Diffraction Effects on Broadband Radiation in Multi-staged Systems

Eric L. Shirley

Sensor Science Division, National Institute of Standards and Technology
Gaithersburg, Maryland

Motivation: To better understand diffraction, towards useful approximations to reduce computational requirements (risk reduction)

Outline:
• Review very basic diffraction theory
• Decomposing systems into source-aperture-detector (SAD) subsystems
• Beyond the SAD problem: itemizing higher-order effects
• On spectral and thermal aspects of diffraction for broadband radiation
• Higher-order effects in practice
• Conclusions

DISCLAIMER: Formulas subject to final proofread/check.
“Master Problem”: end-to-end propagation of light in optical systems

- paraxial, scalar gaussian optics
- unfolded optical systems
- reflective optics treated as lenses/apertures

Mathematical expression:

\[ u(Q) = \frac{U_0}{(i\lambda)^N} \int_{A_1} dx_1 \ldots \int_{A_k} dx_k \ldots \int_{A_N} dx_N \ e^{iq\sum_i \delta_i(x_i)} G(P, x_1) \ldots G(x_N, Q) \]

\[ F(P, Q, \{x_k\}) \]

Free-space propagator:

\[ G(x, x') = \frac{1}{|z' - z|} \exp \left[ iq \left( \frac{(x' - x)^2 + (y' - y)^2}{2|z' - z|} \right) \right] \]
Multi-staged vs. single-staged systems

- Break system down into 1-aperture subsystems
- Non-additivity of effects of subsystems to make total effect
Diffraction effects at 1st order:
- Monochromatic light, point source
- Formula for flux on aperture reaching detector $= \Phi_d/\Phi_a$

**Bkg: JOSA A 21, 1895 (2004)**

For formula:

$$L_B(v, w) = \frac{2}{\pi v (1 - w^2)} - \frac{1}{\pi v^2 (1 - w^2)} \cos(2v) + \frac{w^8 - 20w^6 - 90w^4 - 20w^2 + 1}{4\pi^3 (1 - w^2)^5} - \frac{w^4 - 18w^2 + 1}{4\pi^3 (1 - w^2)^3} \sin(2v) + ...$$

$$L_x(v, w) = (4w/v)[Y_1(v/w, v) \cos(2v(w+1/w)) + Y_2(v/w, v) \sin(2v(w+1/w))]$$

$$Y_1(v, w) = \frac{\nu v [2/(\nu v)]^{1/2}}{2} \left[ \frac{\sin(v - \pi/4)}{v} S_1^{(1)} + \frac{\cos(v - \pi/4)}{v^2} C_1^{(1)} + \frac{\sin(v - \pi/4)}{v^3} S_1^{(1)} + ... \right]$$

$$Y_2(v, w) = - \frac{\nu v [2/(\nu v)]^{1/2}}{2} \left[ \frac{\sin(v + \pi/4)}{v} S_1^{(2)} + \frac{\cos(v + \pi/4)}{v^2} C_2^{(2)} + \frac{\sin(v + \pi/4)}{v^3} S_2^{(2)} + ... \right]$$

$$S_1^{(1)} = \frac{2(w^2 + 1)}{(1 - w^2)^2}$$

$$C_1^{(1)} = \frac{3(w^6 + 3w^4 + 3w^2 + 1)}{(1 - w^2)^4}$$

$$S_1^{(2)} = \frac{15(2w^6 - 195w^4 - 1854w^2 - 1854w^4 - 195w^2 + 1)}{64(1 - w^2)^6}$$

$$S_2^{(1)} = \frac{4w^2}{(1 - w^2)^2}$$

$$C_2^{(2)} = \frac{3(5w^6 + 22w^4 + 5w^2)}{2(1 - w^2)^4}$$

$$S_2^{(2)} = \frac{15(-7w^{10} - 420w^8 - 1194w^6 - 420w^4 - 7w^2)}{32(1 - w^2)^6}$$

Aperture/lens, radius $R_a$

Detector, radius $R_d$

Source

- $d_s$

Large parameter:

$$v = \frac{(2\pi/\lambda)R_aR_d/d_d}{u = (2\pi/\lambda)(1/d_s + 1/d_d - 1/f)R_a^2}$$

$$w = \min(u, v)/\max(u, v)$$
Diffraction effects at 1\textsuperscript{st} order:

- Blackbody radiation
- Point source
- Integration over Planck function replaces \(L_B, L_X\) with \(F_B, F_X\)


\[
\Phi_d / \Phi_a = w^2 + \left( \frac{w^2 A^4}{6\zeta(4)} \right) F_B (A, w) - \left( \frac{A^4}{6\zeta(4)} \right) F_X (A, w)
\]

or

\[
\Phi_d / \Phi_a = 1 - \left( \frac{A^4}{6\zeta(4)} \right) F_B (A, w)
\]

\[
F_B (A, w) \sim \sum_{p=-3}^{\infty} (C_p A^p + L_p A^p \log_e A);
\]

\[
F_X (A, w) \sim -\frac{96w^6(1+3w^2+w^4)}{(1-w^2)^7}
\]

\[
A = \frac{c_2}{(2\pi R_a R_d / d_d)} \cdot \frac{1}{T}
\]

\[
C_{-3} = \frac{4\zeta(3)}{\pi(1-w^2)}
\]

\[
C_{-1} = \frac{(2\gamma + 6\log e - 2)(3w^8 - 60w^6 - 270w^4 - 60w^2 + 3)}{24\pi(1-w^2)^5}
\]
\[
- \frac{9w^8 - 228w^6 - 1354w^4 - 228w^2 + 9}{24\pi(1-w^2)^5}
\]
\[
+ \frac{768(w^6 + w^3)}{24\pi(1-w^2)^5} \log_e \left( \frac{1+w}{1-w} \right)
\]

\[
C_0 = -\frac{24(w^8 + 3w^6 + w^4)}{(1-w^2)^7}
\]

\[
L_{-1} = \frac{w^8 - 20w^6 - 90w^4 - 20w^2 + 1}{4\pi(1-w^2)^5}
\]
Diffraction effects at 1\textsuperscript{st} order:

- Extended source
- Monochromatic or blackbody radiation


Introduce:

$$\sigma = \frac{\min(R_s/d_s, R_d/d_d)}{\max(R_s/d_s, R_d/d_d)}$$

$$g(\sigma, x) = (1+\sigma x)^{-1} \left\{ (1-x^2)[(2+\sigma x)^2 - \sigma^2] \right\}^{1/2}$$

$$\alpha(\sigma, x) = 2\pi R_a (1+\sigma x) \max(R_s/d_s, R_d/d_d)$$

$$C = \pi R_a^2 [\min(R_s/d_s, R_d/d_d)]^2$$

Then

$$\Phi_\lambda(\lambda) = C \int_{-1}^{+1} dx \ g(\sigma, x) L(u, \alpha(\sigma, x)/\lambda) L_\lambda(\lambda)$$

$$\Phi = C \int_{-1}^{+1} dx \ g(\sigma, x) \int_0^{\infty} d\lambda \ L(u, \alpha(\sigma, x)/\lambda) L_\lambda(\lambda)$$

Amenable to efficient Numerical integration
Separating diffraction effects because of 1 element (1\textsuperscript{st}-order), 2 elements (2\textsuperscript{nd}-order), etc.:

\[ u(Q) = \frac{U_0}{(i\lambda)^N} \int_{A_1} dx_1 \ldots \int_{A_k} dx_k \ldots \int_{A_N} dx_N F(P, Q, \{x_k\}) \]

\[ = \frac{U_0}{(i\lambda)^N} \left( \int_{A_1} + \int_{A_1'} - \int_{A_1'} \right) dx_1 \ldots \left( \int_{A_k} + \int_{A_k'} - \int_{A_k'} \right) dx_k \ldots \left( \int_{A_N} + \int_{A_N'} - \int_{A_N'} \right) dx_N F(P, Q, \{x_k\}) \]

Integration over entire plane
→ Free-space propagation through plane of an element (but with focusing effects included)

Small effect, \textit{might} be written as 1-d integral around optical element perimeter
Separating diffraction effects because of 1 element (1\textsuperscript{st}-order), 2 elements (2\textsuperscript{nd}-order), etc.:

\[ u(Q) = \frac{U_0}{(i\lambda)^N} \int_{A_1} dx_1 \cdots \int_{A_k} dx_k \cdots \int_{A_N} dx_N F(P, Q, \{x_k\}) \]

\[ = \frac{U_0}{(i\lambda)^N} \left( \int_{A_1} + \int_{A_1'} - \int_{A_1''} \right) dx_1 \cdots \left( \int_{A_k} + \int_{A_k'} - \int_{A_k''} \right) dx_k \cdots \left( \int_{A_N} + \int_{A_N'} - \int_{A_N''} \right) dx_N F(P, Q, \{x_k\}) \]

Introduce propagator \( G' \) (which may including focusing effects) and reorganize above, according to number of primed regions versus whole planes sampled in each of \( 2^N \) terms:

\[ u(Q) = U_0 G'(P, Q) \]

\[ - \frac{U_0}{i\lambda} \sum_k \int_{A_k'} dx_k G'(P, x_k) G'(x_k, Q) \]

\[ + \frac{U_0}{(i\lambda)^2} \sum_k \sum_{k' < N} \int_{A_k'} \int_{A_{k'}} dx_k dx_{k'} G'(P, x_k) G'(x_k, x_{k'}) G'(x_{k'}, Q) \]

\[ - \ldots \]

Initial terms=easy to calculate; successive terms=more difficult, but vanish more rapidly at small wavelength→a useful expansion
Separating diffraction effects because of 1 element (1\textsuperscript{st}-order), 2 elements (2\textsuperscript{nd}-order), etc.:

\begin{align*}
    u(Q) &= U_0 G'(P, Q) \\
    &\quad - \frac{U_0}{i\lambda} \sum_k \int_{A_k} dx_k G'(P, x_k) G'(x_k, Q) \\
    &\quad + \frac{U_0}{(i\lambda)^2} \sum_{k<N} \sum_{k'>k} \int_{A_k} dx_k \int_{A_{k'}} dx_{k'} G'(P, x_k) G'(x_k, x_{k'}) G'(x_{k'}, Q) \\
    &\quad - \ldots
\end{align*}

This suggests a shorthand:

\begin{align*}
    u(Q) &= u^{(0)}(P, Q) + \sum_k u^{(1)}_k(P, Q) + \sum_{k<N} \sum_{k'>k} u^{(2)}_{k,k'}(P, Q) + \ldots
\end{align*}

Expressible in path-length distributions \( f \), aiding insight & analysis:

\begin{align*}
    u^{(p)}_X(Q) &= \int_{-\infty}^{+\infty} dl \ e^{iql} f^{(p)}_X(P, Q; l)
\end{align*}

A lot can be known about moments of the \( f \) functions.
Spectral and thermal aspects of diffraction:

Note:

\[
\int_0^\infty d\lambda \ L_\lambda(\lambda) \ \{ [u_X^{(p)}(Q)]*u_Y^{(q)}(Q) + u_X^{(p)}(Q)[u_Y^{(q)}(Q)]* \}
\]

\[
= 2 \int_0^\infty d\lambda \ L_\lambda(\lambda) \int_0^\infty dl \int_0^\infty dl' \ \cos[q(l-l')] \ f_X^{(p)}(P,Q;l) \ f_Y^{(q)}(P,Q;l')
\]

Convolutions of path-length distributions weighted by source spectral radiance, \( L_\lambda(\lambda) \).

For a thermal source, \( L_\lambda(\lambda) \) related to:

\[
\beta = \frac{c_2}{2\pi T}
\]

\[
2 \int_0^\infty dq \ q^3 \cos(ql)(e^{\beta q} - 1)^{-1} = 6S(\beta,l)
\]

\[
S(\beta,l) = \sum_{n=1}^\infty [(n\beta + il)^{-4} + (n\beta - il)^{-4}]
\]

At high \( T \), behavior near \( l=l' \) dictates any series expansion of diffraction effects in \( 1/T \):

\[
I(\nu) = \int_0^\infty dl' \ l' S(\beta,l) = -\frac{\pi \nu(\nu-1)(\nu-2)\zeta(3-\nu)}{6 \cos(\pi \nu / 2) \beta^{3-\nu}}
\]

\[
I(0) = 0
\]

\[
I(1) = -\frac{\zeta(2)}{3 \beta^2}
\]

\[
I(2) = -\frac{\pi}{3 \beta}
\]
PRACTICAL CASE ➔ Non-additivity of diffraction effects in ACRIM III:

- 2 apertures:
  - 6.6548 mm radius, 151.8666 mm from cavity entrance
  - 5.1816 mm radius, 67.2084 mm from cavity entrance
- About 1600 ppm extra power because of diffraction (5900K sun)
- 3.9878 mm radius cavity entrance
- Results obtained numerically (full calculation) and using the higher-order BDW approach
Approximate loss (valid at small wavelength, with BB amply overfilling pinhole’s FOV):

\[
\Delta \approx \left( \frac{\lambda d_1}{\pi^2 R_1 R_2} \right) \left( \frac{\lambda d_2}{\pi^2 R_2 R_3 \{1 - [(d_1 + d_2)R_2 / d_1]^2 / R_3^2\}} \right) \cdot \frac{1}{2} \cdot F\left( \frac{3}{2} \cdot \frac{3}{2}; 2; \left( \frac{R_0 / d_0}{R_2 / d_1} \right)^2 \right)
\]

Error in model results before (points) and after (lines) correcting for above loss:
PRACTICAL CASE ➔ Blackbody Calibration Scenario

- Blackbody, 4 mm radius opening, 30 mm from 0.075 mm radius pinhole aperture
- Active-cavity radiometer (ACR) 1116.56 mm from pinhole, 14.97 mm radius
- 15.835 mm radius stray-light-reducing baffle 155.58 mm upstream from ACR

- BB-pinhole-ACR combination alone: multiplicative diffraction factor, \( F_1 \)
- Pinhole-baffle-ACR combination alone: multiplicative diffraction factor, \( F_2 \)

Diffraction factor has correction (\( \Delta \)) because of non-additivity: 

\[
F = 1 + (F_2 - 1) + (F_1 - 1) + \Delta
\]

**Approximate formula for \( \Delta \):**

**Baffle illuminated:**

\[
\Delta \approx (F_1 - 1)(F_2 - 1) \cdot F \left( \frac{3}{2}, \frac{1}{2}; \left( \frac{R_2 / d_1}{R_0 / d_0} \right)^2 \right)
\]

**Baffle not illuminated:**

\[
\Delta \approx -\frac{1}{2} \left( \frac{R_0 / d_0}{R_2 / d_1} \right)^3 (F_1 - 1)(F_2 - 1) F \left( \frac{3}{2}, \frac{3}{2}; \left( \frac{R_0 / d_0}{R_2 / d_1} \right)^2 \right)
\]

**Sample results:**

<table>
<thead>
<tr>
<th>( \lambda ) (mm)</th>
<th>( F_2 - 1 )</th>
<th>( F_1 )</th>
<th>( F ) (numer.)</th>
<th>( \Delta ) (numer.)</th>
<th>( \Delta ) (formula)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0060</td>
<td>0.0010</td>
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<td>0.9393</td>
<td>-0.0001</td>
<td>-0.0001</td>
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<td>0.0026</td>
<td>0.9147</td>
<td>0.9171</td>
<td>-0.0002</td>
<td>-0.0002</td>
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<tr>
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<td>0.9045</td>
<td>0.9073</td>
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<td>-0.0003</td>
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<td>0.8239</td>
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</tr>
<tr>
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<td>0.0128</td>
<td>0.6453</td>
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<td>-0.0046</td>
<td>-0.0046</td>
</tr>
</tbody>
</table>
PRACTICAL CASE ➔ Collimator + Imaging Radiometer Example

- Over-filled radiometer pupil
- Under-filled radiometer FOV

Detector (focal plane) radius $R_d$

Lens radius $R_L$

Pinhole (focal plane) radius $R_{ph}$

Blackbody cavity opening radius $R_{BB}$

Call area of detector, lens, pinhole and BB opening $A, B, C, D$.

Denote remainder area of their planes by $A', B', C', D'$.

$T(XY)$ relates:

\[ T(XY) = \text{power incident on area } X \]

\[ \text{radiance on area } Y. \]

Effective formula:

\[
\frac{T(A'D')}{T(BC)} \approx \left( \frac{\lambda f_1}{\pi^2 R_d R_L} \right) \left( \frac{\lambda d_{BB}}{\pi^2 R_{ph} R_{BB}} \right)
\]
Conclusions

• One can itemize diffraction effects in a way that separates effects order by order.
  • *This can give physical/mathematical insight into the reason for various behaviors.*

• Useful approximations can be found that work to varying degrees.
  • *Further work remains to fully assess their utility.*

• In the few cases studied to date, it appears that, for systems other than the source-aperture-detector cases, a “clean” expansion of effects versus wavelength or temperature is not assured.