psi)(y)],
\end{aligned}$$

showing $[\psi, \phi]$ to also be a derivation and therefore the set of all derivations in $\mathfrak{gl}(\mathfrak{g})$ is closed under the Lie bracket. Now we demonstrate how to calculate a basis for the derivations first by hand and second using Maple.

Example 2.2.27. Let \mathfrak{g} be an abstract Lie algebra with basis $\{e_1, e_2\}$ and structure equation $\{[e_1, e_2] = e_1\}$. To find the derivations of this Lie algebra, we suppose ψ is a derivation of \mathfrak{g} . Then

$$\begin{aligned}
\psi(e_1) &= a_1e_1 + a_2e_2 \\
\psi(e_2) &= a_3e_1 + a_4e_2,
\end{aligned}$$

for some a_i . Using the fact that we demand $\psi([e_1, e_2]) = [\psi(e_1), e_2] + [e_1, \psi(e_2)]$, we have

$$\begin{aligned}
\psi([e_1, e_2]) &= [\psi(e_1), e_2] + [e_1, \psi(e_2)] \\
\psi(e_1) &= [a_1e_1 + a_2e_2, e_2] + [e_1, a_3e_1 + a_4e_2] \\
a_1e_1 + a_2e_2 &= a_1[e_1, e_2] + a_2[e_2, e_2] + a_3[e_1, e_1] + a_4[e_1, e_2] \\
a_1e_1 + a_2e_2 &= a_1[e_1, e_2] + a_4[e_1, e_2] \\
a_1e_1 + a_2e_2 &= (a_1 + a_4)e_1
\end{aligned}$$

from which we see $a_2 = a_4 = 0$ and so $\psi(e_1) = a_1e_1$, $\psi(e_2) = a_3e_1$. Then

$$\psi = \begin{bmatrix} a_1 & a_3 \\ 0 & 0 \end{bmatrix}.$$

This is what a general derivation looks like for the 2-dimensional Lie algebra \mathfrak{g} . Then the set

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

would serve as a basis for the Lie subalgebra of derivations. Observe that Maple gives the same result:

$$> Eq := [[e1, e2] = e1], [e1, e2] :$$

```
> DGsetup(LieAlgebraData(Eq, g)) :
> Derivations(g, "Full");
```

$$\left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right]$$

The derivations can be quite tedious to compute in higher dimensions by hand, which is why we worked with a minimal case example here.

2.3 Isotropy Subalgebras

Now we wish to discuss properties more specific to Lie algebras of infinitesimal generators.

Definition 2.3.1. *The isotropy subgroup of a group action on a manifold M of a Lie group G is the set $G_p = \{g \mid gp = p\}$.*

It is almost trivial to show G_p is a subgroup of G . If G_p is a Lie subgroup, then we may compute its Lie algebra (the Lie algebra of left-invariant vector fields on G). This Lie algebra at the point p is important to us. To determine if G_p is indeed a Lie subgroup of G one may use the following theorem with a proof given in [1]:

Theorem 2.3.2. *An algebraic subgroup H of a Lie group G is closed as a subset of G if and only if H is a regular submanifold of G (and so is a Lie subgroup).*

In our case $H = G_p$, which is a closed subset of G . The Lie algebra Γ of left-invariant vector fields of the isotropy subgroup of a Lie group G is an example of the following important definition:

Definition 2.3.3. *The isotropy subalgebra of a Lie algebra Γ of vector fields at a point p is defined by $\Gamma_p = \{X \in \Gamma \mid X_p = 0\}$.*

Example 2.3.4. We'll work from Example 2.1.13, where

$$\Gamma = \text{span}\{\partial_y, \partial_z, -\partial_x + y\partial_y + z\partial_z, -z\partial_y + y\partial_z, \alpha\partial_x + \partial_w\},$$

for α a constant. Note that the isotropy subalgebra of Γ at 0 would be the span of $\{-z \partial_y + y \partial_z\}$. However, note that the isotropy subalgebra at $(1, 1, 1, 1)$ would be

$$\text{span}\{-z \partial_y + y \partial_z + \partial_y - \partial_z\} = \text{span}\{(1 - z) \partial_y + (y - 1) \partial_z\}.$$

Maple gives just that:

$$> \text{IsotropySubalgebra}(\Gamma, [x = 0, y = 0, z = 0, w = 0]);$$

$$[-zD_y + yD_z]$$

$$> \text{IsotropySubalgebra}(\Gamma, [x = 1, y = 1, z = 1, w = 1]);$$

$$[(-z + 1)D_y + (y - 1)D_z]$$

Again, assuming Γ has already been set up.

Suppose D is a basis to some vector subspace of a Lie algebra \mathfrak{g} . Then a *complementary basis* to D will be a set C in \mathfrak{g} such that $\text{span}\{D, C\} = \mathfrak{g}$. Maple has a command `ComplementaryBasis` which will compute just what we need (and more).

Definition 2.3.5. *Let \mathfrak{m} be a vector subspace of a Lie algebra \mathfrak{g} . Then if $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ for some Lie subalgebra \mathfrak{h} with the property that $[h, m] \in \mathfrak{m}$ for every $h \in \mathfrak{h}$ and $m \in \mathfrak{m}$, then \mathfrak{m} is said to be a reductive complement to \mathfrak{h} in \mathfrak{g} .*

In all our cases, however, we will desire to know whether or not a general complementary basis $C = \mathfrak{m}$ to the isotropy subalgebra $\Gamma_p = \mathfrak{h}$ of a Lie algebra of vector fields Γ is a reductive complement.

Example 2.3.6. Let Γ be the Lie algebra of vector fields given by the span of the following set:

$$X_1 = \cos(y) \partial_x + \tan(x) \sin(y) \partial_y,$$

$$X_2 = \partial_y,$$

$$X_3 = \sin(y) \partial_x - \cos(y) \tan(x) \partial_y.$$

Then the commutator table for Γ is:

$[\cdot, \cdot]$	X_1	X_2	X_3
X_1	\cdot	X_3	$-X_2$
X_2	$-X_3$	\cdot	X_1
X_3	X_2	$-X_1$	\cdot

Observe that the isotropy subalgebra of Γ at 0 is the span of $\Gamma_0 = \{X_3\}$. Then the general complement is $C = \{X_1 + a_1 X_3, X_2 + a_2 X_3\}$. Observe the multiplication table for $[\Gamma_0, C]$:

$[\cdot, \cdot]$	$X_1 + a_1 X_3$	$X_2 + a_2 X_3$
X_3	X_2	$-X_1$

If C is a reductive complement to Γ_0 , then $[\Gamma_0, C] \subset \text{span}(C)$. Then we suppose $-X_1, X_2 \in \text{span}(C)$ and solve for a_1 and a_2 :

$$-X_1 = r(X_1 + a_1 X_3) + s(X_2 + a_2 X_3)$$

$$X_2 = t(X_1 + a_1 X_3) + u(X_2 + a_2 X_3),$$

from which it follows $a_1 = a_2 = 0$. Therefore $C = \text{span}\{X_1, X_2\}$ and is a reductive complement to the isotropy subalgebra Γ_0 . Here is Maple's output:

```
> DGsetup(LieAlgebraData(Gamma, alg)) :
> Gamma := IsotropySubalgebra(Gamma, [x = 0, y = 0, z = 0, w = 0], output = [alg]);
```

[e3]

```
> C := ComplementaryBasis(Gamma, [e1, e2, e3]);
```

[e1, e2]

```
> Query(Gamma, C, "ReductivePair");
```

true

Note that e_1 , e_2 and e_3 refer to the first, second and third vector fields in Γ .

Here is another property related to the idea of a reductive complement which we need.

Definition 2.3.7. *Suppose the subspace \mathfrak{m} of a Lie algebra \mathfrak{g} is a reductive complement to a subalgebra \mathfrak{h} . If $[\mathfrak{m}, \mathfrak{m}] \subset \text{span}(\mathfrak{h})$ then \mathfrak{m} is said to be a symmetric complement to \mathfrak{h} .*

Our interest in the symmetric complement lies only the case where \mathfrak{m} is the complementary basis to the isotropy subalgebra (\mathfrak{h}).

Example 2.3.8. We continue with the Lie algebra of vector fields Γ from Example 2.3.6. All we must check here is the condition $[C, C] \subset \text{span}(\Gamma_0)$; we've already shown that the complementary basis $C = \{X_1, X_2\}$ to the isotropy subalgebra $\Gamma_0 = \{X_3\}$ of Γ is a reductive complement. Observe the multiplication table for $[C, C]$:

$$\begin{array}{c|cc} [\cdot, \cdot] & X_1 & X_2 \\ \hline X_1 & \cdot & X_3 \\ X_2 & -X_3 & \cdot \end{array}$$

Then clearly $[C, C] \subset \text{span}(\Gamma_0)$, making $\mathfrak{m} = \{X_1, X_2\}$ a symmetric complement to Γ_0 . Maple returns the same result:

`> Query(Γ_0 , m , "SymmetricPair");`

true

noting here we are assuming Maple has the input from Example 2.3.6.

Recall that for a Lie algebra \mathfrak{g} over a vector space V , a *Lie algebra representation* for \mathfrak{g} is a Lie algebra homomorphism $\rho : \mathfrak{g} \rightarrow \text{gl}(V)$, where $\text{gl}(V)$ denotes the set of all endomorphisms on V . Let Γ be any finite-dimensional Lie algebra of infinitesimal generators of a smooth action on an n -dimensional smooth manifold M . Let the isotropy subalgebra be Γ_p , for some point $p \in M$. Define $A : \Gamma_p \rightarrow \text{gl}(T_p M)$ to be given by $X \mapsto A(X)$, with $A(X)(Y_p) = [X, Y]|_p$, where $Y \in \mathfrak{X}(M)$ is a smooth vector field such that $Y(p) = Y_p$. The

map A is called the *linear infinitesimal isotropy representation* of Γ_p . Let $X \in \Gamma_p$ and $Y \in \mathfrak{X}(M)$ such that $Y(p) = Y_p$, and observe that

$$\begin{aligned} A(X)(Y_p) &= [X, Y]|_p \\ &= \sum_{i,j=1}^n X^j \frac{\partial Y^i}{\partial x^j} \Big|_p - Y^j \frac{\partial X^i}{\partial x^j} \Big|_p \\ &= \sum_{i,j=1}^n -Y^j \frac{\partial X^i}{\partial x^j} \Big|_p \end{aligned}$$

noting $X^j \frac{\partial Y^i}{\partial x^j} \Big|_p = 0$ since $X \in \Gamma_p$. The linear infinitesimal isotropy representation can then be given by the matrix transformation

$$A(X) = - \left(\frac{\partial X^i}{\partial x^j} \Big|_p \right).$$

Note that it is indeed a representation as it is a Lie algebra homomorphism (noting that the commutator of matrices, $[A, B] = AB - BA$, makes $\mathfrak{gl}(T_p M)$ a Lie algebra) since

$$(2.3) \quad A([X, Y]) = \left(\frac{\partial [X, Y]^i}{\partial x^j} \Big|_p \right)$$

$$(2.4) \quad = \left(\frac{\partial}{\partial x^j} \sum_{k=1}^n \left(X^k \frac{\partial Y^i}{\partial x^k} - Y^k \frac{\partial X^i}{\partial x^k} \right) \Big|_p \right)$$

$$(2.5) \quad = \left(\sum_{k=1}^n \frac{\partial}{\partial x^j} \left(X^k \frac{\partial Y^i}{\partial x^k} - Y^k \frac{\partial X^i}{\partial x^k} \right) \Big|_p \right)$$

$$(2.6) \quad = \left(\sum_{k=1}^n \left(\frac{\partial}{\partial x^j} \left(X^k \frac{\partial Y^i}{\partial x^k} \right) - \frac{\partial}{\partial x^j} \left(Y^k \frac{\partial X^i}{\partial x^k} \right) \right) \Big|_p \right)$$

$$(2.7) \quad = \left(\sum_{k=1}^n \left(\frac{\partial X^k}{\partial x^j} \frac{\partial Y^i}{\partial x^k} + X^k \frac{\partial}{\partial x^j} \frac{\partial Y^i}{\partial x^k} - \frac{\partial Y^k}{\partial x^j} \frac{\partial X^i}{\partial x^k} - Y^k \frac{\partial}{\partial x^j} \frac{\partial X^i}{\partial x^k} \right) \Big|_p \right)$$

$$(2.8) \quad = \left(\sum_{k=1}^n \left(\frac{\partial X^k}{\partial x^j} \frac{\partial Y^i}{\partial x^k} - \frac{\partial Y^k}{\partial x^j} \frac{\partial X^i}{\partial x^k} \right) \Big|_p \right)$$

$$(2.9) \quad = \left(\frac{\partial X^k}{\partial x^j} \Big|_p \right) \left(\frac{\partial Y^i}{\partial x^k} \Big|_p \right) - \left(\frac{\partial Y^k}{\partial x^j} \Big|_p \right) \left(\frac{\partial X^i}{\partial x^k} \Big|_p \right)$$

$$(2.10) \quad = [A(X), A(Y)],$$

where $[A(X), A(Y)]$ is the matrix commutator and equation (2.9) denotes matrix multiplication. Clearly then we get a matrix $A_X = A(X)$ for each $X \in \Gamma_p$ and since A is a Lie

algebra homomorphism, $A(\Gamma_p)$ is a Lie subalgebra of $\mathfrak{gl}(T_p M)$. However, if we suppose further that $X \in \Gamma_p$ is also a Killing vector for a Lorentz metric g , we see from the Lie derivative

$$\mathcal{L}_{X_p}(g_p) = \sum_{i,j=1}^n \left(\sum_{k=1}^n g_{ik}(\mathbf{x}) \Big|_p \frac{\partial X^k}{\partial x^j} \Big|_p + g_{kj}(\mathbf{x}) \Big|_p \frac{\partial X^k}{\partial x^i} \Big|_p + X^k(\mathbf{x}) \Big|_p \frac{\partial g_{ij}}{\partial x^k} \Big|_p \right) dx^i dx^j = 0$$

and since $X \in \Gamma_p$, this becomes

$$\mathcal{L}_{X_p}(g_p) = \sum_{i,j=1}^n \left(\sum_{k=1}^n g_{ik}(\mathbf{x}) \Big|_p \frac{\partial X^k}{\partial x^j} \Big|_p + g_{kj}(\mathbf{x}) \Big|_p \frac{\partial X^k}{\partial x^i} \Big|_p \right) dx^i dx^j = 0.$$

Choose coordinates on M such that $\eta = \left(g_{ij}(\mathbf{x}) \Big|_p \right) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and we see that

$$\eta \left(\frac{\partial X^i}{\partial x^j} \Big|_p \right)^t + \left(\frac{\partial X^i}{\partial x^j} \Big|_p \right) \eta = -\eta A(X)^t - A(X) \eta = 0.$$

and therefore $A(X) \in \{A \in \mathfrak{gl}(T_p M) \mid \eta A^t + A \eta = 0\} = \mathfrak{so}(3,1)$, the tangent algebra of the Lie group $SO(3,1)$, called the *Lorentz Lie algebra*. Thus $A(\Gamma_p)$ is a Lie subalgebra of $\mathfrak{so}(3,1)$.

Now, the subalgebras of $\mathfrak{so}(3,1)$ have been classified by Patera, Winternitz, and Zassenhaus in [9]. Thus the image $A(\Gamma_p)$ gives an identification of the isotropy subalgebra Γ_p with a subalgebra of the Lorentz Lie algebra $\mathfrak{so}(3,1)$. Note the following standard basis for $\mathfrak{so}(3,1)$:

$$Rx = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad Ry = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad Rz = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Kx = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad Ky = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad Kz = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

And here are the subalgebras given in [9]:

$$F1 : \{2Rz, -2Kz, -Ry - Kx, Rx - Ky, Ry - Kx, Rx + Ky\}$$

$$F2 : \{2Rz, -2Kz, -Ry - Kx, Rx - Ky\}$$

$$F3 : \{Rx, Ry, Rz\}$$

$$F4 : \{Rz, Kx, Ky\}$$

$$F5 : \{\cos(\theta)Rz - \sin(\theta)Kz, -Ry - Kx, Rx - Ky\}$$

$$F6 : \{2Rz, -Ry - Kx, Rx - Ky\}$$

$$F7 : \{-2Kz, -Ry - Kx, Rx - Ky\}$$

$$F8 : \{-2Kz, -Ry - Kx\}$$

$$F9 : \{2Rz, -2Kz\}$$

$$F10 : \{-Ry - Kx, Rx - Ky\}$$

$$F11 : \{\cos(\theta)Rz - \sin(\theta)Kz\}$$

$$F12 : \{Rz\}$$

$$F13 : \{Kz\}$$

$$F14 : \{Ry + Kz\}$$

and we let $F15$ denote the trivial isotropy subalgebra. We now have the following definition.

Definition 2.3.9. *The isotropy type at the point p of a Lie algebra of infinitesimal isometries Γ is the identification of the isotropy subalgebra Γ_p under the linear infinitesimal isotropy representation with a subalgebra of $\mathfrak{so}(3,1)$, as classified by Patera, Winternitz, and Zassenhaus in [9], and as labeled above (i.e. $F1, \dots, F15$).*

Example 2.3.10. Let Γ be the Lie algebra of infinitesimal isometries

$$\Gamma = \text{span}\{\partial_x, \partial_z, -\partial_x + y\partial_y, -\partial_x + z\partial_z + \partial_w, -e^{-x}\partial_x + e^{w-x}\partial_z + e^{-x}\partial_w\}.$$

Observe that at $p = (0, 0, 0, 0)$ we have $\Gamma_p = \{y\partial_y, (e^{-x} - 1)\partial_x + (-e^{w-x} + z + 1)\partial_z + (-e^{-x} + 1)\partial_w\}$. Let $X_1 = y\partial_y$ and $X_2 = (e^{-x} - 1)\partial_x + (-e^{w-x} + z + 1)\partial_z + (-e^{-x} + 1)\partial_w$.

Then clearly

$$A_{X_1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then observe for $X_2 = (e^{-x} - 1, 0, -e^{w-x} + z + 1, -e^{-x} + 1)$ we have

$$\begin{aligned} \frac{\partial X_2^1}{\partial x} \Big|_p &= -1, \quad \frac{\partial X_2^2}{\partial x} \Big|_p = 0, \quad \frac{\partial X_2^3}{\partial x} \Big|_p = 1, \quad \frac{\partial X_2^4}{\partial x} \Big|_p = 1, \\ \frac{\partial X_2^1}{\partial y} \Big|_p &= 0, \quad \frac{\partial X_2^2}{\partial y} \Big|_p = 0, \quad \frac{\partial X_2^3}{\partial y} \Big|_p = 0, \quad \frac{\partial X_2^4}{\partial y} \Big|_p = 0, \\ \frac{\partial X_2^1}{\partial z} \Big|_p &= 0, \quad \frac{\partial X_2^2}{\partial z} \Big|_p = 0, \quad \frac{\partial X_2^3}{\partial z} \Big|_p = 1, \quad \frac{\partial X_2^4}{\partial z} \Big|_p = 0, \\ \frac{\partial X_2^1}{\partial w} \Big|_p &= 0, \quad \frac{\partial X_2^2}{\partial w} \Big|_p = 0, \quad \frac{\partial X_2^3}{\partial w} \Big|_p = -1, \quad \frac{\partial X_2^4}{\partial w} \Big|_p = 0, \end{aligned}$$

and thus

$$A_{X_2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix},$$

noting the sign changes. Note that we have $A(\Gamma_p) = \{A_{X_1}, A_{X_2}\}$, a two dimensional Lie subalgebra of $\mathfrak{so}(3,1)$. Then $A(\Gamma_p)$ must be $F8$, $F9$, or $F10$ as these are the only two-dimensional subalgebras of $\mathfrak{so}(3,1)$. We now determine some properties of the three subalgebras $F8$, $F9$, and $F10$ and compare them with those of $A(\Gamma_p)$ to determine which subalgebra it is.

Clearly $A_{X_1}A_{X_2} = A_{X_2}A_{X_1}$ and thus $A(\Gamma_p)$ is Abelian. But $F8$ is non-Abelian while $F9$ and

$F10$ are. That is, $A(\Gamma_p)$ is either $F9$ or $F10$. However, observe that $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector of A_{X_1} and A_{X_2} associated to the eigenvalues $\lambda_{A_{X_1}} = 0$ and $\lambda_{A_{X_2}} = -1$, respectively. That is, A_{X_1} and A_{X_2} have a common eigenvector. A little computation shows that $2Rz$ and $-2Kz$ do not share a common eigenvector while $-Ry - Kx$ and $Rx - Ky$ do. In other

words, $A(\Gamma_p)$ can only be the subalgebra $F10 : \{-Ry - Kx, Rx - Ky\}$.

Indeed this is what Maple returns as well:

```
> P := [x = 0, y = 0, z = 0, w = 0] :
> IsotropySubalgebra(L, P, output = ["Vector", "Representation"]);
```

$$[y \partial_y, (e^{-x} - 1) \partial_x + (-e^{w-x} + z + 1) \partial_z + (-e^{-x} + 1) \partial_w],$$

$$\left[\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \right]$$

```
> IsotropyType(L, P);
```

“F10”

Now, let g be a metric tensor on an n -dimensional differentiable manifold M and $p \in M$. Suppose $U \subset T_p M$ is a k -dimensional subspace. Let $\{Y_1, Y_2, \dots, Y_k\}$ be a basis for U and q be the $k \times k$ matrix with components $q_{ij} = g_p(Y_i, Y_j)$, where $i, j = 1 \dots k$.

Definition 2.3.11. *The subspace U as given above is said to be null if $\det(q) = 0$. In other words, U is null if q is degenerate and we will say the subspace type of U is null. We say the subspace type of U is non-null otherwise.*

Remark 2.3.12. We are interested in the case when the subspace $U \subset T_p M$ is generated by a subset Λ of the Lie algebra Γ of infinitesimal isometries of the metric g such that $U = \text{span}(\Lambda) = \text{span}(\Gamma)$.

Example 2.3.13. We'll take as an example of calculating this property a metric tensor and its Killing vectors from Petrov found in [10]. Let $p = (x, y, z, w)$ and

$$g = dx dw + w \cos^2 z dy dy + \sin(z) dy dw + w dz dz + dw dx + \sin z dw dy,$$

labeled by Petrov as (32.25), with $\varepsilon = 1$ and $\alpha(x^4) = x^4$. Note the matrix representation

$$g = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & w \cos^2 z & 0 & \sin z \\ 0 & 0 & w & 0 \\ 1 & \sin z & 0 & 0 \end{pmatrix}.$$

The Killing vectors of this metric as given by Petrov are

$$\Gamma = \left\{ \begin{aligned} &\partial_y, \sec(z) \cos(y) \partial_x - \tan(z) \cos(y) \partial_y + \sin(y) \partial_z, \\ &-\sec(z) \sin(y) \partial_x + \tan(z) \sin(y) \partial_y + \cos(y) \partial_z, \partial_x \end{aligned} \right\}.$$

We see that a basis of the subspace U spanned by Γ is

$$\Lambda = \left\{ \begin{aligned} &Y_1 = \partial_y, \\ &Y_2 = \sec(z) \cos(y) \partial_x - \tan(z) \cos(y) \partial_y + \sin(y) \partial_z, \\ &Y_3 = -\sec(z) \sin(y) \partial_x + \tan(z) \sin(y) \partial_y + \cos(y) \partial_z \end{aligned} \right\}.$$

Then to find the matrix q defined on U induced by g , we perform the following:

$$\begin{aligned} q_{11} = g(Y_1, Y_1) &= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}^T \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & w \cos^2 z & 0 & \sin z \\ 0 & 0 & w & 0 \\ 1 & \sin z & 0 & 0 \end{pmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \\ &= w (\cos(z))^2 \\ q_{12} = g(Y_1, Y_2) &= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}^T \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & w \cos^2 z & 0 & \sin z \\ 0 & 0 & w & 0 \\ 1 & \sin z & 0 & 0 \end{pmatrix} \begin{bmatrix} \frac{\cos(y)}{\cos(z)} \\ -\frac{\sin(z) \cos(y)}{\cos(z)} \\ \sin(y) \\ 0 \end{bmatrix} \\ &= -\sin(z) \cos(y) w \cos(z) \end{aligned}$$

$$q_{13} = g(Y_1, Y_3) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}^T \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & w \cos^2 z & 0 & \sin z \\ 0 & 0 & w & 0 \\ 1 & \sin z & 0 & 0 \end{pmatrix} \begin{bmatrix} -\frac{\sin(y)}{\cos(z)} \\ \frac{\sin(z) \sin(y)}{\cos(z)} \\ \cos(y) \\ 0 \end{bmatrix}$$

$$= \sin(z) \sin(y) w \cos(z)$$

$$q_{22} = g(Y_2, Y_2) = \begin{bmatrix} \frac{\cos(y)}{\cos(z)} \\ -\frac{\sin(z) \cos(y)}{\cos(z)} \\ \sin(y) \\ 0 \end{bmatrix}^T \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & w \cos^2 z & 0 & \sin z \\ 0 & 0 & w & 0 \\ 1 & \sin z & 0 & 0 \end{pmatrix} \begin{bmatrix} \frac{\cos(y)}{\cos(z)} \\ -\frac{\sin(z) \cos(y)}{\cos(z)} \\ \sin(y) \\ 0 \end{bmatrix}$$

$$= (\sin(z))^2 (\cos(y))^2 w + (\sin(y))^2 w$$

$$q_{23} = g(Y_2, Y_3) = \begin{bmatrix} \frac{\cos(y)}{\cos(z)} \\ -\frac{\sin(z) \cos(y)}{\cos(z)} \\ \sin(y) \\ 0 \end{bmatrix}^T \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & w \cos^2 z & 0 & \sin z \\ 0 & 0 & w & 0 \\ 1 & \sin z & 0 & 0 \end{pmatrix} \begin{bmatrix} -\frac{\sin(y)}{\cos(z)} \\ \frac{\sin(z) \sin(y)}{\cos(z)} \\ \cos(y) \\ 0 \end{bmatrix}$$

$$= -(\sin(z))^2 \sin(y) \cos(y) w + \cos(y) \sin(y) w$$

$$q_{33} = g(Y_3, Y_3) = \begin{bmatrix} -\frac{\sin(y)}{\cos(z)} \\ \frac{\sin(z) \sin(y)}{\cos(z)} \\ \cos(y) \\ 0 \end{bmatrix}^T \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & w \cos^2 z & 0 & \sin z \\ 0 & 0 & w & 0 \\ 1 & \sin z & 0 & 0 \end{pmatrix} \begin{bmatrix} -\frac{\sin(y)}{\cos(z)} \\ \frac{\sin(z) \sin(y)}{\cos(z)} \\ \cos(y) \\ 0 \end{bmatrix}$$

$$= (\sin(z))^2 (\sin(y))^2 w + (\cos(y))^2 w.$$

Note that it is not necessary to compute all combinations as g is symmetric. Then the matrix q is

$$q = \begin{bmatrix} w(\cos(z))^2 & -\sin(z)\cos(y)w\cos(z) & \sin(z)\sin(y)w\cos(z) \\ * & (\sin(z))^2(\cos(y))^2w+(\sin(y))^2w & -(\sin(z))^2\sin(y)\cos(y)w+\cos(y)\sin(y)w \\ * & * & (\sin(z))^2(\sin(y))^2w+(\cos(y))^2w \end{bmatrix},$$

where it is understood that q is a symmetric matrix. Now we need only compute the determinant of q . Observe that $\det(q) = 0$. Therefore q is degenerate and thus $\text{span}(U)$ is a null subspace. That is, $\text{span}(U)$ has subspace type null. Maple simplifies this procedure greatly:

$$\begin{aligned} &> \text{SubspaceType}(g, \Gamma); \\ &\quad 3, \text{“Null”} \end{aligned}$$

where 3 represents the dimension of the subspace U and we are assuming g and Γ have been input already. Note that we only needed to input Γ and we didn't need to find U ourselves.

In the next chapter we will demonstrate the calculation of these properties for a particular example from the database.

CHAPTER 3
EXAMPLE FROM DATABASE

We'll verify the computations comprising entry [33, 19] in the database (see appendix for database). We illustrate how the properties were used and how they are presented in the database.

Here are the vector fields of [33, 19],

$$\begin{aligned} X_1 &= \partial_{x^2}, \\ X_2 &= \cos(x^2) \partial_{x^1} - \frac{\cos(x^1) \sin(x^2)}{\sin(x^1)} \partial_{x^2} + \frac{\sin(x^2)}{\sin(x^1)} \partial_{x^3}, \\ X_3 &= -\sin(x^2) \partial_{x^1} - \frac{\cos(x^1) \cos(x^2)}{\sin(x^1)} \partial_{x^2} + \frac{\cos(x^2)}{\sin(x^1)} \partial_{x^3}, \\ X_4 &= \partial_{x^3}, \\ X_5 &= \partial_{x^4}, \end{aligned}$$

With a little work it is easily seen that the non-trivial Lie brackets are as follows:

$$\begin{aligned} [X_1, X_2] &= -\sin x^2 \partial_{x^1} - \cot x^1 \cos x^2 \partial_{x^2} + \csc x^1 \cos x^2 \partial_{x^3} = X_3 \\ [X_1, X_3] &= -\cos x^2 \partial_{x^1} + \cot x^1 \sin x^2 \partial_{x^2} - \csc x^1 \sin x^2 \partial_{x^3} = -X_2 \\ [X_2, X_3] &= \partial_{x^2} = X_1, \end{aligned}$$

which we show in a multiplication table

$[\cdot, \cdot]$	X_1	X_2	X_3	X_4	X_5
X_1	·	X_3	$-X_2$	·	·
X_2	$-X_3$	·	X_1	·	·
X_3	X_2	$-X_1$	·	·	·
X_4	·	·	·	·	·
X_5	·	·	·	·	·

Let $\Gamma = \text{span}_{\mathbb{R}}\{X_1, X_2, X_3, X_4, X_5\}$, the Lie algebra of Killing vectors. However, we'll also view Γ as a five-dimensional abstract Lie algebra with basis $\{e_1, e_2, e_3, e_4, e_5\}$ and structure equations $\{[e_1, e_2] = e_3, [e_1, e_3] = -e_2, [e_2, e_3] = e_1\}$, and denote it \mathfrak{g} . Obviously the multiplication table of \mathfrak{g} is

$[\cdot, \cdot]$	e_1	e_2	e_3	e_4	e_5
e_1	\cdot	e_3	$-e_2$	\cdot	\cdot
e_2	$-e_3$	\cdot	e_1	\cdot	\cdot
e_3	e_2	$-e_1$	\cdot	\cdot	\cdot
e_4	\cdot	\cdot	\cdot	\cdot	\cdot
e_5	\cdot	\cdot	\cdot	\cdot	\cdot

Observe that $\mathfrak{g}^{(1)} = \text{span}\{e_1, e_2, e_3\}$. Then by the table for \mathfrak{g} , $\mathfrak{g}^{(2)} = \text{span}\{e_1, e_2, e_3\} = \mathfrak{g}^{(3)} = \dots$. From this it follows that \mathfrak{g} is not solvable (hence not nilpotent).

We'll see next if \mathfrak{g} is semi-simple or not. For that we look at the Killing form on \mathfrak{g} and use Cartan's Criterion for Semi-simplicity. Note the following,

$$\text{ade}_1(e_2) = [e_1, e_2] = e_3$$

$$\text{ade}_1(e_3) = [e_1, e_3] = -e_2$$

$$\text{ade}_2(e_3) = [e_2, e_3] = e_1,$$

which is enough to give the matrix representations of the adjoint representation:

$$\text{ade}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{ade}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{ade}_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{ade}_4 = \text{ade}_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We next calculate

$$K(e_1, e_1) = \text{Tr}(\text{ade}_1 \text{ade}_1) = -2$$

$$K(e_1, e_2) = \text{Tr}(\text{ade}_1 \text{ade}_2) = 0$$

$$K(e_1, e_3) = \text{Tr}(\text{ade}_1 \text{ade}_3) = 0$$

$$K(e_2, e_2) = \text{Tr}(\text{ade}_2 \text{ade}_2) = -2$$

$$K(e_2, e_3) = \text{Tr}(\text{ade}_2 \text{ade}_3) = 0$$

$$K(e_3, e_3) = \text{Tr}(\text{ade}_3 \text{ade}_3) = -2$$

giving

$$(3.1) \quad K = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that $\det(K) = 0$, showing K to be degenerate. Then by Cartan's Criterion for Semi-simplicity, \mathfrak{g} is *not* semi-simple. Since \mathfrak{g} is non-Abelian, non-nilpotent, non-solvable, and non-semi-simple, it is a generic Lie algebra.

Now, obviously we have $\dim(\mathfrak{g}^{(0)}) = 5$. As noted previously, $\mathfrak{g}^{(1)} = \mathfrak{g}^{(2)} = \mathfrak{g}^{(3)} = \dots = \text{span}(e_1, e_2, e_3)$ and thus $\dim(\mathfrak{g}^{(1)}) = \dim(\mathfrak{g}^{(2)}) = \dim(\mathfrak{g}^{(3)}) = \dots = 3$. Then the sequence of the dimensions of the derived series is $5, 3, 3, 3, \dots$, which is simply stored in the database as $[5, 3, 3]$.

Exploring further, we see from the multiplication table for \mathfrak{g} that if $\mathfrak{h} = \text{span}(e_1, e_2, e_3)$ and $\mathfrak{l} = \text{span}(e_4, e_5)$, then $\mathfrak{g} = \mathfrak{h} + \mathfrak{l}$, $\mathfrak{h} \cap \mathfrak{l} = \emptyset$, and $[\mathfrak{h}, \mathfrak{l}] = 0$. That is to say, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{l}$, and so \mathfrak{g} is decomposable with the dimensions of the decomposition labeled as $[3, 2]$. Note that \mathfrak{h} is semi-simple by Cartan's Criterion for Semi-simplicity since the killing form is

$$K_{\mathfrak{h}} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

with $\det(K_{\mathfrak{h}}) = -8$. Note that \mathfrak{l} is Abelian (and thus nilpotent and solvable) and is a likely candidate for the radical and nilradical. Since $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{l}$, with \mathfrak{h} semi-simple, we know the radical must be non-trivial (since \mathfrak{l} is non-trivial and nilpotent) and thus is not contained in \mathfrak{h} . Then the radical or nilradical are each at most two-dimensional and contained in \mathfrak{l} . Since $\dim(\mathfrak{l}) = 2$ with \mathfrak{l} nilpotent (noting the nilradical and radical are unique), we conclude

that $\text{Rad}(\mathfrak{g}) = \text{Nil}(\mathfrak{g}) = \mathfrak{l}$. Note in this case $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{l} = \mathfrak{h} \oplus_s \text{Rad}(\mathfrak{g})$, a Levi-decomposition. In addition, observe that $\text{Nil}(\mathfrak{g}) = \mathfrak{l} = \{e_4\} \oplus \{e_5\}$ and thus the nilradical is decomposable and two-dimensional.

Next, we wish to compute the Lie algebra of derivations associated with \mathfrak{g} . Suppose we have a derivation ϕ of \mathfrak{g} . Then $\phi(e_i) = \sum_{j=1}^5 a_{ij}e_j$, for some a_{ij} . We solve for the a_{ij} under the condition that $\phi([e_k, e_l]) = [\phi(e_k), e_l] + [e_k, \phi(e_l)]$. From these we create a matrix and find a basis for the space of all matrices of that form. Here we will only present the results of those computations:

$$\begin{aligned}
 w_1 &= \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, w_2 = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, w_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
 w_4 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, w_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, w_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
 w_7 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.
 \end{aligned}$$

From this we see that the derivations form a seven-dimensional Lie algebra (under the matrix commutator), denoted $\mathcal{D}(\mathfrak{g})$, with structure equations

$$[w_1, w_2] = w_3, [w_1, w_3] = -w_2, [w_2, w_3] = w_1, [w_4, w_5] = -w_5,$$

$$[w_4, w_6] = w_6, [w_5, w_6] = -w_4 + w_7, [w_5, w_7] = -w_5, [w_6, w_7] = w_6,$$

and basis $\{w_1, w_2, w_3, w_4, w_5, w_6, w_7\}$. It can be shown that $\text{Rad}(\mathcal{D}(\mathfrak{g})) = \text{span}(w_4 + w_7)$.

Since $\text{Rad}(\mathcal{D}(\mathfrak{g}))$ is a one-dimensional Lie subalgebra, a decomposition of the radical is not

necessary. This situation is reflected in the database by simply storing an empty pair of braces $[\]$. It should be noted that we make the choice to not decompose Abelian Lie algebras regardless of dimension and store an empty pair of braces $[\]$ in those cases as well.

Note, however, that as a Lie algebra $\mathcal{D}(\mathfrak{g})$ has associated with it a Lie algebra of derivations. Our interest in the “derivations of the derivations” only lies in its dimension. The same type of calculations for $\mathcal{D}(\mathfrak{g})$ shows that the derivations of $\mathcal{D}(\mathfrak{g})$ are the following

$$\begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

also a seven-dimensional Lie algebra.

Now we turn to the geometric properties of the vector fields Γ . Recall the vector fields comprising Γ :

$$\begin{aligned} X_1 &= \partial_{x^2}, \\ X_2 &= \cos(x^2) \partial_{x^1} - \frac{\cos(x^1) \sin(x^2)}{\sin(x^1)} \partial_{x^2} + \frac{\sin(x^2)}{\sin(x^1)} \partial_{x^3}, \\ X_3 &= -\sin(x^2) \partial_{x^1} - \frac{\cos(x^1) \cos(x^2)}{\sin(x^1)} \partial_{x^2} + \frac{\cos(x^2)}{\sin(x^1)} \partial_{x^3}, \\ X_4 &= \partial_{x^3}, \\ X_5 &= \partial_{x^4}. \end{aligned}$$

Observe that a linearly independent subset Λ of Γ such that $\text{span}(\Lambda) = \text{span}(\Gamma)$, using function coefficients, would be

$$\begin{aligned} X_1 &= \partial_{x^2}, \\ X_2 &= \cos(x^2) \partial_{x^1} - \frac{\cos(x^1) \sin(x^2)}{\sin(x^1)} \partial_{x^2} + \frac{\sin(x^2)}{\sin(x^1)} \partial_{x^3}, \\ X_3 &= -\sin(x^2) \partial_{x^1} - \frac{\cos(x^1) \cos(x^2)}{\sin(x^1)} \partial_{x^2} + \frac{\cos(x^2)}{\sin(x^1)} \partial_{x^3}, \\ X_5 &= \partial_{x^4}. \end{aligned}$$

That is to say, the orbit dimension is four.

Now consider the point $p = (\pi/2, 0, 0, 0)$. Then the isotropy subalgebra at p is

$$\begin{aligned} \Gamma_p &= \text{span}\{X_3 - X_4\} \\ &= \text{span}\left\{-\sin(x^2) \partial_{x^1} - \frac{\cos(x^1) \cos(x^2)}{\sin(x^1)} \partial_{x^2} + \frac{-\sin(x^1) + \cos(x^2)}{\sin(x^1)} \partial_{x^3}\right\} \end{aligned}$$

Now note that Γ_p is a one-dimensional Lie subalgebra. Then its isotropy type at p is $F11$, $F12$, $F13$, or $F14$. However, note the matrix representation A_X for the linear infinitesimal isotropy representation $A(X)$. Let X denote $X_3 - X_4$ and observe

$$\begin{aligned} \frac{\partial X^1}{\partial x^1} \Big|_p &= 0, & \frac{\partial X^2}{\partial x^1} \Big|_p &= 1, & \frac{\partial X^3}{\partial x^1} \Big|_p &= 0, & \frac{\partial X^4}{\partial x^1} \Big|_p &= 0, \\ \frac{\partial X^1}{\partial x^2} \Big|_p &= -1, & \frac{\partial X^2}{\partial x^2} \Big|_p &= 0, & \frac{\partial X^3}{\partial x^2} \Big|_p &= 0, & \frac{\partial X^4}{\partial x^2} \Big|_p &= 0, \\ \frac{\partial X^1}{\partial x^3} \Big|_p &= 0, & \frac{\partial X^2}{\partial x^3} \Big|_p &= 0, & \frac{\partial X^3}{\partial x^3} \Big|_p &= 0, & \frac{\partial X^4}{\partial x^3} \Big|_p &= 0, \\ \frac{\partial X^1}{\partial x^4} \Big|_p &= 0, & \frac{\partial X^2}{\partial x^4} \Big|_p &= 0, & \frac{\partial X^3}{\partial x^4} \Big|_p &= 0, & \frac{\partial X^4}{\partial x^4} \Big|_p &= 0, \end{aligned}$$

giving

$$A_X = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

noting the sign change. This is precisely the basis for the subalgebra of $\mathfrak{so}(3, 1)$ labeled as $F12$. That is to say, Γ_p has isotropy type $F12$ at p .

Next, we wish to know if the isotropy subalgebra and a complementary basis to the isotropy subalgebra form a reductive pair. Observe that a complementary basis to Γ_p (viewed as $\text{span}(e_3 - e_4)$) would be $\beta = \{e_1, e_2, e_3, e_5\}$. Let $\mathfrak{m} = \text{span}(\beta)$. Then note that

$$[e_3 - e_4, e_1] = [e_3, e_1] - [e_4, e_1] = e_2 - 0 = e_2 \in \mathfrak{m}$$

$$[e_3 - e_4, e_2] = [e_3, e_2] - [e_4, e_2] = -e_1 - 0 = -e_1 \in \mathfrak{m}$$

$$[e_3 - e_4, e_3] = [e_3, e_3] - [e_4, e_3] = 0 - 0 = 0 \in \mathfrak{m}$$

$$[e_3 - e_4, e_5] = [e_3, e_5] - [e_4, e_5] = 0 - 0 = 0 \in \mathfrak{m},$$

that is, $[\Gamma_p, \mathfrak{m}] \subset \mathfrak{m}$. Therefore \mathfrak{m} is a reductive complement to Γ_p , or in other words, Γ_p and \mathfrak{m} form a reductive pair.

As a reductive pair, it then makes sense to see if they form a symmetric pair. Just observe that $e_1, e_3 \in \mathfrak{m}$ but $[e_3, e_1] = e_2 \notin \Gamma_p$. That is, $[\mathfrak{m}, \mathfrak{m}] \not\subset \Gamma_p$ and therefore \mathfrak{m} is not a symmetric complement to Γ_p .

Lastly, we compute the subspace type of Γ . However, just how to do this was shown in the previous section and we will not go through the many computations here. But, Maple can compute it for us. Here is the metric as given by Petrov:

$$g = k_{11} dx^1 dx^1 + (k_{33} \cos^2 x^1 + k_{11} \sin^2 x^1) dx^2 dx^2 \\ + 2k_{33} \cos x^1 dx^2 dx^3 + k_{33} dx^3 dx^3 + k_{44} dx^4 dx^4$$

We place some assumptions on the constants showing up in the metric so Maple knows they are real:

$$> \text{SubspaceType}(g, \Gamma, [x1 = Pi/2, x2 = 0, x3 = 0, x4 = 0]) \\ \text{assuming } k_{44} > 0, k_{11} > 0, k_{33} > 0, k_{44} > 0;$$

4, “*Riemannian*”

Note that for our purposes, we are only concerned with the fact that the subspace type is non-null.

Here is a summary of the properties we have computed for [33, 19]:

- Algebra Type: Generic
- Derived Series Dimensions: [5, 3, 3]
- Decomposability of \mathfrak{g} : Decomposable
- Decomposability of the nilradical $\text{Nil}(\mathfrak{g})$: Decomposable
- Dimensions of the Decomposition of \mathfrak{g} : [3, 2]
- Orbit Dimension: 4
- Dimension of the Nilradical: 2
- Dimension of the Lie Algebra of Derivations \mathcal{D} of \mathfrak{g} : 7

- Dimension of the Lie Algebra of Derivations constructed from the abstract Lie algebra of \mathcal{D} : 7
- Dimensions of the Decomposition of the Radical of the Derivations: []
- Isotropy Type: F12
- The Isotropy Subalgebra Γ_p is Contained in the Semi-simple Part of the Levi Decomposition: *false*
- The Isotropy Subalgebra Γ_p and a Complementary Basis \mathfrak{m} form a Reductive Pair: *true*
- The Isotropy Subalgebra Γ_p and a Complementary Basis \mathfrak{m} form a Symmetric Pair: *false*
- Subspace Type of Γ : non-null

And here is the database entry for [33, 19]:

```
#####
# [33, 19]
#####
JTable[[33, 19]]:=table();
JTable[[33, 19]]["AlgebraType"] := "Generic";
JTable[[33, 19]]["Indecomposable"] := false;
JTable[[33, 19]]["DecompositionDimensions"] := [3,2];
JTable[[33, 19]]["IsotropySubalgebraReductivePair"] := true;
JTable[[33, 19]]["IsotropySubalgebraSymmetricPair"] := false;
JTable[[33, 19]]["DerivedSeriesDimensions"] := [5,3,3];
JTable[[33, 19]]["OrbitDimension"] := 4;
JTable[[33, 19]]["IsotropyType"] := "F12";
JTable[[33, 19]]["NilradicalDimension"] := 2;
JTable[[33, 19]]["NilradicalIndecomposable"] := false;
JTable[[33, 19]]["DerivationsDimension"]:= 7;
JTable[[33, 19]]["DecompositionOfRadicalOfDerivationsDimensions"] := [];
JTable[[33, 19]]["DerivationsOfDerivationsDimension"]:= 7;
JTable[[33, 19]]["SubspaceType"]:= nonnull;
JTable[[33, 19]]["IsotropySubalgebraInSemisimplePart"]:= false;
JTable[[33, 19]]["VectorFields"]:= [D_x2, cos(x2)*D_x1-cos(x1)*sin(x2)
*D_x2/sin(x1)+sin(x2)*D_x3/sin(x1), -sin(x2)*D_x1-cos(x1)*cos(x2)*
D_x2/sin(x1)+cos(x2)*D_x3/sin(x1), D_x3,D_x4];
```

Following this chapter one can find an application of the database. The actual database can be found in the appendices.

CHAPTER 4
APPLICATION

The Lie algebraic properties that we have tabulated for the five-dimensional Lie algebras found in Petrov [10] leads to new insights regarding Petrov's classification. We will give a few examples that illustrate how the database can be used to conjecture and prove relationships between pairs of Petrov's Lie algebras.

Example 4.0.14. In this example we will demonstrate the existence of a diffeomorphism ϕ whose push-forward ϕ_* maps the vector fields of [33, 25] to those of [33, 26]:

$$\begin{aligned}
 [33, 25] : X_1 &= \partial_{x^2}, \\
 X_2 &= \partial_{x^3}, \\
 X_3 &= -\partial_{x^1} + x^2 \partial_{x^2} + x^3 \partial_{x^3}, \\
 X_4 &= -x^3 \partial_{x^2} + x^2 \partial_{x^3}, \\
 X_5 &= \alpha \partial_{x^1} + \partial_{x^4}, \\
 [33, 26] : Y_1 &= \partial_{y^2}, \\
 Y_2 &= \partial_{y^3}, \\
 Y_3 &= 2 \partial_{y^1} + y^2 \partial_{y^2} + y^3 \partial_{y^3}, \\
 Y_4 &= -y^3 \partial_{y^2} + y^2 \partial_{y^3}, \\
 Y_5 &= \alpha \partial_{y^1} + \partial_{y^4},
 \end{aligned}$$

where α is a constant.

A look at the commutator tables for both Lie algebras shows that [33, 25] and [33, 26] have identical abstract Lie algebras:

$[\cdot, \cdot]$	X_1	X_2	X_3	X_4	X_5	$[\cdot, \cdot]$	Y_1	Y_2	Y_3	Y_4	Y_5
X_1	·	·	X_1	X_2	·	Y_1	·	·	Y_1	Y_2	·
X_2	·	·	X_2	$-X_1$	·	Y_2	·	·	Y_2	$-Y_1$	·
X_3	$-X_1$	$-X_2$	·	·	·	Y_3	$-Y_1$	$-Y_2$	·	·	·
X_4	$-X_2$	X_1	·	·	·	Y_4	$-Y_2$	Y_1	·	·	·
X_5	·	·	·	·	·	Y_5	·	·	·	·	·

We would like to know if these are equivalent Lie algebras of Killing vectors. In other words, we would like to know if there is a transformation ϕ such that $\phi_*(X_i) = Y_i$ and $\phi_*^{-1}(Y_i) = X_i$. As a first step, we define a general mapping $\phi(x^i) = (y^i)$, where

$$\begin{aligned} y^1 &= A_1^1(x^1, x^2, x^3, x^4), \\ y^2 &= A_1^2(x^1, x^2, x^3, x^4), \\ y^3 &= A_1^3(x^1, x^2, x^3, x^4), \\ y^4 &= A_1^4(x^1, x^2, x^3, x^4). \end{aligned}$$

We wish to solve for the unknown functions A_1^i by setting up the equation $\phi_*(X_1) = Y_1$. We set the terms of the left side equal to the corresponding terms on the right side and get the equations

$$\begin{aligned} \frac{\partial}{\partial x^2} A_1^1(x^1, x^2, x^3, x^4) \partial_{y^1} &= 0 \\ \left(-1 + \frac{\partial}{\partial x^2} A_1^2(x^1, x^2, x^3, x^4) \right) \partial_{y^2} &= 0 \\ \frac{\partial}{\partial x^2} A_1^3(x^1, x^2, x^3, x^4) \partial_{y^3} &= 0 \\ \frac{\partial}{\partial x^2} A_1^4(x^1, x^2, x^3, x^4) \partial_{y^4} &= 0. \end{aligned}$$

We solve these and substitute the solutions into ϕ , giving

$$\begin{aligned} y^1 &= A_2^1(x^1, x^3, x^4), \\ y^2 &= x^2 + A_2^2(x^1, x^3, x^4), \\ y^3 &= A_2^3(x^1, x^3, x^4), \\ y^4 &= A_2^4(x^1, x^3, x^4). \end{aligned}$$

Next, we solve $\phi_*(X_2) - Y_2 = 0$, giving

$$\frac{\partial}{\partial x^3} A_2^1(x^1, x^3, x^4) \partial_{y^1} = 0$$

$$\begin{aligned} \frac{\partial}{\partial x^3} A_2^2(x^1, x^3, x^4) \partial_{y^2} &= 0 \\ \left(-1 + \frac{\partial}{\partial x^3} A_2^3(x^1, x^3, x^4)\right) \partial_{y^3} &= 0 \\ \frac{\partial}{\partial x^3} A_2^4(x^1, x^3, x^4) \partial_{y^4} &= 0, \end{aligned}$$

which we again solve for the unknown functions and substitute the solutions into ϕ , now giving:

$$\begin{aligned} y^1 &= A_3^1(x^1, x^4), \\ y^2 &= x^2 + A_3^2(x^1, x^4), \\ y^3 &= x^3 + A_3^3(x^1, x^4), \\ y^4 &= A_3^4(x^1, x^4). \end{aligned}$$

Similarly, for X_3 and Y_3 we have $\phi_*(X_3) - Y_3 = 0$, giving the equations

$$\begin{aligned} \left(-2 - \frac{\partial}{\partial x^1} A_3^1(x^1, x^4)\right) \partial_{y^1} &= 0 \\ \left(-A_3^2(x^1, x^4) - \frac{\partial}{\partial x^1} A_3^2(x^1, x^4)\right) \partial_{y^2} &= 0 \\ \left(-A_3^3(x^1, x^4) - \frac{\partial}{\partial x^1} A_3^3(x^1, x^4)\right) \partial_{y^3} &= 0 \\ -\frac{\partial}{\partial x^1} A_3^4(x^1, x^4) \partial_{y^4} &= 0, \end{aligned}$$

the solutions of which are substituted into ϕ , giving

$$\begin{aligned} y^1 &= -2x^1 + A_4^1(x^4), \\ y^2 &= x^2 + A_4^2(x^4)e^{-x^1}, \\ y^3 &= x^3 + A_4^3(x^4)e^{-x^1}, \\ y^4 &= A_4^4(x^4). \end{aligned}$$

By repeating this process for X_4 and Y_4 , we find that the equation $\phi_*(X_4) - Y_4 = 0$ yields

$$\begin{aligned} A_5^3(x^4) e^{-x^1} \partial_{y^2} &= 0 \\ -A_5^2(x^4) e^{-x^1} \partial_{y^3} &= 0, \end{aligned}$$

the obvious solutions to which we substitute in to ϕ , giving

$$\begin{aligned} y^1 &= -2x^1 + A_5^1(x^4), \\ y^2 &= x^2, \\ y^3 &= x^3, \\ y^4 &= A_5^4(x^4). \end{aligned}$$

Lastly, for X_5 and Y_5 , we solve $\phi_*(X_5) - Y_5 = 0$. This gives the equations

$$\begin{aligned} \left(-3\alpha + \frac{d}{dx^4} A^1(x^4) \right) \partial_{y^1} &= 0 \\ \left(-1 + \frac{d}{dx^4} A^4(x^4) \right) \partial_{y^4} &= 0. \end{aligned}$$

The solutions are substituted into ϕ , giving

$$\begin{aligned} y^1 &= -2x^1 + 3\alpha x^4 + C_1, \\ y^2 &= x^2, \\ y^3 &= x^3, \\ y^4 &= x^4 + C_2. \end{aligned}$$

Here is the inverse transformation $\phi^{-1}(y^i)$

$$\begin{aligned} x^1 &= -\frac{1}{2}y^1 + \frac{3}{2}\alpha y^4 - \frac{3}{2}\alpha C_2 + \frac{1}{2}C_1, \\ x^2 &= y^2, \\ x^3 &= y^3, \\ x^4 &= y^4 - C_2. \end{aligned}$$

Clearly ϕ is smooth with a smooth inverse. It is a simple matter of checking that ϕ does in fact push-forward each X_i to each Y_i , and that ϕ^{-1} will push-forward each Y_i to each X_i . These are diffeomorphic Lie algebras of Killing vectors.

Example 4.0.15. Let X_i , Y_i , and W_i be the generators of $[33, 12, k \neq 0]$, $\{x^i\}$, $[33, 13, k \neq 0]$, $\{y^i\}$, and $[33, 29, 1, k \neq 0]$, $\{w^i\}$, respectively. Then perform the following changes of

basis

$$\{-X_1, X_3, X_5, X_2, X_4\}, \text{ setting } k = 2/a + 1,$$

and

$$\{Y_1, Y_2, -Y_5, Y_3, \frac{k-1}{k+1}Y_4\}, \text{ setting } k = -2/a - 1,$$

and

$$\{W_3, W_2, W_1, W_4, \frac{2}{l-1}W_5\}, \text{ setting } l = 2/a + 1, \text{ noting Petrov gives } k = \frac{l+1}{2}.$$

Then the respective abstract Lie algebras are identical to the abstract Lie algebra \mathfrak{g} , with commutator table

$[\cdot, \cdot]$	e_1	e_2	e_3	e_4	e_5
e_1	\cdot	\cdot	\cdot	\cdot	$(a+2)e_1$
e_2	\cdot	\cdot	\cdot	e_1	$(a+1)e_2$
e_3	\cdot	\cdot	\cdot	e_2	ae_3
e_4	\cdot	$-e_1$	$-e_2$	\cdot	$-e_4$
e_5	$-(a+2)e_1$	$-(a+1)e_2$	$-ae_3$	$-e_4$	\cdot

Therefore the abstract Lie algebras of [33, 12, $k \neq 0$], $\{x^i\}$, [33, 13, $k \neq 0$], $\{y^i\}$, and [33, 29, 1, $k \neq 0$], $\{w^i\}$ are isomorphic. Moreover, a similar analysis as in Example 4.0.14 gives a diffeomorphism ϕ_1 from [33, 12, $k \neq 0$], $\{x^i\}$ to [33, 13, $k \neq 0$], $\{y^i\}$, given by

$$\begin{aligned} y^1 &= -x^1, \\ y^2 &= -x^2 - \frac{1}{2}x^3e^{-x^1}, \\ y^3 &= -x^3e^{-x^1}, \\ y^4 &= -x^4. \end{aligned}$$

Also, in this same manner, we get a diffeomorphism ϕ_2 from [33, 12, $k \neq 0$], $\{x^i\}$ to [33, 29, 1, $k \neq 0$], $\{w^i\}$, given by

$$\begin{aligned} w^1 &= x^2 + \frac{1}{2}x^3e^{-x^1} + \left(k_2 \frac{a}{(1+a)(2+a)} + k_2 \frac{a^2}{(1+a)(2+a)} + \frac{1}{2} \frac{ak_{44}e^{\frac{(2+a)x^4}{a} + x^4}}{k_{12}(1+a)} \right) e^{-\frac{(2+a)x^1}{a}}, \\ w^2 &= -e^{-x^1}, \\ w^3 &= -x^3e^{-x^1}, \\ w^4 &= -x^1 + x^4. \end{aligned}$$

Example 4.0.16. Let X_i be the generators for [33, 12, $k = 0$] and Y_i the generators for [33, 13, $k = 0$], then the following changes of basis

$$\{X_1, -X_3, -X_5, X_2, X_4\}$$

and

$$\{Y_1, Y_2, -Y_5, Y_3, -Y_4\}$$

show that these define isomorphic abstract Lie algebras. Through a similar analysis as in Example 4.0.14, we get a complex diffeomorphism ϕ between [33, 12, $k = 0$] and [33, 13, $k = 0$], given by

$$\begin{aligned} y^1 &= i\pi - x^1, \\ y^2 &= x^2 + \left(\frac{1}{2}\right)e^{-x^1}x^3 + \frac{2k_{44}e^{x^4} - 2k_2k_{12}x^4}{k_{12}}, \\ y^3 &= e^{-x^1}x^3, \\ y^4 &= -x^4. \end{aligned}$$

However, we believe no real diffeomorphism exists between [33, 12, $k \neq 0$] and [33, 13, $k \neq 0$], but we have not been able to find an algebraic or geometric invariant that separates them. This is a difficult problem as can be illustrated by the following simple example.

Example 4.0.17. Observe the vector fields for [33, 38, 1] and [33, 38, -1] and note the only difference between them is the sign of the coefficient on ∂_{y^3} in X_5 and Y_5 :

$$\begin{aligned} [33, 38, 1] : X_1 &= \partial_{x^1}, \\ X_2 &= \partial_{x^3}, \\ X_3 &= -\partial_{x^1} + x^2 \partial_{x^2}, \\ X_4 &= -\lambda \partial_{x^1} + x^3 \partial_{x^3} + \partial_{x^4}, \\ X_5 &= -\lambda e^{-\frac{x^1}{\lambda}} \partial_{x^1} + e^{\frac{x^4 \lambda - x^1}{\lambda}} \partial_{x^3} + e^{-\frac{x^1}{\lambda}} \partial_{x^4}, \end{aligned}$$

$$[33, 38, -1] : Y_1 = \partial_{y^1},$$

$$Y_2 = \partial_{y^3},$$

$$Y_3 = -\partial_{y^1} + y^2 \partial_{y^2},$$

$$Y_4 = -\lambda \partial_{y^1} + y^3 \partial_{y^3} + \partial_{y^4},$$

$$Y_5 = -\lambda e^{-\frac{y^1}{\lambda}} \partial_{y^1} - e^{\frac{y^4 \lambda - y^1}{\lambda}} \partial_{y^3} + e^{-\frac{y^1}{\lambda}} \partial_{y^4},$$

where λ is a constant. The abstract Lie algebras defined by these are identical. Then a complex diffeomorphism is

$$\phi(x) = (y^1 = x^1, y^2 = x^2, y^3 = x^3, y^4 = x^4 + i\pi),$$

but we have not been able to find a real diffeomorphism. We believe a real diffeomorphism does not exist, but we have not been able to find an algebraic or geometric invariant that separates them.

CHAPTER 5

FUTURE RESEARCH

The database was created for the five-dimensional Lie algebras of Killing vectors found in Chapter 5 of Petrov [10]. However, this could be extended to include the three, four, and six-dimensional Lie algebras of Killing vectors also found in Petrov.

In addition, as we saw in Chapter 4, some Lie algebras of Killing vectors in Petrov [10] are identical to one another. The following questions are natural at this point.

- Exactly what pairs of Petrov's Lie algebras of Killing vectors define isomorphic abstract Lie algebras?
- Among those with isomorphic abstract Lie algebras, which pairs of Killing vectors have diffeomorphisms between them?
- Given a pair of Lie algebras of Killing vectors from Petrov's classification for which there exists a diffeomorphism ϕ such that ϕ_* maps one space of Killing vectors to the other, does there exist a diffeomorphism ψ such that ψ^* maps one corresponding metric to the other?

Once it is known exactly which Lie algebras of Killing vectors are distinct and which are identical, we may create a classifier program that when given a Lorentz metric g in four dimensions, it would 1) compute the Lie algebra of Killing vectors Γ of g , 2) compute the same properties used to create the database for Γ , 3) then make a comparison of Γ 's properties against the database to determine which of Petrov's Lie algebras share identical values as Γ , 4) once those Lie algebras are determined, the classifier would then return the reference numbers (found in Petrov [10]) that are associated with those Lie algebras in the database.

Then, a natural next step would be to create another program that provides a change of basis and diffeomorphisms in an attempt to answer the three questions above.

We have not attempted to independently verify the completion of Petrov's list of Lie

algebras of Killing vectors. Once these additional steps are taken, the hope is to publish the results and findings.

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APPENDIX

APPENDIX A

DATABASE

The following is the database of properties calculated for the five-dimensional Lie algebras of Killing vectors found in Petrov, [10].

```
PetrovMainTable := proc()
local JTable;
JTable := table();
#####
#####
# 5-dimensional
#####
# [33, 1]
#####
JTable[[33, 1]]:=table();
JTable[[33, 1]]["AlgebraType"] := "Nilpotent";
JTable[[33, 1]]["Indecomposable"] := true;
JTable[[33, 1]]["DecompositionDimensions"] := [5];
JTable[[33, 1]]["IsotropySubalgebraReductivePair"] := true;
JTable[[33, 1]]["IsotropySubalgebraSymmetricPair"] := true;
JTable[[33, 1]]["DerivedSeriesDimensions"] := [5,1,0];
JTable[[33, 1]]["OrbitDimension"] := 3;
JTable[[33, 1]]["IsotropyType"] := "F10";
JTable[[33, 1]]["NilradicalDimension"] := 5;
JTable[[33, 1]]["NilradicalIndecomposable"] := true;
JTable[[33, 1]]["DerivationsDimension"]:= 15;
JTable[[33, 1]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [5];
JTable[[33, 1]]["DerivationsOfDerivationsDimension"]:= 15;
JTable[[33, 1]]["SubspaceType"]:= nonnull;
JTable[[33, 1]]["IsotropySubalgebraInSemisimplePart"]:= NA;
JTable[[33, 1]]["VectorFields"]:= [D_x1, D_x2, D_x3, x2*D_x1+phi(x4)
*D_x2+psi(x4)*D_x3, x3*D_x1+psi(x4)*D_x2+nu(x4)*D_x3];
#####
# [33, 5, 1]
#####
JTable[[33, 5, 1]]:=table();
JTable[[33, 5, 1]]["AlgebraType"] := "Solvable";
JTable[[33, 5, 1]]["Indecomposable"] := true;
JTable[[33, 5, 1]]["DecompositionDimensions"] := [5];
JTable[[33, 5, 1]]["IsotropySubalgebraReductivePair"] := false;
JTable[[33, 5, 1]]["IsotropySubalgebraSymmetricPair"] := NA;
JTable[[33, 5, 1]]["DerivedSeriesDimensions"] := [5,3,1,0];
JTable[[33, 5, 1]]["OrbitDimension"] := 4;
JTable[[33, 5, 1]]["IsotropyType"] := "F14";
JTable[[33, 5, 1]]["NilradicalDimension"] := 3;
JTable[[33, 5, 1]]["NilradicalIndecomposable"] := true;
JTable[[33, 5, 1]]["DerivationsDimension"]:= 5;
JTable[[33, 5, 1]]["DecompositionOfRadicalOfDerivationsDimensions"]
```

```

:= [5];
JTable[[33, 5, 1]]["DerivationsOfDerivationsDimension"]:= 5;
JTable[[33, 5, 1]]["SubspaceType"]:= nonnull;
JTable[[33, 5, 1]]["IsotropySubalgebraInSemisimplePart"]:= NA;
JTable[[33, 5, 1]]["VectorFields"]:= [D_x2, D_x3, x3*D_x2-exp(x1)*D_x3,
-2*D_x1/(k-1)+2*k*x2*D_x2/(k-1)+x3*D_x3, (alpha-beta1)*D_x1+
(x2*alpha+x2*beta1+exp(-x1*k))*D_x2+alpha*x3*D_x3+D_x4];
#####
# [33, 5, -1]
#####
#JTable[[33, 5, -1]]:=table();
#JTable[[33, 5, -1]]["AlgebraType"] := "Solvable";
#JTable[[33, 5, -1]]["Indecomposable"] := true;
#JTable[[33, 5, -1]]["DecompositionDimensions"] := [5];
#JTable[[33, 5, -1]]["IsotropySubalgebraReductivePair"] := false;
#JTable[[33, 5, -1]]["IsotropySubalgebraSymmetricPair"] := NA;
#JTable[[33, 5, -1]]["DerivedSeriesDimensions"] := [5,3,1,0];
#JTable[[33, 5, -1]]["OrbitDimension"] := 4;
#JTable[[33, 5, -1]]["IsotropyType"] := "F14";
#JTable[[33, 5, -1]]["NilradicalDimension"] := 3;
#JTable[[33, 5, -1]]["NilradicalIndecomposable"] := true;
#JTable[[33, 5, -1]]["DerivationsDimension"]:= 5;
#JTable[[33, 5, -1]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [5];
#JTable[[33, 5, -1]]["DerivationsOfDerivationsDimension"]:= 5;
#JTable[[33, 5, -1]]["SubspaceType"]:= nonnull;
#JTable[[33, 5, -1]]["IsotropySubalgebraInSemisimplePart"]:= NA;
#JTable[[33, 5, -1]]["VectorFields"]:= [D_x2, D_x3, x3*D_x2-exp(x1)
*D_x3, -2*D_x1/(k-1)+2*k*x2*D_x2/(k-1)+x3*D_x3, (alpha-beta1)*D_x1
+(x2*alpha+x2*beta1-exp(-x1*k))*D_x2+alpha*x3*D_x3+D_x4];
#####
# [33, 5, 0]
#####
#JTable[[33, 5, 0]]:=table();
#JTable[[33, 5, 0]]["AlgebraType"] := "Solvable";
#JTable[[33, 5, 0]]["Indecomposable"] := true;
#JTable[[33, 5, 0]]["DecompositionDimensions"] := [5];
#JTable[[33, 5, 0]]["IsotropySubalgebraReductivePair"] := false;
#JTable[[33, 5, 0]]["IsotropySubalgebraSymmetricPair"] := NA;
#JTable[[33, 5, 0]]["DerivedSeriesDimensions"] := [5,3,1,0];
#JTable[[33, 5, 0]]["OrbitDimension"] := 4;
#JTable[[33, 5, 0]]["IsotropyType"] := "F14";
#JTable[[33, 5, 0]]["NilradicalDimension"] := 3;
#JTable[[33, 5, 0]]["NilradicalIndecomposable"] := true;
#JTable[[33, 5, 0]]["DerivationsDimension"]:= 5;
#JTable[[33, 5, 0]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [5];
#JTable[[33, 5, 0]]["DerivationsOfDerivationsDimension"]:= 5;
#JTable[[33, 5, 0]]["SubspaceType"]:= nonnull;
#JTable[[33, 5, 0]]["IsotropySubalgebraInSemisimplePart"]:= NA;
#JTable[[33, 5, 0]]["VectorFields"]:= [D_x2, D_x3, x3*D_x2-exp(x1)

```

```

*D_x3, -2*D_x1/(k-1)+2*k*x2*D_x2/(k-1)+x3*D_x3, (alpha-beta1)*D_x1
+(x2*alpha+x2*beta1)*D_x2+alpha*x3*D_x3+D_x4];
#####
# [33, 5]
#####
#JTable[[33, 5]]:=table();
#JTable[[33, 5]]["AlgebraType"] := "Solvable";
#JTable[[33, 5]]["Indecomposable"] := true;
#JTable[[33, 5]]["DecompositionDimensions"] := [5];
#JTable[[33, 5]]["IsotropySubalgebraReductivePair"] := false;
#JTable[[33, 5]]["IsotropySubalgebraSymmetricPair"] := NA;
#JTable[[33, 5]]["DerivedSeriesDimensions"] := [5,3,1,0];
#JTable[[33, 5]]["OrbitDimension"] := 4;
#JTable[[33, 5]]["IsotropyType"] := "F14";
#JTable[[33, 5]]["NilradicalDimension"] := 3;
#JTable[[33, 5]]["NilradicalIndecomposable"] := true;
#JTable[[33, 5]]["DerivationsDimension"]:= 5;
#JTable[[33, 5]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [5];
#JTable[[33, 5]]["DerivationsOfDerivationsDimension"]:= 5;
#JTable[[33, 5]]["SubspaceType"]:= nonnull;
#JTable[[33, 5]]["IsotropySubalgebraInSemisimplePart"]:= NA;
#JTable[[33, 5]]["VectorFields"]:= [D_x2, D_x3, x3*D_x2-exp(x1)*D_x3,
-2*D_x1/(k-1)+2*k*x2*D_x2/(k-1)+x3*D_x3, (alpha-beta1)*D_x1+(x2*alpha
+x2*beta1+epsilon*exp(-x1*k))*D_x2+alpha*x3*D_x3+D_x4];
#####
# [33, 6, 1]
#####
JTable[[33, 6, 1]]:=table();
JTable[[33, 6, 1]]["AlgebraType"] := "Solvable";
JTable[[33, 6, 1]]["Indecomposable"] := true;
JTable[[33, 6, 1]]["DecompositionDimensions"] := [5];
JTable[[33, 6, 1]]["IsotropySubalgebraReductivePair"] := false;
JTable[[33, 6, 1]]["IsotropySubalgebraSymmetricPair"] := NA;
JTable[[33, 6, 1]]["DerivedSeriesDimensions"] := [5,3,1,0];
JTable[[33, 6, 1]]["OrbitDimension"] := 4;
JTable[[33, 6, 1]]["IsotropyType"] := "F14";
JTable[[33, 6, 1]]["NilradicalDimension"] := 4;
JTable[[33, 6, 1]]["NilradicalIndecomposable"] := true;
JTable[[33, 6, 1]]["DerivationsDimension"]:= 6;
JTable[[33, 6, 1]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [6];
JTable[[33, 6, 1]]["DerivationsOfDerivationsDimension"]:= 6;
JTable[[33, 6, 1]]["SubspaceType"]:= nonnull;
JTable[[33, 6, 1]]["IsotropySubalgebraInSemisimplePart"]:= NA;
JTable[[33, 6, 1]]["VectorFields"]:= [D_x2, D_x3, x3*D_x2-x1*D_x3,
D_x1+2*x2*D_x2+x3*D_x3, D_x1+exp(2*x1)*D_x2+D_x4];
#####
# [33, 6, -1]
#####
#JTable[[33, 6, -1]]:=table();

```

```

#JTable[[33, 6, -1]]["AlgebraType"] := "Solvable";
#JTable[[33, 6, -1]]["Indecomposable"] := true;
#JTable[[33, 6, -1]]["DecompositionDimensions"] := [5];
#JTable[[33, 6, -1]]["IsotropySubalgebraReductivePair"] := false;
#JTable[[33, 6, -1]]["IsotropySubalgebraSymmetricPair"] := NA;
#JTable[[33, 6, -1]]["DerivedSeriesDimensions"] := [5,3,1,0];
#JTable[[33, 6, -1]]["OrbitDimension"] := 4;
#JTable[[33, 6, -1]]["IsotropyType"] := "F14";
#JTable[[33, 6, -1]]["NilradicalDimension"] := 4;
#JTable[[33, 6, -1]]["NilradicalIndecomposable"] := true;
#JTable[[33, 6, -1]]["DerivationsDimension"]:= 6;
#JTable[[33, 6, -1]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [6];
#JTable[[33, 6, -1]]["DerivationsOfDerivationsDimension"]:= 6;
#JTable[[33, 6, -1]]["SubspaceType"]:= nonnull;
#JTable[[33, 6, -1]]["IsotropySubalgebraInSemisimplePart"]:= NA;
#JTable[[33, 6, -1]]["VectorFields"]:= [D_x2, D_x3, x3*D_x2-x1*D_x3,
D_x1+2*x2*D_x2+x3*D_x3, D_x1-exp(2*x1)*D_x2+D_x4];
#####
# [33, 6, 0]
#####
#JTable[[33, 6, 0]]:=table();
#JTable[[33, 6, 0]]["AlgebraType"] := "Solvable";
#JTable[[33, 6, 0]]["Indecomposable"] := true;
#JTable[[33, 6, 0]]["DecompositionDimensions"] := [5];
#JTable[[33, 6, 0]]["IsotropySubalgebraReductivePair"] := false;
#JTable[[33, 6, 0]]["IsotropySubalgebraSymmetricPair"] := NA;
#JTable[[33, 6, 0]]["DerivedSeriesDimensions"] := [5,3,1,0];
#JTable[[33, 6, 0]]["OrbitDimension"] := 4;
#JTable[[33, 6, 0]]["IsotropyType"] := "F14";
#JTable[[33, 6, 0]]["NilradicalDimension"] := 4;
#JTable[[33, 6, 0]]["NilradicalIndecomposable"] := true;
#JTable[[33, 6, 0]]["DerivationsDimension"]:= 6;
#JTable[[33, 6, 0]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [6];
#JTable[[33, 6, 0]]["DerivationsOfDerivationsDimension"]:= 6;
#JTable[[33, 6, 0]]["SubspaceType"]:= nonnull;
#JTable[[33, 6, 0]]["IsotropySubalgebraInSemisimplePart"]:= NA;
#JTable[[33, 6, 0]]["VectorFields"]:= [D_x2, D_x3, x3*D_x2-x1*D_x3,
D_x1+2*x2*D_x2+x3*D_x3, D_x1+D_x4];
#####
# [33, 6]
#####
#JTable[[33, 6]]:=table();
#JTable[[33, 6]]["AlgebraType"] := "Solvable";
#JTable[[33, 6]]["Indecomposable"] := true;
#JTable[[33, 6]]["DecompositionDimensions"] := [5];
#JTable[[33, 6]]["IsotropySubalgebraReductivePair"] := false;
#JTable[[33, 6]]["IsotropySubalgebraSymmetricPair"] := NA;
#JTable[[33, 6]]["DerivedSeriesDimensions"] := [5,3,1,0];
#JTable[[33, 6]]["OrbitDimension"] := 4;

```



```

#JTable[[33, 6]]["IsotropyType"] := "F14";
#JTable[[33, 6]]["NilradicalDimension"] := 4;
#JTable[[33, 6]]["NilradicalIndecomposable"] := true;
#JTable[[33, 6]]["DerivationsDimension"]:= 6;
#JTable[[33, 6]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [6];
#JTable[[33, 6]]["DerivationsOfDerivationsDimension"]:= 6;
#JTable[[33, 6]]["SubspaceType"]:= nonnull;
#JTable[[33, 6]]["IsotropySubalgebraInSemisimplePart"]:= NA;
#JTable[[33, 6]]["VectorFields"]:= [D_x2, D_x3, x3*D_x2-x1*D_x3,
D_x1+2*x2*D_x2
+x3*D_x3, D_x1+epsilon*exp(2*x1)*D_x2+D_x4];
#####
# [33, 7, 1]
#####
JTable[[33, 7, 1]]:=table();
JTable[[33, 7, 1]]["AlgebraType"] := "Solvable";
JTable[[33, 7, 1]]["Indecomposable"] := true;
JTable[[33, 7, 1]]["DecompositionDimensions"] := [5];
JTable[[33, 7, 1]]["IsotropySubalgebraReductivePair"] := false;
JTable[[33, 7, 1]]["IsotropySubalgebraSymmetricPair"] := NA;
JTable[[33, 7, 1]]["DerivedSeriesDimensions"] := [5,3,1,0];
JTable[[33, 7, 1]]["OrbitDimension"] := 4;
JTable[[33, 7, 1]]["IsotropyType"] := "F14";
JTable[[33, 7, 1]]["NilradicalDimension"] := 3;
JTable[[33, 7, 1]]["NilradicalIndecomposable"] := true;
JTable[[33, 7, 1]]["DerivationsDimension"]:= 5;
JTable[[33, 7, 1]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [5];
JTable[[33, 7, 1]]["DerivationsOfDerivationsDimension"]:= 5;
JTable[[33, 7, 1]]["SubspaceType"]:= nonnull;
JTable[[33, 7, 1]]["IsotropySubalgebraInSemisimplePart"]:= NA;
JTable[[33, 7, 1]]["VectorFields"]:= [D_x2, D_x3, x3*D_x2-(1/2)
*sqrt(4-q^2)*tan((1/2)*sqrt(4-q^2)*x1)*D_x3, -D_x1+(q*x2+(1/2)*x3^2)
*D_x2-(1/2)*x3*(-q+sqrt(4-q^2)*tan((1/2)*sqrt(4-q^2)*x1))*D_x3,
(exp(-q*x1)+2*x2)*D_x2+x3*D_x3+D_x4];
#####
# [33, 7, -1]
#####
#JTable[[33, 7, -1]]:=table();
#JTable[[33, 7, -1]]["AlgebraType"] := "Solvable";
#JTable[[33, 7, -1]]["Indecomposable"] := true;
#JTable[[33, 7, -1]]["DecompositionDimensions"] := [5];
#JTable[[33, 7, -1]]["IsotropySubalgebraReductivePair"] := false;
#JTable[[33, 7, -1]]["IsotropySubalgebraSymmetricPair"] := NA;
#JTable[[33, 7, -1]]["DerivedSeriesDimensions"] := [5,3,1,0];
#JTable[[33, 7, -1]]["OrbitDimension"] := 4;
#JTable[[33, 7, -1]]["IsotropyType"] := "F14";
#JTable[[33, 7, -1]]["NilradicalDimension"] := 3;
#JTable[[33, 7, -1]]["NilradicalIndecomposable"] := true;
#JTable[[33, 7, -1]]["DerivationsDimension"]:= 5;

```

```

#JTable[[33, 7, -1]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [5];
#JTable[[33, 7, -1]]["DerivationsOfDerivationsDimension"]:= 5;
#JTable[[33, 7, -1]]["SubspaceType"]:= nonnull;
#JTable[[33, 7, -1]]["IsotropySubalgebraInSemisimplePart"]:= NA;
#JTable[[33, 7, -1]]["VectorFields"]:= [D_x2, D_x3, x3*D_x2-(1/2)
*sqrt(4-q^2)*tan((1/2)*sqrt(4-q^2)*x1)*D_x3,-D_x1+(q*x2+(1/2)*x3^2)
*D_x2-(1/2)*x3*(-q+sqrt(4-q^2)*tan((1/2)*sqrt(4-q^2)*x1))*D_x3,
(-exp(-q*x1)+2*x2)*D_x2+x3*D_x3+D_x4];
#####
# [33, 7, 0]
#####
#JTable[[33, 7, 0]]:=table();
#JTable[[33, 7, 0]]["AlgebraType"] := "Solvable";
#JTable[[33, 7, 0]]["Indecomposable"] := true;
#JTable[[33, 7, 0]]["DecompositionDimensions"] := [5];
#JTable[[33, 7, 0]]["IsotropySubalgebraReductivePair"] := false;
#JTable[[33, 7, 0]]["IsotropySubalgebraSymmetricPair"] := NA;
#JTable[[33, 7, 0]]["DerivedSeriesDimensions"] := [5,3,1,0];
#JTable[[33, 7, 0]]["OrbitDimension"] := 4;
#JTable[[33, 7, 0]]["IsotropyType"] := "F14";
#JTable[[33, 7, 0]]["NilradicalDimension"] := 3;
#JTable[[33, 7, 0]]["NilradicalIndecomposable"] := true;
#JTable[[33, 7, 0]]["DerivationsDimension"]:= 5;
#JTable[[33, 7, 0]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [5];
#JTable[[33, 7, 0]]["DerivationsOfDerivationsDimension"]:= 5;
#JTable[[33, 7, 0]]["SubspaceType"]:= nonnull;
#JTable[[33, 7, 0]]["IsotropySubalgebraInSemisimplePart"]:= NA;
#JTable[[33, 7, 0]]["VectorFields"]:= [D_x2, D_x3, x3*D_x2-(1/2)
*sqrt(4-q^2)*tan((1/2)*sqrt(4-q^2)*x1)*D_x3,-D_x1+(q*x2+(1/2)*x3^2)
*D_x2-(1/2)*x3*(-q+sqrt(4-q^2)*tan((1/2)*sqrt(4-q^2)*x1))*D_x3,
2*x2*D_x2+x3*D_x3+D_x4];
#####
# [33, 7]
#####
#JTable[[33, 7]]:=table();
#JTable[[33, 7]]["AlgebraType"] := "Solvable";
#JTable[[33, 7]]["Indecomposable"] := true;
#JTable[[33, 7]]["DecompositionDimensions"] := [5];
#JTable[[33, 7]]["IsotropySubalgebraReductivePair"] := false;
#JTable[[33, 7]]["IsotropySubalgebraSymmetricPair"] := NA;
#JTable[[33, 7]]["DerivedSeriesDimensions"] := [5,3,1,0];
#JTable[[33, 7]]["OrbitDimension"] := 4;
#JTable[[33, 7]]["IsotropyType"] := "F14";
#JTable[[33, 7]]["NilradicalDimension"] := 3;
#JTable[[33, 7]]["NilradicalIndecomposable"] := true;
#JTable[[33, 7]]["DerivationsDimension"]:= 5;
#JTable[[33, 7]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [5];
#JTable[[33, 7]]["DerivationsOfDerivationsDimension"]:= 5;

```

```

#JTable[[33, 7]]["SubspaceType"]:= nonnull;
#JTable[[33, 7]]["IsotropySubalgebraInSemisimplePart"]:= NA;
#JTable[[33, 7]]["VectorFields"]:= [D_x2, D_x3, x3*D_x2-(1/2)
*sqrt(4-q^2)*tan((1/2)*sqrt(4-q^2)*x1)*D_x3, -D_x1+(q*x2+(1/2)*x3^2)
*D_x2-(1/2)*x3*(-q+sqrt(4-q^2)*tan((1/2)*sqrt(4-q^2)*x1))*D_x3,
(epsilon*exp(-q*x1)+2*x2)*D_x2+x3*D_x3+D_x4];
#####
# [33, 8, 1]
#####
JTable[[33, 8, 1]]:=table();
JTable[[33, 8, 1]]["AlgebraType"] := "Solvable";
JTable[[33, 8, 1]]["Indecomposable"] := true;
JTable[[33, 8, 1]]["DecompositionDimensions"] := [5];
JTable[[33, 8, 1]]["IsotropySubalgebraReductivePair"] := false;
JTable[[33, 8, 1]]["IsotropySubalgebraSymmetricPair"] := NA;
JTable[[33, 8, 1]]["DerivedSeriesDimensions"] := [5,3,1,0];
JTable[[33, 8, 1]]["OrbitDimension"] := 4;
JTable[[33, 8, 1]]["IsotropyType"] := "F14";
JTable[[33, 8, 1]]["NilradicalDimension"] := 3;
JTable[[33, 8, 1]]["NilradicalIndecomposable"] := true;
JTable[[33, 8, 1]]["DerivationsDimension"]:= 5;
JTable[[33, 8, 1]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [5];
JTable[[33, 8, 1]]["DerivationsOfDerivationsDimension"]:= 5;
JTable[[33, 8, 1]]["SubspaceType"]:= nonnull;
JTable[[33, 8, 1]]["IsotropySubalgebraInSemisimplePart"]:= NA;
JTable[[33, 8, 1]]["VectorFields"]:= [D_x2, D_x3, x3*D_x2-exp(x1)*D_x3,
2*D_x1+x3*D_x3, (alpha-beta1)*D_x1+(x2*alpha+x2*beta1+x1)*D_x2+alpha
*x3*D_x3+D_x4];
#####
# [33, 8, -1]
#####
#JTable[[33, 8, -1]]:=table();
#JTable[[33, 8, -1]]["AlgebraType"] := "Solvable";
#JTable[[33, 8, -1]]["Indecomposable"] := true;
#JTable[[33, 8, -1]]["DecompositionDimensions"] := [5];
#JTable[[33, 8, -1]]["IsotropySubalgebraReductivePair"] := false;
#JTable[[33, 8, -1]]["IsotropySubalgebraSymmetricPair"] := NA;
#JTable[[33, 8, -1]]["DerivedSeriesDimensions"] := [5,3,1,0];
#JTable[[33, 8, -1]]["OrbitDimension"] := 4;
#JTable[[33, 8, -1]]["IsotropyType"] := "F14";
#JTable[[33, 8, -1]]["NilradicalDimension"] := 3;
#JTable[[33, 8, -1]]["NilradicalIndecomposable"] := true;
#JTable[[33, 8, -1]]["DerivationsDimension"]:= 5;
#JTable[[33, 8, -1]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [5];
#JTable[[33, 8, -1]]["DerivationsOfDerivationsDimension"]:= 5;
#JTable[[33, 8, -1]]["SubspaceType"]:= nonnull;
#JTable[[33, 8, -1]]["IsotropySubalgebraInSemisimplePart"]:= NA;
#JTable[[33, 8, -1]]["VectorFields"]:= [D_x2, D_x3, x3*D_x2-exp(x1)
*D_x3, 2*D_x1+x3*D_x3, (alpha-beta1)*D_x1+(x2*alpha+x2*beta1-x1)*D_x2

```

```

+alpha*x3*D_x3+D_x4];
#####
# [33, 8, 0]
#####
#JTable[[33, 8, 0]]:=table();
#JTable[[33, 8, 0]]["AlgebraType"] := "Solvable";
#JTable[[33, 8, 0]]["Indecomposable"] := true;
#JTable[[33, 8, 0]]["DecompositionDimensions"] := [5];
#JTable[[33, 8, 0]]["IsotropySubalgebraReductivePair"] := false;
#JTable[[33, 8, 0]]["IsotropySubalgebraSymmetricPair"] := NA;
#JTable[[33, 8, 0]]["DerivedSeriesDimensions"] := [5,3,1,0];
#JTable[[33, 8, 0]]["OrbitDimension"] := 4;
#JTable[[33, 8, 0]]["IsotropyType"] := "F14";
#JTable[[33, 8, 0]]["NilradicalDimension"] := 3;
#JTable[[33, 8, 0]]["NilradicalIndecomposable"] := true;
#JTable[[33, 8, 0]]["DerivationsDimension"]:= 5;
#JTable[[33, 8, 0]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [5];
#JTable[[33, 8, 0]]["DerivationsOfDerivationsDimension"]:= 5;
#JTable[[33, 8, 0]]["SubspaceType"]:= nonnull;
#JTable[[33, 8, 0]]["IsotropySubalgebraInSemisimplePart"]:= NA;
#JTable[[33, 8, 0]]["VectorFields"]:= [D_x2, D_x3, x3*D_x2-exp(x1)
*D_x3, 2*D_x1+x3*D_x3, (alpha-beta1)*D_x1+(x2*alpha+x2*beta1)*D_x2
+alpha*x3*D_x3+D_x4];
#####
# [33, 8]
#####
#JTable[[33, 8]]:=table();
#JTable[[33, 8]]["AlgebraType"] := "Solvable";
#JTable[[33, 8]]["Indecomposable"] := true;
#JTable[[33, 8]]["DecompositionDimensions"] := [5];
#JTable[[33, 8]]["IsotropySubalgebraReductivePair"] := false;
#JTable[[33, 8]]["IsotropySubalgebraSymmetricPair"] := NA;
#JTable[[33, 8]]["DerivedSeriesDimensions"] := [5,3,1,0];
#JTable[[33, 8]]["OrbitDimension"] := 4;
#JTable[[33, 8]]["IsotropyType"] := "F14";
#JTable[[33, 8]]["NilradicalDimension"] := 3;
#JTable[[33, 8]]["NilradicalIndecomposable"] := true;
#JTable[[33, 8]]["DerivationsDimension"]:= 5;
#JTable[[33, 8]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [5];
#JTable[[33, 8]]["DerivationsOfDerivationsDimension"]:= 5;
#JTable[[33, 8]]["SubspaceType"]:= nonnull;
#JTable[[33, 8]]["IsotropySubalgebraInSemisimplePart"]:= NA;
#JTable[[33, 8]]["VectorFields"]:= [D_x2, D_x3, x3*D_x2-exp(x1)*D_x3,
2*D_x1+x3*D_x3, (alpha-beta1)*D_x1+(x2*alpha+x2*beta1+x1)*D_x2+alpha
*x3*D_x3+D_x4];
#####
# [33, 12, k<>0]
#####
JTable[[33, 12, k<>0]]:=table();

```

```

JTable[[33, 12, k<>0]]["AlgebraType"] := "Solvable";
JTable[[33, 12, k<>0]]["Indecomposable"] := true;
JTable[[33, 12, k<>0]]["DecompositionDimensions"] := [5];
JTable[[33, 12, k<>0]]["IsotropySubalgebraReductivePair"] := false;
JTable[[33, 12, k<>0]]["IsotropySubalgebraSymmetricPair"] := NA;
JTable[[33, 12, k<>0]]["DerivedSeriesDimensions"] := [5,4,2,0];
JTable[[33, 12, k<>0]]["OrbitDimension"] := 4;
JTable[[33, 12, k<>0]]["IsotropyType"] := "F14";
JTable[[33, 12, k<>0]]["NilradicalDimension"] := 4;
JTable[[33, 12, k<>0]]["NilradicalIndecomposable"] := true;
JTable[[33, 12, k<>0]]["DerivationsDimension"]:= 6;
JTable[[33, 12, k<>0]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [6];
JTable[[33, 12, k<>0]]["DerivationsOfDerivationsDimension"]:= 6;
JTable[[33, 12, k<>0]]["SubspaceType"]:= nonnull;
JTable[[33, 12, k<>0]]["IsotropySubalgebraInSemisimplePart"]:= NA;
JTable[[33, 12, k<>0]]["VectorFields"]:= [D_x2, D_x3, x3*D_x2-exp(x1)
*D_x3, -2*D_x1/(k-1)+2*k*x2*D_x2/(k-1)+x3*D_x3, exp(x1)*D_x1-(1/2)
*(x3^2*k12+k+x3^2*k12+2*exp(-x1*k+x1+x4*k+x4)*k44-2*exp(-x1*(k-1))
*k2*k12*k-2*exp(-x1*(k-1))*k2*k12)*D_x2/(k12*(k+1))+exp(x1)*x3*D_x3
+exp(x1)*D_x4];
#####
# [33, 12, k=0]
#####
JTable[[33, 12, k=0]]:=table();
JTable[[33, 12, k=0]]["AlgebraType"] := "Solvable";
JTable[[33, 12, k=0]]["Indecomposable"] := true;
JTable[[33, 12, k=0]]["DecompositionDimensions"] := [5];
JTable[[33, 12, k=0]]["IsotropySubalgebraReductivePair"] := false;
JTable[[33, 12, k=0]]["IsotropySubalgebraSymmetricPair"] := NA;
JTable[[33, 12, k=0]]["DerivedSeriesDimensions"] := [5,4,2,0];
JTable[[33, 12, k=0]]["OrbitDimension"] := 4;
JTable[[33, 12, k=0]]["IsotropyType"] := "F14";
JTable[[33, 12, k=0]]["NilradicalDimension"] := 4;
JTable[[33, 12, k=0]]["NilradicalIndecomposable"] := true;
JTable[[33, 12, k=0]]["DerivationsDimension"]:= 6;
JTable[[33, 12, k=0]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [6];
JTable[[33, 12, k=0]]["DerivationsOfDerivationsDimension"]:= 7;
JTable[[33, 12, k=0]]["SubspaceType"]:= nonnull;
JTable[[33, 12, k=0]]["IsotropySubalgebraInSemisimplePart"]:= NA;
JTable[[33, 12, k=0]]["VectorFields"]:= [D_x2, D_x3, x3*D_x2-exp(x1)
*D_x3, 2*D_x1+x3*D_x3, exp(x1)*D_x1-(1/2)*(x3^2*k12+2*exp(x1+x4)*k44
-2*exp(x1)*k2*k12)*D_x2/k12+exp(x1)*x3*D_x3+exp(x1)*D_x4];
#####
# [33, 13, k<>0]
#####
JTable[[33, 13, k<>0]]:=table();
JTable[[33, 13, k<>0]]["AlgebraType"] := "Solvable";
JTable[[33, 13, k<>0]]["Indecomposable"] := true;
JTable[[33, 13, k<>0]]["DecompositionDimensions"] := [5];

```

```

JTable[[33, 13, k<>0]]["IsotropySubalgebraReductivePair"] := false;
JTable[[33, 13, k<>0]]["IsotropySubalgebraSymmetricPair"] := NA;
JTable[[33, 13, k<>0]]["DerivedSeriesDimensions"] := [5,4,2,0];
JTable[[33, 13, k<>0]]["OrbitDimension"] := 4;
JTable[[33, 13, k<>0]]["IsotropyType"] := "F14";
JTable[[33, 13, k<>0]]["NilradicalDimension"] := 4;
JTable[[33, 13, k<>0]]["NilradicalIndecomposable"] := true;
JTable[[33, 13, k<>0]]["DerivationsDimension"]:= 6;
JTable[[33, 13, k<>0]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [6];
JTable[[33, 13, k<>0]]["DerivationsOfDerivationsDimension"]:= 6;
JTable[[33, 13, k<>0]]["SubspaceType"]:= nonnull;
JTable[[33, 13, k<>0]]["IsotropySubalgebraInSemisimplePart"]:= NA;
JTable[[33, 13, k<>0]]["VectorFields"]:= [D_x2, D_x3, x3*D_x2-exp(x1)
*D_x3, -2*D_x1/(k-1)+2*k*x2*D_x2/(k-1)+x3*D_x3,exp(-x1)*D_x1+
(exp(-x1*k-x1+x4*k-x4)*k44+exp(-x1*(k+1))*k2*k12*k-exp(-x1*(k+1))
*k2*k12)*D_x2/((k-1)*k12)+exp(-x1)*D_x4];
#####
# [33, 13, k=0]
#####
JTable[[33, 13, k=0]]:=table();
JTable[[33, 13, k=0]]["AlgebraType"] := "Solvable";
JTable[[33, 13, k=0]]["Indecomposable"] := true;
JTable[[33, 13, k=0]]["DecompositionDimensions"] := [5];
JTable[[33, 13, k=0]]["IsotropySubalgebraReductivePair"] := false;
JTable[[33, 13, k=0]]["IsotropySubalgebraSymmetricPair"] := NA;
JTable[[33, 13, k=0]]["DerivedSeriesDimensions"] := [5,4,2,0];
JTable[[33, 13, k=0]]["OrbitDimension"] := 4;
JTable[[33, 13, k=0]]["IsotropyType"] := "F14";
JTable[[33, 13, k=0]]["NilradicalDimension"] := 4;
JTable[[33, 13, k=0]]["NilradicalIndecomposable"] := true;
JTable[[33, 13, k=0]]["DerivationsDimension"]:= 6;
JTable[[33, 13, k=0]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [6];
JTable[[33, 13, k=0]]["DerivationsOfDerivationsDimension"]:= 7;
JTable[[33, 13, k=0]]["SubspaceType"]:= nonnull;
JTable[[33, 13, k=0]]["IsotropySubalgebraInSemisimplePart"]:= NA;
JTable[[33, 13, k=0]]["VectorFields"]:= [D_x2, D_x3,x3*D_x2-exp(x1)
*D_x3, 2*D_x1+x3*D_x3,exp(-x1)*D_x1-(exp(-x1-x4)*k44-exp(-x1)*k2*k12)
*D_x2/k12+exp(-x1)*D_x4];
#####
# [33, 14, k<>0]
#####
JTable[[33, 14, k<>0]]:=table();
JTable[[33, 14, k<>0]]["AlgebraType"] := "Solvable";
JTable[[33, 14, k<>0]]["Indecomposable"] := true;
JTable[[33, 14, k<>0]]["DecompositionDimensions"] := [5];
JTable[[33, 14, k<>0]]["IsotropySubalgebraReductivePair"] := true;
JTable[[33, 14, k<>0]]["IsotropySubalgebraSymmetricPair"] := false;
JTable[[33, 14, k<>0]]["DerivedSeriesDimensions"] := [5,3,0];
JTable[[33, 14, k<>0]]["OrbitDimension"] := 4;

```

```

JTable[[33, 14, k<>0]]["IsotropyType"] := "F14";
JTable[[33, 14, k<>0]]["NilradicalDimension"] := 4;
JTable[[33, 14, k<>0]]["NilradicalIndecomposable"] := true;
JTable[[33, 14, k<>0]]["DerivationsDimension"]:= 6;
JTable[[33, 14, k<>0]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [6];
JTable[[33, 14, k<>0]]["DerivationsOfDerivationsDimension"]:= 8;
JTable[[33, 14, k<>0]]["SubspaceType"]:= nonnull;
JTable[[33, 14, k<>0]]["IsotropySubalgebraInSemisimplePart"]:= NA;
JTable[[33, 14, k<>0]]["VectorFields"]:= [D_x2, D_x3, x3*D_x2-exp(x1)
*D_x3, D_x1+x2*D_x2+x3*D_x3, exp(-x1)*D_x1+(x1*k-(1/2)*exp(-2*x4))
*D_x2+exp(-x1)*D_x4];
#####
# [33, 14, k=0]
#####
JTable[[33, 14, k=0]]:=table();
JTable[[33, 14, k=0]]["AlgebraType"] := "Solvable";
JTable[[33, 14, k=0]]["Indecomposable"] := true;
JTable[[33, 14, k=0]]["DecompositionDimensions"] := [5];
JTable[[33, 14, k=0]]["IsotropySubalgebraReductivePair"] := true;
JTable[[33, 14, k=0]]["IsotropySubalgebraSymmetricPair"] := false;
JTable[[33, 14, k=0]]["DerivedSeriesDimensions"] := [5,3,0];
JTable[[33, 14, k=0]]["OrbitDimension"] := 4;
JTable[[33, 14, k=0]]["IsotropyType"] := "F14";
JTable[[33, 14, k=0]]["NilradicalDimension"] := 4;
JTable[[33, 14, k=0]]["NilradicalIndecomposable"] := true;
JTable[[33, 14, k=0]]["DerivationsDimension"]:= 7;
JTable[[33, 14, k=0]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [7];
JTable[[33, 14, k=0]]["DerivationsOfDerivationsDimension"]:= 7;
JTable[[33, 14, k=0]]["SubspaceType"]:= nonnull;
JTable[[33, 14, k=0]]["IsotropySubalgebraInSemisimplePart"]:= NA;
JTable[[33, 14, k=0]]["VectorFields"]:= [D_x2, D_x3, x3*D_x2-exp(x1)
*D_x3, D_x1+x2*D_x2+x3*D_x3, exp(-x1)*D_x1-(1/2)*exp(-2*x4)*D_x2
+exp(-x1)*D_x4];
#####
# [33, 16]
#####
JTable[[33, 16]]:=table();
JTable[[33, 16]]["AlgebraType"] := "Generic";
JTable[[33, 16]]["Indecomposable"] := false;
JTable[[33, 16]]["DecompositionDimensions"] := [3,2];
JTable[[33, 16]]["IsotropySubalgebraReductivePair"] := true;
JTable[[33, 16]]["IsotropySubalgebraSymmetricPair"] := false;
JTable[[33, 16]]["DerivedSeriesDimensions"] := [5,3,3];
JTable[[33, 16]]["OrbitDimension"] := 4;
JTable[[33, 16]]["IsotropyType"] := "F14";
JTable[[33, 16]]["NilradicalDimension"] := 2;
JTable[[33, 16]]["NilradicalIndecomposable"] := false;
JTable[[33, 16]]["DerivationsDimension"]:= 7;
JTable[[33, 16]]["DecompositionOfRadicalOfDerivationsDimensions"]

```

```

:= [];
JTable[[33, 16]]["DerivationsOfDerivationsDimension"]:= 7;
JTable[[33, 16]]["SubspaceType"]:= nonnull;
JTable[[33, 16]]["IsotropySubalgebraInSemisimplePart"]:= false;
JTable[[33, 16]]["VectorFields"]:= [D_x1, exp(x3)*D_x2,D_x3, D_x4,
exp(-x3)*D_x1-x2^2*exp(-x3)*D_x2-2*exp(-x3)*x2*D_x3];
#####
# [33, 17, 1] epsilon = 1
#####
JTable[[33, 17, 1]]:=table();
JTable[[33, 17, 1]]["AlgebraType"] := "Generic";
JTable[[33, 17, 1]]["Indecomposable"] := false;
JTable[[33, 17, 1]]["DecompositionDimensions"] := [3,2];
JTable[[33, 17, 1]]["IsotropySubalgebraReductivePair"] := true;
JTable[[33, 17, 1]]["IsotropySubalgebraSymmetricPair"] := false;
JTable[[33, 17, 1]]["DerivedSeriesDimensions"] := [5,3,3];
JTable[[33, 17, 1]]["OrbitDimension"] := 4;
JTable[[33, 17, 1]]["IsotropyType"] := "F13";
JTable[[33, 17, 1]]["NilradicalDimension"] := 2;
JTable[[33, 17, 1]]["NilradicalIndecomposable"] := false;
JTable[[33, 17, 1]]["DerivationsDimension"]:= 7;
JTable[[33, 17, 1]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [];
JTable[[33, 17, 1]]["DerivationsOfDerivationsDimension"]:= 7;
JTable[[33, 17, 1]]["SubspaceType"]:= nonnull;
JTable[[33, 17, 1]]["IsotropySubalgebraInSemisimplePart"]:= false;
JTable[[33, 17, 1]]["VectorFields"]:= [D_x2, x2*D_x2+D_x3, -exp(x3)
*D_x1+(x2^2+exp(2*x3))*D_x2+2*x2*D_x3,D_x1, D_x4];
#####
# [33, 17, -1] epsilon = -1
#####
JTable[[33, 17, -1]]:=table();
JTable[[33, 17, -1]]["AlgebraType"] := "Generic";
JTable[[33, 17, -1]]["Indecomposable"] := false;
JTable[[33, 17, -1]]["DecompositionDimensions"] := [3,2];
JTable[[33, 17, -1]]["IsotropySubalgebraReductivePair"] := true;
JTable[[33, 17, -1]]["IsotropySubalgebraSymmetricPair"] := false;
JTable[[33, 17, -1]]["DerivedSeriesDimensions"] := [5,3,3];
JTable[[33, 17, -1]]["OrbitDimension"] := 4;
JTable[[33, 17, -1]]["IsotropyType"] := "F12";
JTable[[33, 17, -1]]["NilradicalDimension"] := 2;
JTable[[33, 17, -1]]["NilradicalIndecomposable"] := false;
JTable[[33, 17, -1]]["DerivationsDimension"]:= 7;
JTable[[33, 17, -1]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [];
JTable[[33, 17, -1]]["DerivationsOfDerivationsDimension"]:= 7;
JTable[[33, 17, -1]]["SubspaceType"]:= nonnull;
JTable[[33, 17, -1]]["IsotropySubalgebraInSemisimplePart"]:= false;
JTable[[33, 17, -1]]["VectorFields"]:= [D_x2, x2*D_x2+D_x3, -exp(x3)
*D_x1+(x2^2-exp(2*x3))*D_x2+2*x2*D_x3, D_x1, D_x4];
#####

```



```

# [33, 18, 0] epsilon=0
#####
JTable[[33, 18, 0]]:=table();
JTable[[33, 18, 0]]["AlgebraType"] := "Nilpotent";
JTable[[33, 18, 0]]["Indecomposable"] := true;
JTable[[33, 18, 0]]["DecompositionDimensions"] := [5];
JTable[[33, 18, 0]]["IsotropySubalgebraReductivePair"] := true;
JTable[[33, 18, 0]]["IsotropySubalgebraSymmetricPair"] := false;
JTable[[33, 18, 0]]["DerivedSeriesDimensions"] := [5,2,0];
JTable[[33, 18, 0]]["OrbitDimension"] := 4;
JTable[[33, 18, 0]]["IsotropyType"] := "F14";
JTable[[33, 18, 0]]["NilradicalDimension"] := 5;
JTable[[33, 18, 0]]["NilradicalIndecomposable"] := true;
JTable[[33, 18, 0]]["DerivationsDimension"]:= 10;
JTable[[33, 18, 0]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [10];
JTable[[33, 18, 0]]["DerivationsOfDerivationsDimension"]:= 11;
JTable[[33, 18, 0]]["SubspaceType"]:= nonnull;
JTable[[33, 18, 0]]["IsotropySubalgebraInSemisimplePart"]:= NA;
JTable[[33, 18, 0]]["VectorFields"]:= [D_x1, D_x2, D_x3,-x3*D_x2+x1*
D_x3, x1*D_x2+D_x4];
#####
# [33, 18, 1] epsilon=1
#####
JTable[[33, 18, 1]]:=table();
JTable[[33, 18, 1]]["AlgebraType"] := "Solvable";
JTable[[33, 18, 1]]["Indecomposable"] := true;
JTable[[33, 18, 1]]["DecompositionDimensions"] := [5];
JTable[[33, 18, 1]]["IsotropySubalgebraReductivePair"] := true;
JTable[[33, 18, 1]]["IsotropySubalgebraSymmetricPair"] := false;
JTable[[33, 18, 1]]["DerivedSeriesDimensions"] := [5,3,0];
JTable[[33, 18, 1]]["OrbitDimension"] := 4;
JTable[[33, 18, 1]]["IsotropyType"] := "F14";
JTable[[33, 18, 1]]["NilradicalDimension"] := 4;
JTable[[33, 18, 1]]["NilradicalIndecomposable"] := true;
JTable[[33, 18, 1]]["DerivationsDimension"]:= 6;
JTable[[33, 18, 1]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [6];
JTable[[33, 18, 1]]["DerivationsOfDerivationsDimension"]:= 8;
JTable[[33, 18, 1]]["SubspaceType"]:= nonnull;
JTable[[33, 18, 1]]["IsotropySubalgebraInSemisimplePart"]:= NA;
JTable[[33, 18, 1]]["VectorFields"]:= [D_x1, D_x2, D_x3,-x3*D_x2+x1*
D_x3, x1*D_x1+(x1+x2)*D_x2+x3*D_x3+D_x4];
#####
# [33, 19]
#####
JTable[[33, 19]]:=table();
JTable[[33, 19]]["AlgebraType"] := "Generic";
JTable[[33, 19]]["Indecomposable"] := false;
JTable[[33, 19]]["DecompositionDimensions"] := [3,2];
JTable[[33, 19]]["IsotropySubalgebraReductivePair"] := true;

```

```

JTable[[33, 19]]["IsotropySubalgebraSymmetricPair"] := false;
JTable[[33, 19]]["DerivedSeriesDimensions"] := [5,3,3];
JTable[[33, 19]]["OrbitDimension"] := 4;
JTable[[33, 19]]["IsotropyType"] := "F12";
JTable[[33, 19]]["NilradicalDimension"] := 2;
JTable[[33, 19]]["NilradicalIndecomposable"] := false;
JTable[[33, 19]]["DerivationsDimension"] := 7;
JTable[[33, 19]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [];
JTable[[33, 19]]["DerivationsOfDerivationsDimension"] := 7;
JTable[[33, 19]]["SubspaceType"] := nonnull;
JTable[[33, 19]]["IsotropySubalgebraInSemisimplePart"] := false;
JTable[[33, 19]]["VectorFields"] := [D_x2, cos(x2)*D_x1-cos(x1)*sin(x2)
*D_x2/sin(x1)+sin(x2)*D_x3/sin(x1), -sin(x2)*D_x1-cos(x1)*cos(x2)
*D_x2/sin(x1)+cos(x2)*D_x3/sin(x1), D_x3, D_x4];
#####
# [33, 20]
#####
JTable[[33, 20]]:=table();
JTable[[33, 20]]["AlgebraType"] := "Generic";
JTable[[33, 20]]["Indecomposable"] := false;
JTable[[33, 20]]["DecompositionDimensions"] := [3,2];
JTable[[33, 20]]["IsotropySubalgebraReductivePair"] := true;
JTable[[33, 20]]["IsotropySubalgebraSymmetricPair"] := false;
JTable[[33, 20]]["DerivedSeriesDimensions"] := [5,3,3];
JTable[[33, 20]]["OrbitDimension"] := 4;
JTable[[33, 20]]["IsotropyType"] := "F12";
JTable[[33, 20]]["NilradicalDimension"] := 2;
JTable[[33, 20]]["NilradicalIndecomposable"] := false;
JTable[[33, 20]]["DerivationsDimension"] := 7;
JTable[[33, 20]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [];
JTable[[33, 20]]["DerivationsOfDerivationsDimension"] := 7;
JTable[[33, 20]]["SubspaceType"] := nonnull;
JTable[[33, 20]]["IsotropySubalgebraInSemisimplePart"] := false;
JTable[[33, 20]]["VectorFields"] := [D_x2,cos(x2)*D_x1/cos(x3)-sin(x3)
*cos(x2)*D_x2/cos(x3)+sin(x2)*D_x3, -sin(x2)*D_x1/cos(x3)+sin(x3)
*sin(x2)*D_x2/cos(x3)+cos(x2)*D_x3, D_x1, D_x4];
#####
# [33, 21, c<>0]
#####
JTable[[33, 21, c<>0]]:=table();
JTable[[33, 21, c<>0]]["AlgebraType"] := "Solvable";
JTable[[33, 21, c<>0]]["Indecomposable"] := true;
JTable[[33, 21, c<>0]]["DecompositionDimensions"] := [5];
JTable[[33, 21, c<>0]]["IsotropySubalgebraReductivePair"] := true;
JTable[[33, 21, c<>0]]["IsotropySubalgebraSymmetricPair"] := false;
JTable[[33, 21, c<>0]]["DerivedSeriesDimensions"] := [5,3,1,0];
JTable[[33, 21, c<>0]]["OrbitDimension"] := 4;
JTable[[33, 21, c<>0]]["IsotropyType"] := "F13";
JTable[[33, 21, c<>0]]["NilradicalDimension"] := 3;

```

```

JTable[[33, 21, c<>0]]["NilradicalIndecomposable"] := true;
JTable[[33, 21, c<>0]]["DerivationsDimension"]:= 5;
JTable[[33, 21, c<>0]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [5];
JTable[[33, 21, c<>0]]["DerivationsOfDerivationsDimension"]:= 5;
JTable[[33, 21, c<>0]]["SubspaceType"]:= nonnull;
JTable[[33, 21, c<>0]]["IsotropySubalgebraInSemisimplePart"]:= NA;
JTable[[33, 21, c<>0]]["VectorFields"]:= [D_x2, D_x3,-D_x1+x3*D_x2,
-x1*D_x1+x3*D_x3,x1*c*D_x1+c*x2*D_x2+D_x4];
#####
# [33, 21, c=0]
#####
JTable[[33, 21, c=0]]:=table();
JTable[[33, 21, c=0]]["AlgebraType"] := "Solvable";
JTable[[33, 21, c=0]]["Indecomposable"] := false;
JTable[[33, 21, c=0]]["DecompositionDimensions"] := [5];
JTable[[33, 21, c=0]]["IsotropySubalgebraReductivePair"] := true;
JTable[[33, 21, c=0]]["IsotropySubalgebraSymmetricPair"] := false;
JTable[[33, 21, c=0]]["DerivedSeriesDimensions"] := [5,3,1,0];
JTable[[33, 21, c=0]]["OrbitDimension"] := 4;
JTable[[33, 21, c=0]]["IsotropyType"] := "F13";
JTable[[33, 21, c=0]]["NilradicalDimension"] := 4;
JTable[[33, 21, c=0]]["NilradicalIndecomposable"] := false;
JTable[[33, 21, c=0]]["DerivationsDimension"]:= 8;
JTable[[33, 21, c=0]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [8];
JTable[[33, 21, c=0]]["DerivationsOfDerivationsDimension"]:= 9;
JTable[[33, 21, c=0]]["SubspaceType"]:= nonnull;
JTable[[33, 21, c=0]]["IsotropySubalgebraInSemisimplePart"]:= NA;
JTable[[33, 21, c=0]]["VectorFields"]:= [D_x2, D_x3,-D_x1+x3*D_x2,
-x1*D_x1+x3*D_x3, D_x4];
#####
# [33, 22]
#####
JTable[[33, 22]]:=table();
JTable[[33, 22]]["AlgebraType"] := "Solvable";
JTable[[33, 22]]["Indecomposable"] := true;
JTable[[33, 22]]["DecompositionDimensions"] := [5];
JTable[[33, 22]]["IsotropySubalgebraReductivePair"] := true;
JTable[[33, 22]]["IsotropySubalgebraSymmetricPair"] := false;
JTable[[33, 22]]["DerivedSeriesDimensions"] := [5,3,1,0];
JTable[[33, 22]]["OrbitDimension"] := 4;
JTable[[33, 22]]["IsotropyType"] := "F12";
JTable[[33, 22]]["NilradicalDimension"] := 3;
JTable[[33, 22]]["NilradicalIndecomposable"] := true;
JTable[[33, 22]]["DerivationsDimension"]:= 5;
JTable[[33, 22]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [5];
JTable[[33, 22]]["DerivationsOfDerivationsDimension"]:= 5;
JTable[[33, 22]]["SubspaceType"]:= nonnull;
JTable[[33, 22]]["IsotropySubalgebraInSemisimplePart"]:= NA;

```

```

JTable[[33, 22]]["VectorFields"]:= [D_x2, D_x3, -D_x1+x3*D_x2,
-x3*D_x1+((1/2)*x3^2-(1/2)*x1^2)*D_x2+x1*D_x3,x1*D_x1+2*x2*D_x2
+x3*D_x3+D_x4];
#####
# [33, 23]
#####
JTable[[33, 23]]:=table();
JTable[[33, 23]]["AlgebraType"] := "Solvable";
JTable[[33, 23]]["Indecomposable"] := false;
JTable[[33, 23]]["DecompositionDimensions"] := [4,1];
JTable[[33, 23]]["IsotropySubalgebraReductivePair"] := true;
JTable[[33, 23]]["IsotropySubalgebraSymmetricPair"] := false;
JTable[[33, 23]]["DerivedSeriesDimensions"] := [5,3,1,0];
JTable[[33, 23]]["OrbitDimension"] := 4;
JTable[[33, 23]]["IsotropyType"] := "F12";
JTable[[33, 23]]["NilradicalDimension"] := 4;
JTable[[33, 23]]["NilradicalIndecomposable"] := false;
JTable[[33, 23]]["DerivationsDimension"]:= 8;
JTable[[33, 23]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [8];
JTable[[33, 23]]["DerivationsOfDerivationsDimension"]:= 9;
JTable[[33, 23]]["SubspaceType"]:= nonnull;
JTable[[33, 23]]["IsotropySubalgebraInSemisimplePart"]:= NA;
JTable[[33, 23]]["VectorFields"]:= [D_x2, D_x3, -D_x1+x3*D_x2, -x3
*D_x1+((1/2)*x3^2-(1/2)*x1^2)*D_x2+x1*D_x3, D_x4];
#####
# [33, 24]
#####
JTable[[33, 24]]:=table();
JTable[[33, 24]]["AlgebraType"] := "Solvable";
JTable[[33, 24]]["Indecomposable"] := false;
JTable[[33, 24]]["DecompositionDimensions"] := [2,2,1];
JTable[[33, 24]]["IsotropySubalgebraReductivePair"] := true;
JTable[[33, 24]]["IsotropySubalgebraSymmetricPair"] := false;
JTable[[33, 24]]["DerivedSeriesDimensions"] := [5,2,0];
JTable[[33, 24]]["OrbitDimension"] := 4;
JTable[[33, 24]]["IsotropyType"] := "F13";
JTable[[33, 24]]["NilradicalDimension"] := 3;
JTable[[33, 24]]["NilradicalIndecomposable"] := false;
JTable[[33, 24]]["DerivationsDimension"]:= 7;
JTable[[33, 24]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [3,2,2];
JTable[[33, 24]]["DerivationsOfDerivationsDimension"]:= 10;
JTable[[33, 24]]["SubspaceType"]:= nonnull;
JTable[[33, 24]]["IsotropySubalgebraInSemisimplePart"]:= NA;
JTable[[33, 24]]["VectorFields"]:= [D_x2, D_x3, -D_x1+x2*D_x2,-D_x1
+x3*D_x3, (alpha+1)*D_x1+D_x4];
#####
# [33, 25]
#####
JTable[[33, 25]]:=table();

```

```

JTable[[33, 25]]["AlgebraType"] := "Solvable";
JTable[[33, 25]]["Indecomposable"] := false;
JTable[[33, 25]]["DecompositionDimensions"] := [4,1];
JTable[[33, 25]]["IsotropySubalgebraReductivePair"] := true;
JTable[[33, 25]]["IsotropySubalgebraSymmetricPair"] := false;
JTable[[33, 25]]["DerivedSeriesDimensions"] := [5,2,0];
JTable[[33, 25]]["OrbitDimension"] := 4;
JTable[[33, 25]]["IsotropyType"] := "F12";
JTable[[33, 25]]["NilradicalDimension"] := 3;
JTable[[33, 25]]["NilradicalIndecomposable"] := false;
JTable[[33, 25]]["DerivationsDimension"] := 7;
JTable[[33, 25]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [4,3];
JTable[[33, 25]]["DerivationsOfDerivationsDimension"] := 10;
JTable[[33, 25]]["SubspaceType"] := nonnull;
JTable[[33, 25]]["IsotropySubalgebraInSemisimplePart"] := NA;
JTable[[33, 25]]["VectorFields"] := [D_x2, D_x3, -D_x1+x2*D_x2+x3*D_x3,
-x3*D_x2+x2*D_x3, alpha*D_x1+D_x4];
#####
# [33, 26]
#####
JTable[[33, 26]]:=table();
JTable[[33, 26]]["AlgebraType"] := "Solvable";
JTable[[33, 26]]["Indecomposable"] := false;
JTable[[33, 26]]["DecompositionDimensions"] := [4,1];
JTable[[33, 26]]["IsotropySubalgebraReductivePair"] := true;
JTable[[33, 26]]["IsotropySubalgebraSymmetricPair"] := false;
JTable[[33, 26]]["DerivedSeriesDimensions"] := [5,2,0];
JTable[[33, 26]]["OrbitDimension"] := 4;
JTable[[33, 26]]["IsotropyType"] := "F12";
JTable[[33, 26]]["NilradicalDimension"] := 3;
JTable[[33, 26]]["NilradicalIndecomposable"] := false;
JTable[[33, 26]]["DerivationsDimension"] := 7;
JTable[[33, 26]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [4,3];
JTable[[33, 26]]["DerivationsOfDerivationsDimension"] := 10;
JTable[[33, 26]]["SubspaceType"] := nonnull;
JTable[[33, 26]]["IsotropySubalgebraInSemisimplePart"] := NA;
JTable[[33, 26]]["VectorFields"] := [D_x2, D_x3, 2*D_x1+x2*D_x2+x3
*D_x3, -x3*D_x2+x2*D_x3, alpha*D_x1+D_x4];
#####
# [33, 27]
#####
JTable[[33, 27]]:=table();
JTable[[33, 27]]["AlgebraType"] := "Solvable";
JTable[[33, 27]]["Indecomposable"] := false;
JTable[[33, 27]]["DecompositionDimensions"] := [2,2,1];
JTable[[33, 27]]["IsotropySubalgebraReductivePair"] := true;
JTable[[33, 27]]["IsotropySubalgebraSymmetricPair"] := false;
JTable[[33, 27]]["DerivedSeriesDimensions"] := [5,2,0];
JTable[[33, 27]]["OrbitDimension"] := 4;

```

```

JTable[[33, 27]]["IsotropyType"] := "F13";
JTable[[33, 27]]["NilradicalDimension"] := 3;
JTable[[33, 27]]["NilradicalIndecomposable"] := false;
JTable[[33, 27]]["DerivationsDimension"] := 7;
JTable[[33, 27]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [3,2,2];
JTable[[33, 27]]["DerivationsOfDerivationsDimension"] := 10;
JTable[[33, 27]]["SubspaceType"] := nonnull;
JTable[[33, 27]]["IsotropySubalgebraInSemisimplePart"] := NA;
JTable[[33, 27]]["VectorFields"] := [D_x2, D_x3, D_x1+x3*D_x3, D_x1+x2
*D_x2, alpha*D_x1+D_x4];
#####
# [33, 28, 0, k<0, l<0]
#####
JTable[[33, 28, 0, k<0, l<0]]:=table();
JTable[[33, 28, 0, k<0, l<0]]["AlgebraType"] := "Solvable";
JTable[[33, 28, 0, k<0, l<0]]["Indecomposable"] := true;
JTable[[33, 28, 0, k<0, l<0]]["DecompositionDimensions"] := [5];
JTable[[33, 28, 0, k<0, l<0]]["IsotropySubalgebraReductivePair"]
:= true;
JTable[[33, 28, 0, k<0, l<0]]["IsotropySubalgebraSymmetricPair"]
:= false;
JTable[[33, 28, 0, k<0, l<0]]["DerivedSeriesDimensions"]
:= [5,3,0];
JTable[[33, 28, 0, k<0, l<0]]["OrbitDimension"] := 4;
JTable[[33, 28, 0, k<0, l<0]]["IsotropyType"] := "F13";
JTable[[33, 28, 0, k<0, l<0]]["NilradicalDimension"] := 3;
JTable[[33, 28, 0, k<0, l<0]]["NilradicalIndecomposable"] := false;
JTable[[33, 28, 0, k<0, l<0]]["DerivationsDimension"] := 6;
JTable[[33, 28, 0, k<0, l<0]]["DecompositionOfRadicalOfDerivations
Dimensions"] := [2,2,2];
JTable[[33, 28, 0, k<0, l<0]]["DerivationsOfDerivations
Dimension"] := 6;
JTable[[33, 28, 0, k<0, l<0]]["SubspaceType"] := nonnull;
JTable[[33, 28, 0, k<0, l<0]]["IsotropySubalgebraInSemisimplePart"]
:= NA;
JTable[[33, 28, 0, k<0, l<0]]["VectorFields"] := [D_x2,D_x3, -D_x1,
x2*D_x2-x3*D_x3,l*x1*D_x1+x3*k*D_x3+D_x4];
#####
# [33, 28, 1, k<0, l<0]
#####
JTable[[33, 28, 1, k<0, l<0]]:=table();
JTable[[33, 28, 1, k<0, l<0]]["AlgebraType"] := "Solvable";
JTable[[33, 28, 1, k<0, l<0]]["Indecomposable"] := true;
JTable[[33, 28, 1, k<0, l<0]]["DecompositionDimensions"] := [5];
JTable[[33, 28, 1, k<0, l<0]]["IsotropySubalgebraReductivePair"]
:= true;
JTable[[33, 28, 1, k<0, l<0]]["IsotropySubalgebraSymmetricPair"]
:= false;
JTable[[33, 28, 1, k<0, l<0]]["DerivedSeriesDimensions"]
:= [5,3,0];

```

```

JTable[[33, 28, 1, k<>0, l<>0]]["OrbitDimension"] := 4;
JTable[[33, 28, 1, k<>0, l<>0]]["IsotropyType"] := "F13";
JTable[[33, 28, 1, k<>0, l<>0]]["NilradicalDimension"] := 3;
JTable[[33, 28, 1, k<>0, l<>0]]["NilradicalIndecomposable"] := false;
JTable[[33, 28, 1, k<>0, l<>0]]["DerivationsDimension"]:= 6;
JTable[[33, 28, 1, k<>0, l<>0]]["DecompositionOfRadicalOfDerivations
Dimensions"] := [2,2,2];
JTable[[33, 28, 1, k<>0, l<>0]]["DerivationsOfDerivationsDimension"]
:= 6;
JTable[[33, 28, 1, k<>0, l<>0]]["SubspaceType"]:= nonnull;
JTable[[33, 28, 1, k<>0, l<>0]]["IsotropySubalgebraInSemisimplePart"]
:= NA;
JTable[[33, 28, 1, k<>0, l<>0]]["VectorFields"]:= [D_x2, D_x3, -D_x1,
x2*D_x2-x3*D_x3, l*x1*D_x1+x2*D_x2+x3*k*D_x3+D_x4];
#####
# [33, 28, 0, k=0, l<>0]
#####
JTable[[33, 28, 0, k=0, l<>0]]:=table();
JTable[[33, 28, 0, k=0, l<>0]]["AlgebraType"] := "Solvable";
JTable[[33, 28, 0, k=0, l<>0]]["Indecomposable"] := false;
JTable[[33, 28, 0, k=0, l<>0]]["DecompositionDimensions"] := [3,2];
JTable[[33, 28, 0, k=0, l<>0]]["IsotropySubalgebraReductivePair"]
:= true;
JTable[[33, 28, 0, k=0, l<>0]]["IsotropySubalgebraSymmetricPair"]
:= false;
JTable[[33, 28, 0, k=0, l<>0]]["DerivedSeriesDimensions"]
:= [5,3,0];
JTable[[33, 28, 0, k=0, l<>0]]["OrbitDimension"] := 4;
JTable[[33, 28, 0, k=0, l<>0]]["IsotropyType"] := "F13";
JTable[[33, 28, 0, k=0, l<>0]]["NilradicalDimension"] := 3;
JTable[[33, 28, 0, k=0, l<>0]]["NilradicalIndecomposable"] := false;
JTable[[33, 28, 0, k=0, l<>0]]["DerivationsDimension"]:= 6;
JTable[[33, 28, 0, k=0, l<>0]]["DecompositionOfRadicalOfDerivations
Dimensions"] := [2,2,2];
JTable[[33, 28, 0, k=0, l<>0]]["DerivationsOfDerivations
Dimension"]:= 6;
JTable[[33, 28, 0, k=0, l<>0]]["SubspaceType"]:= nonnull;
JTable[[33, 28, 0, k=0, l<>0]]["IsotropySubalgebraInSemisimplePart"]
:= NA;
JTable[[33, 28, 0, k=0, l<>0]]["VectorFields"]:= [D_x2, D_x3,-D_x1,
x2*D_x2-x3*D_x3, l*x1*D_x1+D_x4];
#####
# [33, 28, 0, k=0, l=0]
#####
JTable[[33, 28, 0, k=0, l=0]]:=table();
JTable[[33, 28, 0, k=0, l=0]]["AlgebraType"] := "Solvable";
JTable[[33, 28, 0, k=0, l=0]]["Indecomposable"] := false;
JTable[[33, 28, 0, k=0, l=0]]["DecompositionDimensions"] := [3,2];
JTable[[33, 28, 0, k=0, l=0]]["IsotropySubalgebraReductivePair"]
:= true;
JTable[[33, 28, 0, k=0, l=0]]["IsotropySubalgebraSymmetricPair"]

```

```

:= true;
JTable[[33, 28, 0, k=0, l=0]]["DerivedSeriesDimensions"] := [5,2,0];
JTable[[33, 28, 0, k=0, l=0]]["OrbitDimension"] := 4;
JTable[[33, 28, 0, k=0, l=0]]["IsotropyType"] := "F13";
JTable[[33, 28, 0, k=0, l=0]]["NilradicalDimension"] := 4;
JTable[[33, 28, 0, k=0, l=0]]["NilradicalIndecomposable"] := false;
JTable[[33, 28, 0, k=0, l=0]]["DerivationsDimension"]:= 10;
JTable[[33, 28, 0, k=0, l=0]]["DecompositionOfRadicalOfDerivations
Dimensions"] := [3,2,2];
JTable[[33, 28, 0, k=0, l=0]]["DerivationsOfDerivationsDimension"]
:= 10;
JTable[[33, 28, 0, k=0, l=0]]["SubspaceType"]:= nonnull;
JTable[[33, 28, 0, k=0, l=0]]["IsotropySubalgebraInSemisimplePart"]
:= NA;
JTable[[33, 28, 0, k=0, l=0]]["VectorFields"]:= [D_x2, D_x3,-D_x1,
x2*D_x2-x3*D_x3, D_x4];
#####
# [33, 28, 0, k<>0, l=0]
#####
JTable[[33, 28, 0, k<>0, l=0]]:=table();
JTable[[33, 28, 0, k<>0, l=0]]["AlgebraType"] := "Solvable";
JTable[[33, 28, 0, k<>0, l=0]]["Indecomposable"] := false;
JTable[[33, 28, 0, k<>0, l=0]]["DecompositionDimensions"] := [2,2,1];
JTable[[33, 28, 0, k<>0, l=0]]["IsotropySubalgebraReductivePair"]
:= true;
JTable[[33, 28, 0, k<>0, l=0]]["IsotropySubalgebraSymmetricPair"]
:= true;
JTable[[33, 28, 0, k<>0, l=0]]["DerivedSeriesDimensions"] := [5,2,0];
JTable[[33, 28, 0, k<>0, l=0]]["OrbitDimension"] := 4;
JTable[[33, 28, 0, k<>0, l=0]]["IsotropyType"] := "F13";
JTable[[33, 28, 0, k<>0, l=0]]["NilradicalDimension"] := 3;
JTable[[33, 28, 0, k<>0, l=0]]["NilradicalIndecomposable"] := false;
JTable[[33, 28, 0, k<>0, l=0]]["DerivationsDimension"]:= 7;
JTable[[33, 28, 0, k<>0, l=0]]["DecompositionOfRadicalOfDerivations
Dimensions"] := [3,2,2];
JTable[[33, 28, 0, k<>0, l=0]]["DerivationsOfDerivations
Dimension"]:= 10;
JTable[[33, 28, 0, k<>0, l=0]]["SubspaceType"]:= nonnull;
JTable[[33, 28, 0, k<>0, l=0]]["IsotropySubalgebraInSemisimplePart"]
:= NA;
JTable[[33, 28, 0, k<>0, l=0]]["VectorFields"]:= [D_x2, D_x3, -D_x1,
x2*D_x2-x3*D_x3, x3*k*D_x3+D_x4];
#####
# [33, 28, 1, k<>0, l=0]
#####
JTable[[33, 28, 1, k<>0, l=0]]:=table();
JTable[[33, 28, 1, k<>0, l=0]]["AlgebraType"] := "Solvable";
JTable[[33, 28, 1, k<>0, l=0]]["Indecomposable"] := false;
JTable[[33, 28, 1, k<>0, l=0]]["DecompositionDimensions"] := [2,2,1];
JTable[[33, 28, 1, k<>0, l=0]]["IsotropySubalgebraReductivePair"]
:= true;

```



```

JTable[[33, 28, 1, k<>0, l=0]]["IsotropySubalgebraSymmetricPair"]
:= true;
JTable[[33, 28, 1, k<>0, l=0]]["DerivedSeriesDimensions"] := [5,2,0];
JTable[[33, 28, 1, k<>0, l=0]]["OrbitDimension"] := 4;
JTable[[33, 28, 1, k<>0, l=0]]["IsotropyType"] := "F13";
JTable[[33, 28, 1, k<>0, l=0]]["NilradicalDimension"] := 3;
JTable[[33, 28, 1, k<>0, l=0]]["NilradicalIndecomposable"] := false;
JTable[[33, 28, 1, k<>0, l=0]]["DerivationsDimension"]:= 7;
JTable[[33, 28, 1, k<>0, l=0]]["DecompositionOfRadicalOfDerivations
Dimensions"] := [3,2,2];
JTable[[33, 28, 1, k<>0, l=0]]["DerivationsOfDerivationsDimension"]
:= 10;
JTable[[33, 28, 1, k<>0, l=0]]["SubspaceType"]:= nonnull;
JTable[[33, 28, 1, k<>0, l=0]]["IsotropySubalgebraInSemisimplePart"]
:= NA;
JTable[[33, 28, 1, k<>0, l=0]]["VectorFields"]:= [D_x2, D_x3,-D_x1,
x2*D_x2-x3*D_x3, x2*D_x2+x3*k*D_x3+D_x4];
#####
# [33, 28, 1, k=0, l=0]
#####
JTable[[33, 28, 1, k=0, l=0]]:=table();
JTable[[33, 28, 1, k=0, l=0]]["AlgebraType"] := "Solvable";
JTable[[33, 28, 1, k=0, l=0]]["Indecomposable"] := false;
JTable[[33, 28, 1, k=0, l=0]]["DecompositionDimensions"] := [2,2,1];
JTable[[33, 28, 1, k=0, l=0]]["IsotropySubalgebraReductivePair"]
:= true;
JTable[[33, 28, 1, k=0, l=0]]["IsotropySubalgebraSymmetricPair"]
:= true;
JTable[[33, 28, 1, k=0, l=0]]["DerivedSeriesDimensions"] := [5,2,0];
JTable[[33, 28, 1, k=0, l=0]]["OrbitDimension"] := 4;
JTable[[33, 28, 1, k=0, l=0]]["IsotropyType"] := "F13";
JTable[[33, 28, 1, k=0, l=0]]["NilradicalDimension"] := 3;
JTable[[33, 28, 1, k=0, l=0]]["NilradicalIndecomposable"] := false;
JTable[[33, 28, 1, k=0, l=0]]["DerivationsDimension"]:= 7;
JTable[[33, 28, 1, k=0, l=0]]["DecompositionOfRadicalOfDerivations
Dimensions"] := [3,2,2];
JTable[[33, 28, 1, k=0, l=0]]["DerivationsOfDerivationsDimension"]
:= 10;
JTable[[33, 28, 1, k=0, l=0]]["SubspaceType"]:= nonnull;
JTable[[33, 28, 1, k=0, l=0]]["IsotropySubalgebraInSemisimplePart"]
:= NA;
JTable[[33, 28, 1, k=0, l=0]]["VectorFields"]:= [D_x2, D_x3,-D_x1,
x2*D_x2-x3*D_x3, x2*D_x2+D_x4];
#####
# [33, 29, 1, k<>0]
#####
JTable[[33, 29, 1, k<>0]]:=table();
JTable[[33, 29, 1, k<>0]]["AlgebraType"] := "Solvable";
JTable[[33, 29, 1, k<>0]]["Indecomposable"] := true;
JTable[[33, 29, 1, k<>0]]["DecompositionDimensions"] := [5];
JTable[[33, 29, 1, k<>0]]["IsotropySubalgebraReductivePair"]

```

```

:= false;
JTable[[33, 29, 1, k<>0]]["IsotropySubalgebraSymmetricPair"] := NA;
JTable[[33, 29, 1, k<>0]]["DerivedSeriesDimensions"] := [5,4,2,0];
JTable[[33, 29, 1, k<>0]]["OrbitDimension"] := 4;
JTable[[33, 29, 1, k<>0]]["IsotropyType"] := "F14";
JTable[[33, 29, 1, k<>0]]["NilradicalDimension"] := 4;
JTable[[33, 29, 1, k<>0]]["NilradicalIndecomposable"] := true;
JTable[[33, 29, 1, k<>0]]["DerivationsDimension"]:= 6;
JTable[[33, 29, 1, k<>0]]["DecompositionOfRadicalOfDerivations
Dimensions"] := [6];
JTable[[33, 29, 1, k<>0]]["DerivationsOfDerivationsDimension"]:= 6;
JTable[[33, 29, 1, k<>0]]["SubspaceType"]:= nonnull;
JTable[[33, 29, 1, k<>0]]["IsotropySubalgebraInSemisimplePart"]:= NA;
JTable[[33, 29, 1, k<>0]]["VectorFields"]:= [D_x2, D_x3,-D_x1, -x3
*D_x1+x2*D_x3, 1*x1*D_x1+x2*D_x2+(1/2)*x3*(1+1)*D_x3+D_x4];
#####
# [33, 29, 0, k<>0]
#####
JTable[[33, 29, 0, k<>0]]:=table();
JTable[[33, 29, 0, k<>0]]["AlgebraType"] := "Solvable";
JTable[[33, 29, 0, k<>0]]["Indecomposable"] := true;
JTable[[33, 29, 0, k<>0]]["DecompositionDimensions"] := [5];
JTable[[33, 29, 0, k<>0]]["IsotropySubalgebraReductivePair"]
:= false;
JTable[[33, 29, 0, k<>0]]["IsotropySubalgebraSymmetricPair"] := NA;
JTable[[33, 29, 0, k<>0]]["DerivedSeriesDimensions"] := [5,3,1,0];
JTable[[33, 29, 0, k<>0]]["OrbitDimension"] := 4;
JTable[[33, 29, 0, k<>0]]["IsotropyType"] := "F14";
JTable[[33, 29, 0, k<>0]]["NilradicalDimension"] := 4;
JTable[[33, 29, 0, k<>0]]["NilradicalIndecomposable"] := true;
JTable[[33, 29, 0, k<>0]]["DerivationsDimension"]:= 6;
JTable[[33, 29, 0, k<>0]]["DecompositionOfRadicalOfDerivations
Dimensions"] := [6];
JTable[[33, 29, 0, k<>0]]["DerivationsOfDerivationsDimension"]:= 6;
JTable[[33, 29, 0, k<>0]]["SubspaceType"]:= nonnull;
JTable[[33, 29, 0, k<>0]]["IsotropySubalgebraInSemisimplePart"]:= NA;
JTable[[33, 29, 0, k<>0]]["VectorFields"]:= [D_x2, D_x3,-D_x1, -x3*
D_x1+x2*D_x3, 1*x1*D_x1+(1/2)*x3*1*D_x3+D_x4];
#####
# [33, 29, 0, k=0]
#####
JTable[[33, 29, 0, k=0]]:=table();
JTable[[33, 29, 0, k=0]]["AlgebraType"] := "Nilpotent";
JTable[[33, 29, 0, k=0]]["Indecomposable"] := false;
JTable[[33, 29, 0, k=0]]["DecompositionDimensions"] := [4,1];
JTable[[33, 29, 0, k=0]]["IsotropySubalgebraReductivePair"] := true;
JTable[[33, 29, 0, k=0]]["IsotropySubalgebraSymmetricPair"] := false;
JTable[[33, 29, 0, k=0]]["DerivedSeriesDimensions"] := [5,2,0];
JTable[[33, 29, 0, k=0]]["OrbitDimension"] := 4;
JTable[[33, 29, 0, k=0]]["IsotropyType"] := "F14";
JTable[[33, 29, 0, k=0]]["NilradicalDimension"] := 5;

```

```

JTable[[33, 29, 0, k=0]]["NilradicalIndecomposable"] := false;
JTable[[33, 29, 0, k=0]]["DerivationsDimension"]:= 11;
JTable[[33, 29, 0, k=0]]["DecompositionOfRadicalOfDerivations
Dimensions"] := [11];
JTable[[33, 29, 0, k=0]]["DerivationsOfDerivationsDimension"]:= 11;
JTable[[33, 29, 0, k=0]]["SubspaceType"]:= nonnull;
JTable[[33, 29, 0, k=0]]["IsotropySubalgebraInSemisimplePart"]:= NA;
JTable[[33, 29, 0, k=0]]["VectorFields"]:= [D_x2, D_x3, -D_x1, -x3*
D_x1+x2*D_x3, D_x4];
#####
# [33, 30]
#####
JTable[[33, 30]]:=table();
JTable[[33, 30]]["AlgebraType"] := "Solvable";
JTable[[33, 30]]["Indecomposable"] := true;
JTable[[33, 30]]["DecompositionDimensions"] := [5];
JTable[[33, 30]]["IsotropySubalgebraReductivePair"] := false;
JTable[[33, 30]]["IsotropySubalgebraSymmetricPair"] := NA;
JTable[[33, 30]]["DerivedSeriesDimensions"] := [5,4,2,0];
JTable[[33, 30]]["OrbitDimension"] := 4;
JTable[[33, 30]]["IsotropyType"] := "F14";
JTable[[33, 30]]["NilradicalDimension"] := 4;
JTable[[33, 30]]["NilradicalIndecomposable"] := true;
JTable[[33, 30]]["DerivationsDimension"]:= 7;
JTable[[33, 30]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [7];
JTable[[33, 30]]["DerivationsOfDerivationsDimension"]:= 7;
JTable[[33, 30]]["SubspaceType"]:= nonnull;
JTable[[33, 30]]["IsotropySubalgebraInSemisimplePart"]:= NA;
JTable[[33, 30]]["VectorFields"]:= [D_x2, D_x3, -D_x1,x3*D_x2-x1*D_x3,
x1*D_x1+3*x2*D_x2+2*x3*D_x3+D_x4];
#####
# [33, 31, k<>0, l<>0]
#####
JTable[[33, 31, k<>0, l<>0]]:=table();
JTable[[33, 31, k<>0, l<>0]]["AlgebraType"] := "Solvable";
JTable[[33, 31, k<>0, l<>0]]["Indecomposable"] := true;
JTable[[33, 31, k<>0, l<>0]]["DecompositionDimensions"] := [5];
JTable[[33, 31, k<>0, l<>0]]["IsotropySubalgebraReductivePair"]
:= true;
JTable[[33, 31, k<>0, l<>0]]["IsotropySubalgebraSymmetricPair"]
:= false;
JTable[[33, 31, k<>0, l<>0]]["DerivedSeriesDimensions"] := [5,3,0];
JTable[[33, 31, k<>0, l<>0]]["OrbitDimension"] := 4;
JTable[[33, 31, k<>0, l<>0]]["IsotropyType"] := "F12";
JTable[[33, 31, k<>0, l<>0]]["NilradicalDimension"] := 3;
JTable[[33, 31, k<>0, l<>0]]["NilradicalIndecomposable"] := false;
JTable[[33, 31, k<>0, l<>0]]["DerivationsDimension"]:= 6;
JTable[[33, 31, k<>0, l<>0]]["DecompositionOfRadicalOfDerivations
Dimensions"] := [4,2];
JTable[[33, 31, k<>0, l<>0]]["DerivationsOfDerivationsDimension"]

```

```

:= 6;
JTable[[33, 31, k<>0, l<>0]]["SubspaceType"]:= nonnull;
JTable[[33, 31, k<>0, l<>0]]["IsotropySubalgebraInSemisimplePart"]
:= NA;
JTable[[33, 31, k<>0, l<>0]]["VectorFields"]:= [D_x2, D_x3,-D_x1, -x3
*D_x2+x2*D_x3, l*x1*D_x1+x2*k*D_x2+x3*k*D_x3+D_x4];
#####
# [33, 31, k<>0, l=0]
#####
JTable[[33, 31, k<>0, l=0]]:=table();
JTable[[33, 31, k<>0, l=0]]["AlgebraType"] := "Solvable";
JTable[[33, 31, k<>0, l=0]]["Indecomposable"] := false;
JTable[[33, 31, k<>0, l=0]]["DecompositionDimensions"] := [4,1];
JTable[[33, 31, k<>0, l=0]]["IsotropySubalgebraReductivePair"]
:= true;
JTable[[33, 31, k<>0, l=0]]["IsotropySubalgebraSymmetricPair"]
:= false;
JTable[[33, 31, k<>0, l=0]]["DerivedSeriesDimensions"] := [5,2,0];
JTable[[33, 31, k<>0, l=0]]["OrbitDimension"] := 4;
JTable[[33, 31, k<>0, l=0]]["IsotropyType"] := "F12";
JTable[[33, 31, k<>0, l=0]]["NilradicalDimension"] := 3;
JTable[[33, 31, k<>0, l=0]]["NilradicalIndecomposable"] := false;
JTable[[33, 31, k<>0, l=0]]["DerivationsDimension"]:= 7;
JTable[[33, 31, k<>0, l=0]]["DecompositionOfRadicalOfDerivations
Dimensions"] := [4,3];
JTable[[33, 31, k<>0, l=0]]["DerivationsOfDerivationsDimension"]
:= 10;
JTable[[33, 31, k<>0, l=0]]["SubspaceType"]:= nonnull;
JTable[[33, 31, k<>0, l=0]]["IsotropySubalgebraInSemisimplePart"]
:= NA;
JTable[[33, 31, k<>0, l=0]]["VectorFields"]:= [D_x2, D_x3,-D_x1, -x3
*D_x2+x2*D_x3, x2*k*D_x2+x3*k*D_x3+D_x4];
#####
# [33, 31, k=0, l<>0]
#####
JTable[[33, 31, k=0, l<>0]]:=table();
JTable[[33, 31, k=0, l<>0]]["AlgebraType"] := "Solvable";
JTable[[33, 31, k=0, l<>0]]["Indecomposable"] := false;
JTable[[33, 31, k=0, l<>0]]["DecompositionDimensions"] := [3,2];
JTable[[33, 31, k=0, l<>0]]["IsotropySubalgebraReductivePair"]
:= true;
JTable[[33, 31, k=0, l<>0]]["IsotropySubalgebraSymmetricPair"]
:= false;
JTable[[33, 31, k=0, l<>0]]["DerivedSeriesDimensions"] := [5,3,0];
JTable[[33, 31, k=0, l<>0]]["OrbitDimension"] := 4;
JTable[[33, 31, k=0, l<>0]]["IsotropyType"] := "F12";
JTable[[33, 31, k=0, l<>0]]["NilradicalDimension"] := 3;
JTable[[33, 31, k=0, l<>0]]["NilradicalIndecomposable"] := false;
JTable[[33, 31, k=0, l<>0]]["DerivationsDimension"]:= 6;
JTable[[33, 31, k=0, l<>0]]["DecompositionOfRadicalOfDerivations
Dimensions"] := [4,2];

```

```

JTable[[33, 31, k=0, l<>0]]["DerivationsOfDerivationsDimension"]:= 6;
JTable[[33, 31, k=0, l<>0]]["SubspaceType"]:= nonnull;
JTable[[33, 31, k=0, l<>0]]["IsotropySubalgebraInSemisimplePart"]
:= NA;
JTable[[33, 31, k=0, l<>0]]["VectorFields"]:= [D_x2, D_x3,-D_x1, -x3
*D_x2+x2*D_x3, l*x1*D_x1+D_x4];
#####
# [33, 31, k=0, l=0]
#####
JTable[[33, 31, k=0, l=0]]:=table();
JTable[[33, 31, k=0, l=0]]["AlgebraType"] := "Solvable";
JTable[[33, 31, k=0, l=0]]["Indecomposable"] := false;
JTable[[33, 31, k=0, l=0]]["DecompositionDimensions"] := [3,2];
JTable[[33, 31, k=0, l=0]]["IsotropySubalgebraReductivePair"]
:= true;
JTable[[33, 31, k=0, l=0]]["IsotropySubalgebraSymmetricPair"]
:= false;
JTable[[33, 31, k=0, l=0]]["DerivedSeriesDimensions"] := [5,2,0];
JTable[[33, 31, k=0, l=0]]["OrbitDimension"] := 4;
JTable[[33, 31, k=0, l=0]]["IsotropyType"] := "F12";
JTable[[33, 31, k=0, l=0]]["NilradicalDimension"] := 4;
JTable[[33, 31, k=0, l=0]]["NilradicalIndecomposable"] := false;
JTable[[33, 31, k=0, l=0]]["DerivationsDimension"]:= 10;
JTable[[33, 31, k=0, l=0]]["DecompositionOfRadicalOfDerivations
Dimensions"] := [4,3];
JTable[[33, 31, k=0, l=0]]["DerivationsOfDerivationsDimension"]
:= 10;
JTable[[33, 31, k=0, l=0]]["SubspaceType"]:= nonnull;
JTable[[33, 31, k=0, l=0]]["IsotropySubalgebraInSemisimplePart"]
:= NA;
JTable[[33, 31, k=0, l=0]]["VectorFields"]:= [D_x2, D_x3,-D_x1, -x3
*D_x2+x2*D_x3, D_x4];
#####
# [33, 36]
#####
JTable[[33, 36]]:=table();
JTable[[33, 36]]["AlgebraType"] := "Generic";
JTable[[33, 36]]["Indecomposable"] := true;
JTable[[33, 36]]["DecompositionDimensions"] := [5];
JTable[[33, 36]]["IsotropySubalgebraReductivePair"] := true;
JTable[[33, 36]]["IsotropySubalgebraSymmetricPair"] := false;
JTable[[33, 36]]["DerivedSeriesDimensions"] := [5,5];
JTable[[33, 36]]["OrbitDimension"] := 4;
JTable[[33, 36]]["IsotropyType"] := "F11";
JTable[[33, 36]]["NilradicalDimension"] := 2;
JTable[[33, 36]]["NilradicalIndecomposable"] := false;
JTable[[33, 36]]["DerivationsDimension"]:= 6;
JTable[[33, 36]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [3];
JTable[[33, 36]]["DerivationsOfDerivationsDimension"]:= 6;
JTable[[33, 36]]["SubspaceType"]:= nonnull;

```

```

JTable[[33, 36]]["IsotropySubalgebraInSemisimplePart"]:= false;
JTable[[33, 36]]["VectorFields"]:= [D_x2, D_x3, -D_x1+x3*D_x2, -(1/2)
*x1*D_x1+(1/2)*x2*D_x2+x3*D_x3+D_x4, (-x2-x1*x3)*D_x1-(-x2*x3*lambda1
+x1*exp(2*x4))*D_x2/lambda1+(x3^2*lambda1+exp(2*x4))*D_x3/lambda1
+2*x3*D_x4];
#####
# [33, 37]
#####
JTable[[33, 37]]:=table();
JTable[[33, 37]]["AlgebraType"] := "Generic";
JTable[[33, 37]]["Indecomposable"] := true;
JTable[[33, 37]]["DecompositionDimensions"] := [5];
JTable[[33, 37]]["IsotropySubalgebraReductivePair"] := false;
JTable[[33, 37]]["IsotropySubalgebraSymmetricPair"] := NA;
JTable[[33, 37]]["DerivedSeriesDimensions"] := [5,5];
JTable[[33, 37]]["OrbitDimension"] := 4;
JTable[[33, 37]]["IsotropyType"] := "F14";
JTable[[33, 37]]["NilradicalDimension"] := 2;
JTable[[33, 37]]["NilradicalIndecomposable"] := false;
JTable[[33, 37]]["DerivationsDimension"]:= 6;
JTable[[33, 37]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [3];
JTable[[33, 37]]["DerivationsOfDerivationsDimension"]:= 6;
JTable[[33, 37]]["SubspaceType"]:= nonnull;
JTable[[33, 37]]["IsotropySubalgebraInSemisimplePart"]:= false;
JTable[[33, 37]]["VectorFields"]:= [D_x2, D_x3, -D_x1+x3*D_x2, -(1/2)
*x1*D_x1+(1/2)*x2*D_x2+x3*D_x3+D_x4, (-x2-x1*x3+lambda1*exp((1/2)
*x4))*D_x1+x3*x2*D_x2+x3^2*D_x3+2*x3*D_x4];
#####
# [33, 38, 1]
#####
JTable[[33, 38, 1]]:=table();
JTable[[33, 38, 1]]["AlgebraType"] := "Solvable";
JTable[[33, 38, 1]]["Indecomposable"] := false;
JTable[[33, 38, 1]]["DecompositionDimensions"] := [2,2,1];
JTable[[33, 38, 1]]["IsotropySubalgebraReductivePair"] := true;
JTable[[33, 38, 1]]["IsotropySubalgebraSymmetricPair"] := true;
JTable[[33, 38, 1]]["DerivedSeriesDimensions"] := [5,2,0];
JTable[[33, 38, 1]]["OrbitDimension"] := 4;
JTable[[33, 38, 1]]["IsotropyType"] := "F13";
JTable[[33, 38, 1]]["NilradicalDimension"] := 3;
JTable[[33, 38, 1]]["NilradicalIndecomposable"] := false;
JTable[[33, 38, 1]]["DerivationsDimension"]:= 7;
JTable[[33, 38, 1]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [3,2,2];
JTable[[33, 38, 1]]["DerivationsOfDerivationsDimension"]:= 10;
JTable[[33, 38, 1]]["SubspaceType"]:= nonnull;
JTable[[33, 38, 1]]["IsotropySubalgebraInSemisimplePart"]:= NA;
JTable[[33, 38, 1]]["VectorFields"]:= [D_x1, D_x3, -D_x1+x2*D_x2,
-lambda1*D_x1+x3*D_x3+D_x4, -lambda1*exp(-x1/lambda1)*D_x1
+exp((x4*lambda1-x1)/lambda1)*D_x3+exp(-x1/lambda1)*D_x4];

```

```

#####
# [33, 38, -1]
#####
#JTable[[33, 38, -1]]:=table();
#JTable[[33, 38, -1]]["AlgebraType"] := "Solvable";
#JTable[[33, 38, -1]]["Indecomposable"] := false;
#JTable[[33, 38, -1]]["DecompositionDimensions"] := [2,2,1];
#JTable[[33, 38, -1]]["IsotropySubalgebraReductivePair"] := true;
#JTable[[33, 38, -1]]["IsotropySubalgebraSymmetricPair"] := true;
#JTable[[33, 38, -1]]["DerivedSeriesDimensions"] := [5,2,0];
#JTable[[33, 38, -1]]["OrbitDimension"] := 4;
#JTable[[33, 38, -1]]["IsotropyType"] := "F13";
#JTable[[33, 38, -1]]["NilradicalDimension"] := 3;
#JTable[[33, 38, -1]]["NilradicalIndecomposable"] := false;
#JTable[[33, 38, -1]]["DerivationsDimension"]:= 7;
#JTable[[33, 38, -1]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [3,2,2];
#JTable[[33, 38, -1]]["DerivationsOfDerivationsDimension"]:= 10;
#JTable[[33, 38, -1]]["SubspaceType"]:= nonnull;
#JTable[[33, 38, -1]]["IsotropySubalgebraInSemisimplePart"]:= NA;
#JTable[[33, 38, -1]]["VectorFields"]:= [D_x1, D_x3,-D_x1+x2*D_x2,
-lambda1*D_x1+x3*D_x3+D_x4, -lambda1*exp(-x1/lambda1)*D_x1-exp((x4*
lambda1-x1)/lambda1)*D_x3+exp(-x1/lambda1)*D_x4];
#####
# [33, 38]
#####
#JTable[[33, 38]]:=table();
#JTable[[33, 38]]["AlgebraType"] := "Solvable";
#JTable[[33, 38]]["Indecomposable"] := false;
#JTable[[33, 38]]["DecompositionDimensions"] := [2,2,1];
#JTable[[33, 38]]["IsotropySubalgebraReductivePair"] := true;
#JTable[[33, 38]]["IsotropySubalgebraSymmetricPair"] := true;
#JTable[[33, 38]]["DerivedSeriesDimensions"] := [5,2,0];
#JTable[[33, 38]]["OrbitDimension"] := 4;
#JTable[[33, 38]]["IsotropyType"] := "F13";
#JTable[[33, 38]]["NilradicalDimension"] := 3;
#JTable[[33, 38]]["NilradicalIndecomposable"] := false;
#JTable[[33, 38]]["DerivationsDimension"]:= 7;
#JTable[[33, 38]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [3,2,2];
#JTable[[33, 38]]["DerivationsOfDerivationsDimension"]:= 10;
#JTable[[33, 38]]["SubspaceType"]:= nonnull;
#JTable[[33, 38]]["IsotropySubalgebraInSemisimplePart"]:= NA;
#JTable[[33, 38]]["VectorFields"]:= [D_x1, D_x3, -D_x1+x2*D_x2,
-lambda1*D_x1+x3*D_x3+D_x4,-lambda1*exp(-x1/lambda1)*D_x1+epsilon*
exp((x4*lambda1-x1)/lambda1)*D_x3+exp(-x1/lambda1)*D_x4];
#####
# [33, 39, 1]
#####
JTable[[33, 39, 1]]:=table();
JTable[[33, 39, 1]]["AlgebraType"] := "Solvable";

```

```

JTable[[33, 39, 1]]["Indecomposable"] := false;
JTable[[33, 39, 1]]["DecompositionDimensions"] := [2,2,1];
JTable[[33, 39, 1]]["IsotropySubalgebraReductivePair"] := true;
JTable[[33, 39, 1]]["IsotropySubalgebraSymmetricPair"] := true;
JTable[[33, 39, 1]]["DerivedSeriesDimensions"] := [5,2,0];
JTable[[33, 39, 1]]["OrbitDimension"] := 4;
JTable[[33, 39, 1]]["IsotropyType"] := "F13";
JTable[[33, 39, 1]]["NilradicalDimension"] := 3;
JTable[[33, 39, 1]]["NilradicalIndecomposable"] := false;
JTable[[33, 39, 1]]["DerivationsDimension"]:= 7;
JTable[[33, 39, 1]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [3,2,2];
JTable[[33, 39, 1]]["DerivationsOfDerivationsDimension"]:= 10;
JTable[[33, 39, 1]]["SubspaceType"]:= nonnull;
JTable[[33, 39, 1]]["IsotropySubalgebraInSemisimplePart"]:= NA;
JTable[[33, 39, 1]]["VectorFields"]:= [D_x2, D_x3, -D_x1,x2*D_x2+D_x4,
-exp(-x1)*D_x1+exp(x4-x1)*D_x2+exp(-x1)*D_x3/lambda1+exp(-x1)*D_x4];
#####
# [33, 39, -1]
#####
#JTable[[33, 39, -1]]:=table();
#JTable[[33, 39, -1]]["AlgebraType"] := "Solvable";
#JTable[[33, 39, -1]]["Indecomposable"] := false;
#JTable[[33, 39, -1]]["DecompositionDimensions"] := [2,2,1];
#JTable[[33, 39, -1]]["IsotropySubalgebraReductivePair"] := true;
#JTable[[33, 39, -1]]["IsotropySubalgebraSymmetricPair"] := true;
#JTable[[33, 39, -1]]["DerivedSeriesDimensions"] := [5,2,0];
#JTable[[33, 39, -1]]["OrbitDimension"] := 4;
#JTable[[33, 39, -1]]["IsotropyType"] := "F13";
#JTable[[33, 39, -1]]["NilradicalDimension"] := 3;
#JTable[[33, 39, -1]]["NilradicalIndecomposable"] := false;
#JTable[[33, 39, -1]]["DerivationsDimension"]:= 7;
#JTable[[33, 39, -1]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [3,2,2];
#JTable[[33, 39, -1]]["DerivationsOfDerivationsDimension"]:= 10;
#JTable[[33, 39, -1]]["SubspaceType"]:= nonnull;
#JTable[[33, 39, -1]]["IsotropySubalgebraInSemisimplePart"]:= NA;
#JTable[[33, 39, -1]]["VectorFields"]:= [D_x2, D_x3,-D_x1, x2*D_x2
+D_x4, -exp(-x1)*D_x1-exp(x4-x1)*D_x2+exp(-x1)*D_x3/lambda1+exp(-x1)
*D_x4];
#####
# [33, 39]
#####
#JTable[[33, 39]]:=table();
#JTable[[33, 39]]["AlgebraType"] := "Solvable";
#JTable[[33, 39]]["Indecomposable"] := false;
#JTable[[33, 39]]["DecompositionDimensions"] := [2,2,1];
#JTable[[33, 39]]["IsotropySubalgebraReductivePair"] := true;
#JTable[[33, 39]]["IsotropySubalgebraSymmetricPair"] := true;
#JTable[[33, 39]]["DerivedSeriesDimensions"] := [5,2,0];
#JTable[[33, 39]]["OrbitDimension"] := 4;

```



```

#JTable[[33, 39]]["IsotropyType"] := "F13";
#JTable[[33, 39]]["NilradicalDimension"] := 3;
#JTable[[33, 39]]["NilradicalIndecomposable"] := false;
#JTable[[33, 39]]["DerivationsDimension"]:= 7;
#JTable[[33, 39]]["DecompositionOfRadicalOfDerivationsDimensions"]
:= [3,2,2];
#JTable[[33, 39]]["DerivationsOfDerivationsDimension"]:= 10;
#JTable[[33, 39]]["SubspaceType"]:= nonnull;
#JTable[[33, 39]]["IsotropySubalgebraInSemisimplePart"]:= NA;
#JTable[[33, 39]]["VectorFields"]:= [D_x2, D_x3, -D_x1,x2*D_x2+D_x4,
-exp(-x1)*D_x1+epsilon*exp(x4-x1)*D_x2+exp(-x1)*D_x3/lambda1+exp(-x1)
*D_x4];
eval(JTable);
end proc:

```