Diffraction Effects on Broadband Radiation in Multi-staged Systems

Eric L. Shirley

Sensor Science Division, National Institute of Standards and Technology
Gaithersburg, Maryland

Motivation: To better understand diffraction, towards useful approximations to reduce computational requirements (risk reduction)

Outline:

- Review very basic diffraction theory
- Decomposing systems into source-aperture-detector (SAD) subsystems
- Beyond the SAD problem: itemizing higher-order effects
- On spectral and thermal aspects of diffraction for broadband radiation
- Higher-order effects in practice
- Conclusions

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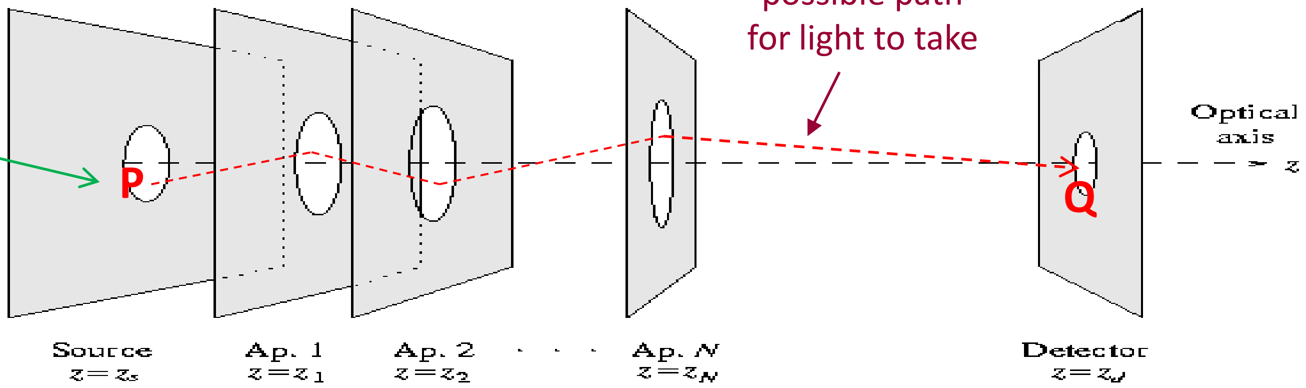
DISCLAIMER: Formulas subject to final proofread/check.

"Master Problem": end-to-end propagation of light in optical systems

- paraxial, scalar gaussian optics
- unfolded optical systems
- reflective optics treated as lenses/apertures



Source point

possible path for light to take

$$u(\mathbf{Q}) = \frac{U_0}{(i\lambda)^N} \int_{A_1} \mathrm{d}\mathbf{x_1} \ldots \int_{A_k} \mathrm{d}\mathbf{x}_k \ldots \int_{A_N} \mathrm{d}\mathbf{x}_N \underbrace{e^{iq\sum_i \delta l_i(\mathbf{x}_i)} G(\mathbf{P},\mathbf{x}_1)\ldots G(\mathbf{x}_N,\mathbf{Q})}_{F(\mathbf{P},\mathbf{Q},\{\mathbf{x}_k\})}$$

Light field

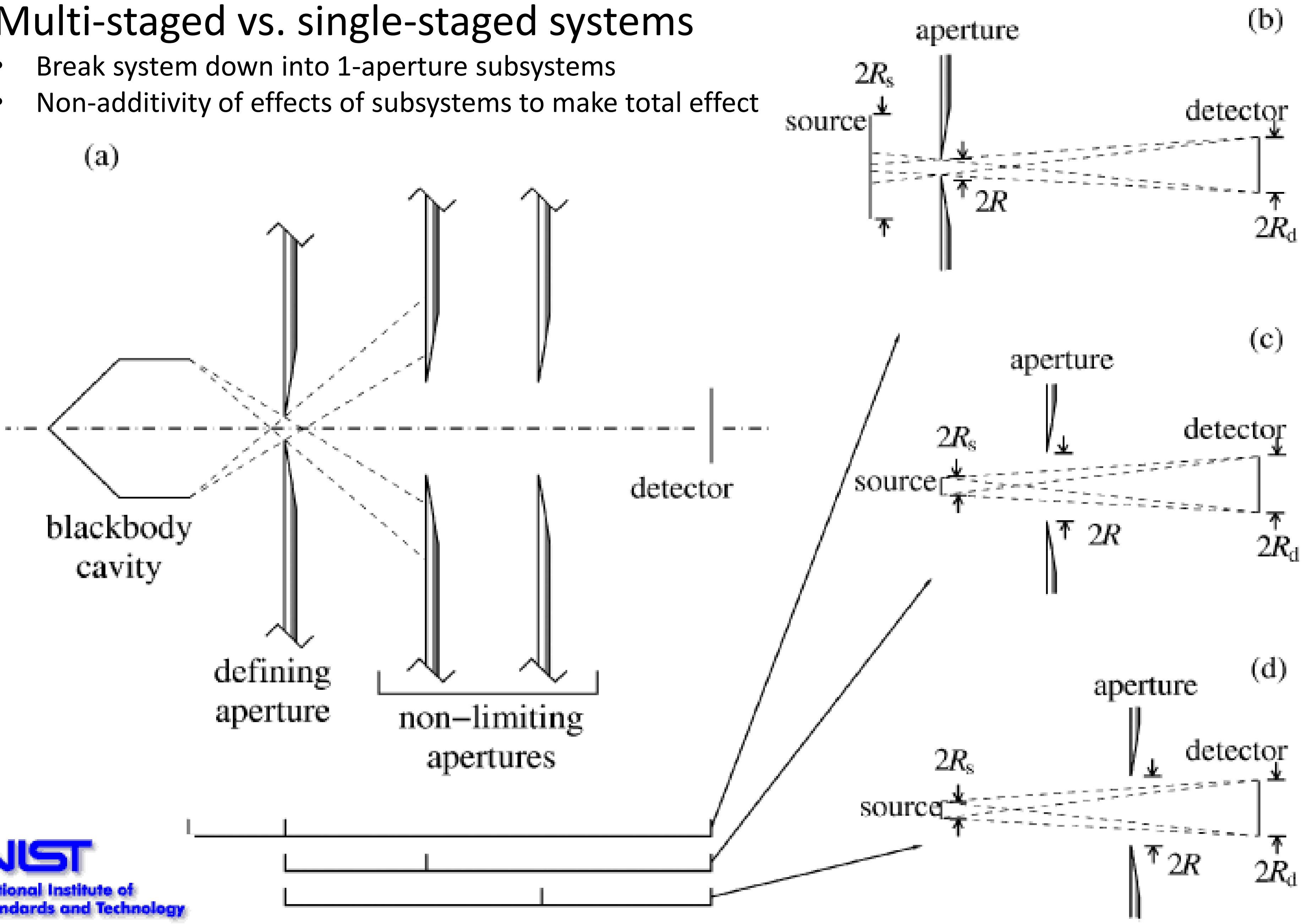
Focusing effects

Free-space propagator

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{|z'-z|} \exp\left[iq\left(|z'-z| + \frac{(x'-x)^2 + (y'-y)^2}{2|z'-z|} \right)\right]$$

Multi-staged vs. single-staged systems

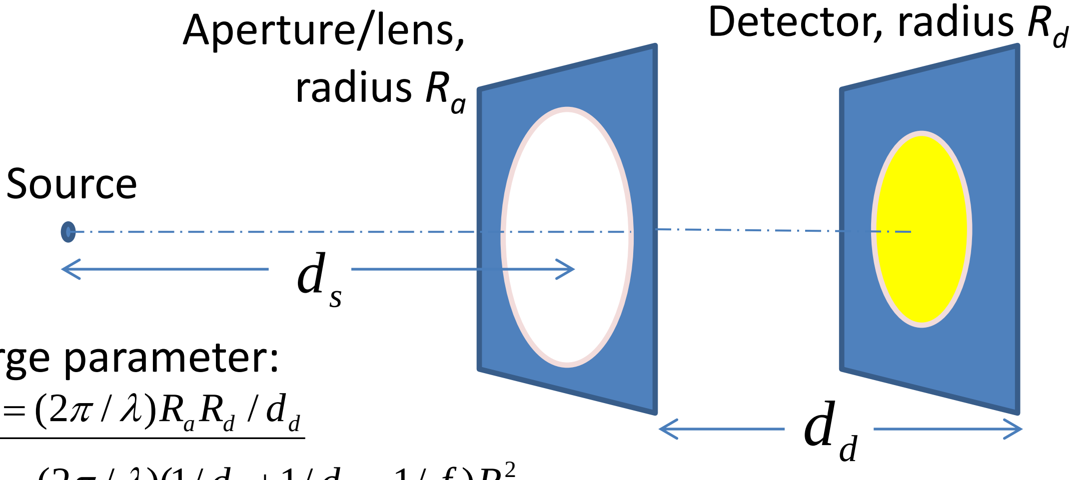
- Break system down into 1-aperture subsystems
- Non-additivity of effects of subsystems to make total effect



Diffraction effects at 1st order:

- Monochromatic light, point source
- Formula for flux on aperture reaching detector=Φ_d/Φ_a)

Bkg: JOSA A 21, 1895 (2004)



Large parameter:

$$\underline{v = (2\pi/\lambda) R_a R_d / d_d}$$

$$u = (2\pi/\lambda)(1/d_s + 1/d_d - 1/f)R_a^2$$

$$w = \min(u,v)/\max(u,v)$$

$$\Phi_d/\Phi_a = L(u,v) = w^2[1 + L_B(v,w)] - L_X(v,w)$$

or

$$\Phi_d/\Phi_a = L(u,v) = 1 - L_B(v,w)$$

$$L_B(v,w) = \frac{2}{\pi v(1-w^2)} - \frac{1}{\pi v^2(1-w^2)}\cos(2v) + \frac{w^8 - 20w^6 - 90w^4 - 20w^2 + 1}{4\pi v^3(1-w^2)^5} - \frac{w^4 - 18w^2 + 1}{4\pi v^3(1-w^2)^3}\sin(2v) + \ldots$$

$$L_X(v,w) = (4w/v)[Y_1(v/w,v)\cos(2v(w+1/w)) + Y_2(v/w,v)\sin(2v(w+1/w))]$$

$$Y_1(v,w) = +\frac{vw[2/(\pi v)]^{1/2}}{2}\left[\frac{\sin(v-\pi/4)}{v}S_1^{(1)} + \frac{\cos(v-\pi/4)}{v^2}C_2^{(1)} + \frac{\sin(v-\pi/4)}{v^3}S_3^{(1)} + \ldots\right]$$

$$Y_2(v,w) = -\frac{v[2/(\pi v)]^{1/2}}{2}\left[\frac{\sin(v+\pi/4)}{v}S_1^{(2)} + \frac{\cos(v+\pi/4)}{v^2}C_2^{(2)} + \frac{\sin(v+\pi/4)}{v^3}S_3^{(2)} + \ldots\right],$$

$$S_1^{(1)} = \frac{2(w^2+1)}{(1-w^2)^2} \qquad S_1^{(2)} = \frac{4w^2}{(1-w^2)^2}$$

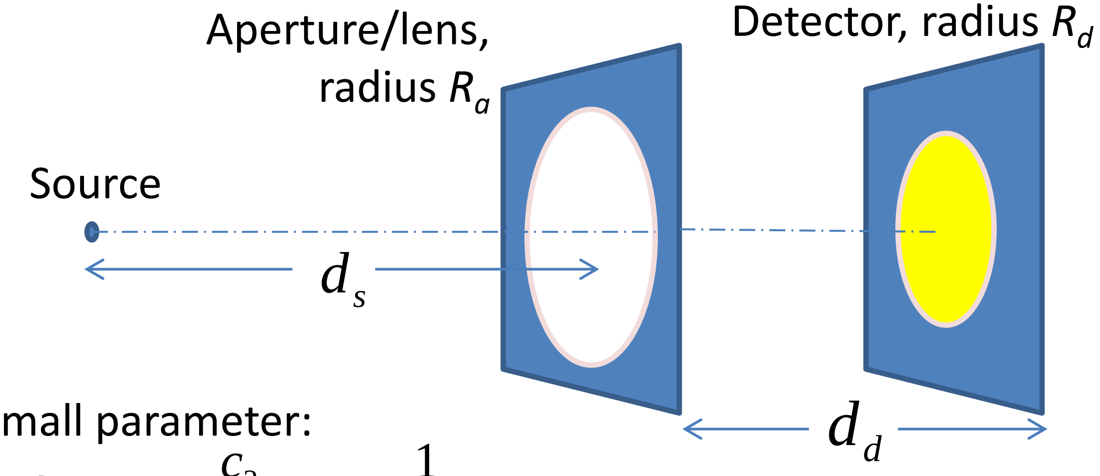
$$C_2^{(1)} = \frac{3(w^6 + 31w^4 + 31w^2 + 1)}{4(1-w^2)^4} \qquad C_2^{(2)} = \frac{3(5w^6 + 22w^4 + 5w^2)}{2(1-w^2)^4}$$

$$S_3^{(1)} = \frac{15(w^{10} - 195w^8 - 1854w^6 - 1854w^4 - 195w^2 + 1)}{64(1-w^2)^6} \qquad S_3^{(2)} = \frac{15(-7w^{10} - 420w^8 - 1194w^6 - 420w^4 - 7w^2)}{32(1-w^2)^6}$$

Diffraction effects at 1st order:

- Blackbody radiation
- Point source
- Integration over Planck function replaces L_B, L_X with F_B, F_X

Bkg: JOSA A 21, 1895 (2004)



Small parameter:

$$A = \frac{c_2}{(2\pi R_a R_d / d_d)} \cdot \frac{1}{T}$$

$$\Phi_d / \Phi_a = w^2 + \left(\frac{w^2 A^4}{6\zeta(4)}\right) F_B(A,w) - \left(\frac{A^4}{6\zeta(4)}\right) F_X(A,w)$$

or

$$\Phi_d / \Phi_a = 1 - \left(\frac{A^4}{6\zeta(4)}\right) F_B(A,w)$$

$$F_B(A,w) \sim \sum_{p=-3}^{\infty} (\mathbf{C}_p A^p + \mathbf{L}_p A^p \log_e A);$$

$$F_X(A,w) \sim -\frac{96w^6(1+3w^2+w^4)}{(1-w^2)^7}$$

$$\mathbf{C}_{-3} = \frac{4\zeta(3)}{\pi(1-w^2)}$$

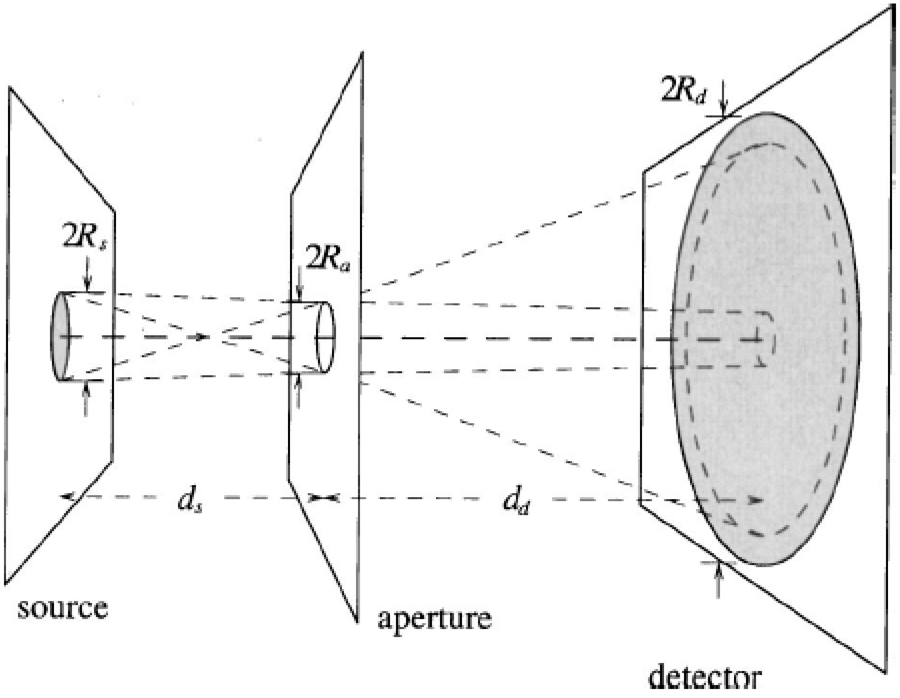
$$\mathbf{C}_{-1} = \frac{(2\gamma + 6\log_e 2)(3w^8 - 60w^6 - 270w^4 - 60w^2 + 3)}{24\pi(1-w^2)^5} - \frac{9w^8 - 228w^6 - 1354w^4 - 228w^2 + 9}{24\pi(1-w^2)^5} + \frac{768(w^5 + w^3)}{24\pi(1-w^2)^5} \log_e\left(\frac{1+w}{1-w}\right)$$

$$\mathbf{C}_0 = -\frac{24(w^8 + 3w^6 + w^4)}{(1-w^2)^7}$$

$$\mathbf{L}_{-1} = \frac{w^8 - 20w^6 - 90w^4 - 20w^2 + 1}{4\pi(1-w^2)^5}$$

Diffraction effects at 1st order:

- Extended source
- Monochromatic or blackbody radiation



Bkg: AO 37, 6581 (1998);
JOSA A 21, 1895 (2004)

Introduce:

$$\sigma = \frac{\min(R_s/d_s, R_d/d_d)}{\max(R_s/d_s, R_d/d_d)}$$

$$g(\sigma,x) = (1+\sigma x)^{-1}\{(1-x^2)[(2+\sigma x)^2-\sigma^2]\}^{1/2}$$
$$\alpha(\sigma,x) = 2\pi R_a(1+\sigma x)\max(R_s/d_s, R_d/d_d)$$

$$C = \pi R_a^2[\min(R_s/d_s, R_d/d_d)]^2$$

Then

$$\Phi_\lambda(\lambda) = C\int_{-1}^{+1}\mathrm{d}x\, g(\sigma,x)L(u,\alpha(\sigma,x)/\lambda)L_\lambda(\lambda)$$

Amenable to efficient
Numerical integration

$$\Phi = C\int_{-1}^{+1}\mathrm{d}x\, g(\sigma,x)\int_0^\infty \mathrm{d}\lambda\, L(u,\alpha(\sigma,x)/\lambda)L_\lambda(\lambda)$$

Separating diffraction effects because of 1 element (1st-order), 2 elements (2nd-order), etc.:

Bkg: JMO 54, 515 (2007)

$$u(\mathbf{Q}) = \frac{U_0}{(i\lambda)^N}\int_{A_1}\mathrm{d}\mathbf{x}_1\ldots\int_{A_k}\mathrm{d}\mathbf{x}_k\ldots\int_{A_N}\mathrm{d}\mathbf{x}_N\,F(\mathbf{P},\mathbf{Q},\{\mathbf{x}_k\})$$

$$= \frac{U_0}{(i\lambda)^N}\left(\underbrace{\int_{A_1} + \int_{A'_1}} - \underbrace{\int_{A'_1}}\right)\mathrm{d}\mathbf{x}_1\ldots\left(\underbrace{\int_{A_k} + \int_{A'_k}} - \underbrace{\int_{A'_k}}\right)\mathrm{d}\mathbf{x}_k\ldots\left(\underbrace{\int_{A_N} + \int_{A'_N}} - \underbrace{\int_{A'_N}}\right)\mathrm{d}\mathbf{x}_N\,F(\mathbf{P},\mathbf{Q},\{\mathbf{x}_k\})$$

Integration over entire plane

→Free-space propagation through plane of an element (but with focusing effects included)

Small effect, *might* be written as 1-d integral around optical element perimeter

Separating diffraction effects because of 1 element (1st-order), 2 elements (2nd-order), etc.:

$$u(\mathbf{Q}) = \frac{U_0}{(i\lambda)^N} \int_{A_1} \mathrm{d}\mathbf{x}_1 \ldots \int_{A_k} \mathrm{d}\mathbf{x}_k \ldots \int_{A_N} \mathrm{d}\mathbf{x}_N F(\mathbf{P},\mathbf{Q},\{\mathbf{x}_k\})$$

$$= \frac{U_0}{(i\lambda)^N} \left(\underbrace{\int_{A_1} + \int_{A_1'}}_{} \underbrace{-\int_{A_1'}}_{} \right) \mathrm{d}\mathbf{x}_1 \ldots \left(\underbrace{\int_{A_k} + \int_{A_k'}}_{} \underbrace{-\int_{A_k'}}_{} \right) \mathrm{d}\mathbf{x}_k \ldots \left(\underbrace{\int_{A_N} + \int_{A_N'}}_{} \underbrace{-\int_{A_N'}}_{} \right) \mathrm{d}\mathbf{x}_N F(\mathbf{P},\mathbf{Q},\{\mathbf{x}_k\})$$

Introduce propagator G’ (which may including focusing effects) and reorganize above, according to number of **primed regions** versus **whole planes** sampled in each of 2^N terms:

$$\begin{aligned} u(\mathbf{Q}) &= U_0 G'(\mathbf{P},\mathbf{Q}) \\ &\quad - \frac{U_0}{i\lambda} \sum_k \int_{A_k'} \mathrm{d}\mathbf{x}_k G'(\mathbf{P},\mathbf{x}_k) G'(\mathbf{x}_k,\mathbf{Q}) \\ &\quad + \frac{U_0}{(i\lambda)^2} \sum_{k<N} \sum_{k'>k} \int_{A_k'} \mathrm{d}\mathbf{x}_k \int_{A_{k'}'} \mathrm{d}\mathbf{x}_{k'} G'(\mathbf{P},\mathbf{x}_k) G'(\mathbf{x}_k,\mathbf{x}_{k'}) G'(\mathbf{x}_k,\mathbf{Q}) \\ &\quad - \ldots \end{aligned}$$

Initial terms=easy to calculate; successive terms=more difficult, but vanish more rapidly at small wavelength→a useful expansion

Separating diffraction effects because of 1 element (1st-order), 2 elements (2nd-order), etc.:

$$\begin{aligned} u(\mathbf{Q}) &= U_0 G'(\mathbf{P},\mathbf{Q}) \\ &-\frac{U_0}{i\lambda}\sum_k \int_{A'_k} \mathrm{d}\mathbf{x}_k G'(\mathbf{P},\mathbf{x}_k) G'(\mathbf{x}_k,\mathbf{Q}) \\ &+\frac{U_0}{(i\lambda)^2}\sum_{k<N}\sum_{k'>k} \int_{A'_k} \mathrm{d}\mathbf{x}_k \int_{A'_{k'}} \mathrm{d}\mathbf{x}_{k'} G'(\mathbf{P},\mathbf{x}_k) G'(\mathbf{x}_k,\mathbf{x}_{k'}) G'(\mathbf{x}_{k'},\mathbf{Q}) \\ &-\ldots \end{aligned}$$

This suggests a shorthand:

$$u(\mathbf{Q}) = u^{(0)}(\mathbf{P},\mathbf{Q}) + \sum_k u_k^{(1)}(\mathbf{P},\mathbf{Q}) + \sum_{k<N}\sum_{k'>k} u_{k,k'}^{(2)}(\mathbf{P},\mathbf{Q}) + \ldots$$

Expressible in path-length distributions f, aiding insight & analysis:

$$u_X^{(p)}(\mathbf{Q}) = \int_{-\infty}^{+\infty} \mathrm{d}l\, e^{iql} f_X^{(p)}(\mathbf{P},\mathbf{Q};l)$$

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A lot can be known about moments of the *f* functions.

Spectral and thermal aspects of diffraction:

Note:

$$\int_0^\infty \mathrm{d}\lambda\, L_\lambda(\lambda)\,\{[u_X^{(p)}(\mathbf{Q})]^* u_Y^{(q)}(\mathbf{Q}) + u_X^{(p)}(\mathbf{Q})[u_Y^{(q)}(\mathbf{Q})]^*\}$$

$$= 2\int_0^\infty \mathrm{d}\lambda\, L_\lambda(\lambda)\int_{-\infty}^{+\infty} \mathrm{d}l \int_{-\infty}^{+\infty} \mathrm{d}l' \cos[q(l-l')]\; f_X^{(p)}(\mathbf{P},\mathbf{Q};l)\, f_Y^{(q)}(\mathbf{P},\mathbf{Q};l')$$

Convolutions of path-length distributions weighted by source spectral radiance, $L_\lambda(\lambda)$.

For a thermal source, $L_\lambda(\lambda)$ related to:

$$\beta = \frac{c_2}{2\pi T}$$

$$2\int_0^\infty \mathrm{d}q\, q^3 \cos(ql)(e^{\beta q}-1)^{-1} = 6S(\beta,l)$$

$$S(\beta,l) = \sum_{n=1}^{\infty}[(n\beta+il)^{-4}+(n\beta-il)^{-4}]$$

At high *T*, behavior near *l*=*l’* dictates any series expansion of diffraction effects in 1/*T*:

$$I(\nu) = \int_0^\infty \mathrm{d}l\, l^\nu S(\beta,l) = -\frac{\pi\nu(\nu-1)(\nu-2)\zeta(3-\nu)}{6\cos(\pi\nu/2)\beta^{3-\nu}}$$

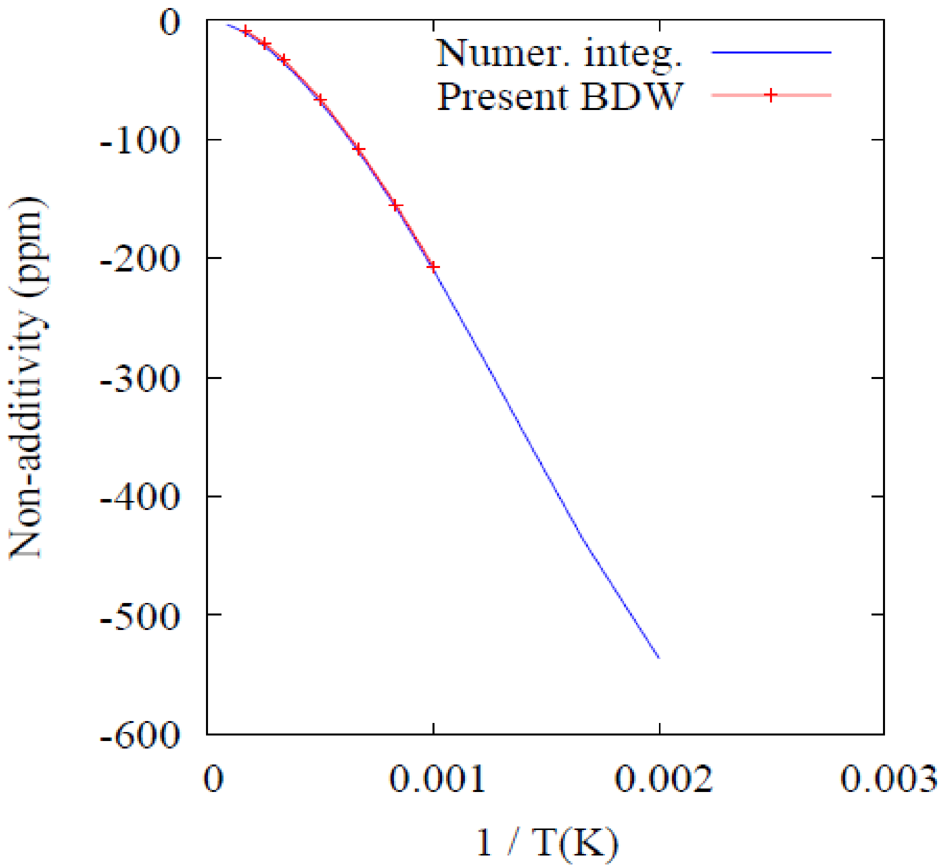
$$I(0) = 0$$

$$I(1) = -\frac{\zeta(2)}{3\beta^2}$$

$$I(2) = -\frac{\pi}{3\beta}$$

PRACTICAL CASE → Non-additivity of diffraction effects in ACRIM III:

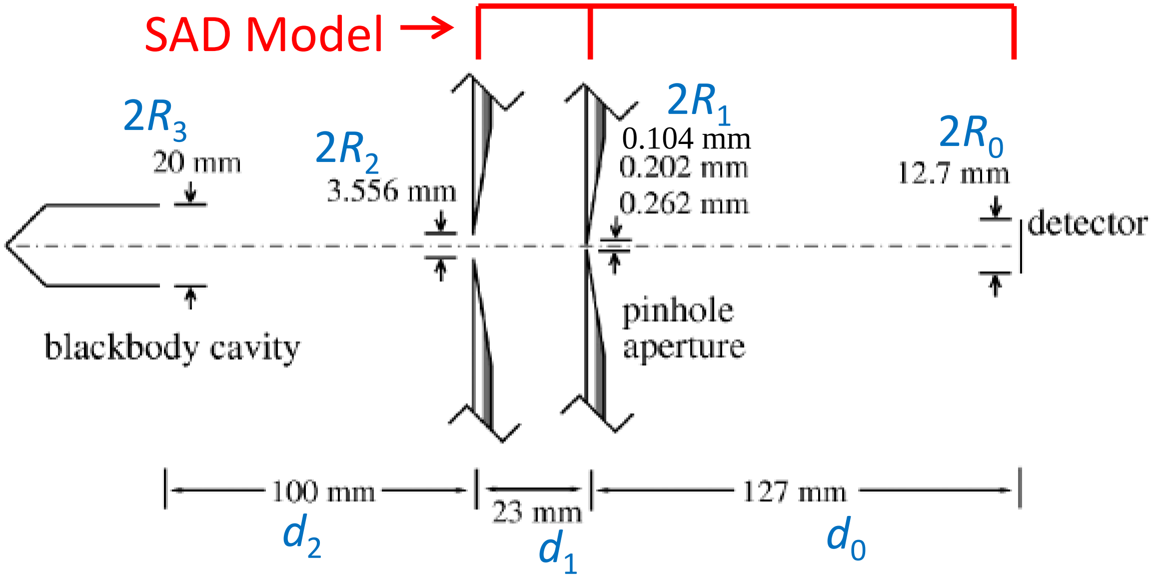
- 2 apertures:
- 6.6548 mm radius, 151.8666 mm from cavity entrance
- 5.1816 mm radius, 67.2084 mm from cavity entrance
- About 1600 ppm extra power because of diffraction (5900K sun)
- 3.9878 mm radius cavity entrance
- Results obtained numerically (full calculation) and using the higher-order BDW approach



PRACTICAL CASE→

2nd-order loss due to separation between blackbody & 3.556 mm aperture.

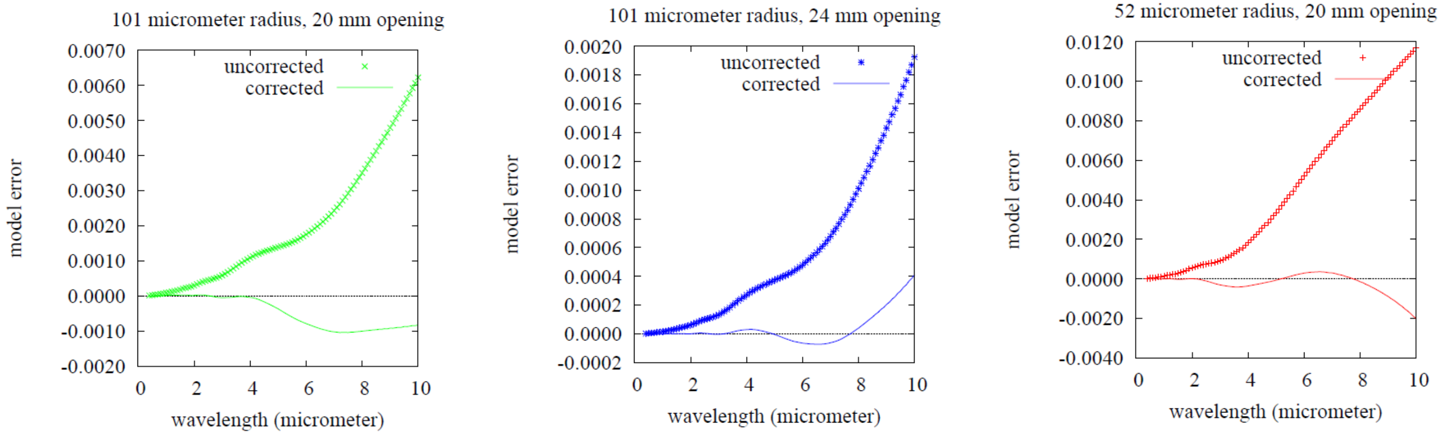
Blackbody fills pinhole's FOV through aperture, but only barely.



Approximate loss (valid at small wavelength, with BB amply overfilling pinhole's FOV):

$$\Delta \approx \left(\frac{\lambda d_1}{\pi^2 R_1 R_2}\right)\left(\frac{\lambda d_2}{\pi^2 R_2 R_3\{1-[(d_1+d_2)R_2/d_1]^2/R_3^2\}}\right)\cdot\frac{1}{2}\cdot F\left(\frac{3}{2},\frac{3}{2};2;\left(\frac{R_0/d_0}{R_2/d_1}\right)^2\right)$$

Error in model results before (points) and after (lines) correcting for above loss:



PRACTICAL CASE →Blackbody Calibration Scenario

- Blackbody, 4 mm radius opening, 30 mm from 0.075 mm radius pinhole aperture
- Active-cavity radiometer (ACR) 1116.56 mm from pinhole, 14.97 mm radius
- 15.835 mm radius stray-light-reducing baffle 155.58 mm upstream from ACR

- ➢ BB-pinhole-ACR combination alone: multiplicative diffraction factor, F_1
- ➢ Pinhole-baffle-ACR combination alone: multiplicative diffraction factor, F_2

Diffraction factor has correction (Δ) because of non-additivity: $F = 1 + (F_2 - 1) + (F_1 - 1) + \Delta$

Approximate formula for Δ:

Baffle illuminated:

$$\Delta \approx (F_1 - 1)(F_2 - 1) \cdot F\left(\frac{3}{2}, \frac{1}{2}; 1; \left(\frac{R_2 / d_1}{R_0 / d_0}\right)^2\right)$$

Baffle not illuminated:

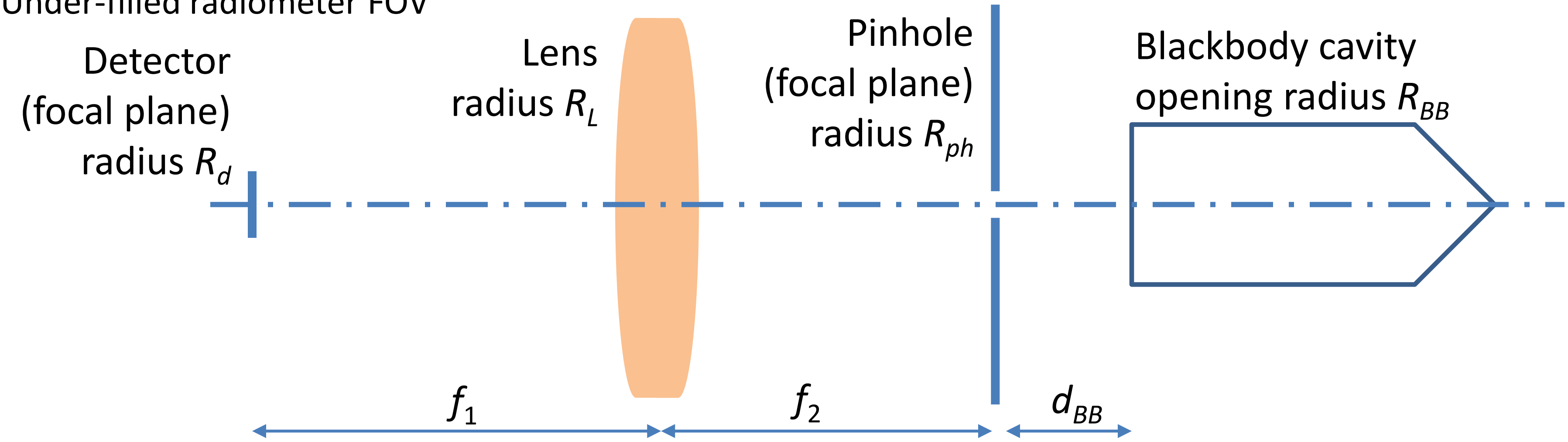
$$\Delta \approx -\frac{1}{2} \cdot \left(\frac{R_0 / d_0}{R_2 / d_1}\right)^3 (F_1 - 1)(F_2 - 1) F\left(\frac{3}{2}, \frac{3}{2}; 2; \left(\frac{R_0 / d_0}{R_2 / d_1}\right)^2\right)$$

Sample results:

λ(mm)	$F_2 - 1$	F_1	F (numer.)	Δ (numer.)	Δ (formula)
0.0060	0.0010	0.9384	0.9393	-0.0001	-0.0001
0.0080	0.0026	0.9147	0.9171	-0.0002	-0.0002
0.0100	0.0031	0.9045	0.9073	-0.0003	-0.0003
0.0200	-0.0003	0.8242	0.8239	0.0000	0.0001
0.0300	0.0128	0.6453	0.6535	-0.0046	-0.0046

PRACTICAL CASE →Collimator + Imaging Radiometer Example

- Over-filled radiometer pupil
- Under-filled radiometer FOV



Detector (focal plane) radius R_d

Lens radius R_L

Pinhole (focal plane) radius R_{ph}

Blackbody cavity opening radius R_{BB}

f_1

f_2

d_{BB}

Call area of detector, lens, pinhole and BB opening *A*, *B*, *C*, *D*.

Denote remainder area of their planes by *A'*, *B'*, *C'*, *D'*.

T(*XY*) relates:

{ **power** incident on area *X*
radiance on area *Y*.

$$T(BC) = T(AD) + T(AD') + T(A'D) + T(A'D')$$

$$\begin{aligned} T(AD) &= T(BC) - T(AD') - T(A'D) - T(A'D') \\ &= T(BC) + \underline{T(A'D')} - [T(AD') + \underline{T(A'D')}] - [T(A'D) - T(A'D')] \\ &= T(BC) - T(BD') - T(A'C) + T(A'D') \end{aligned}$$

trivial

1st-order

2nd-order & small

Effective formula: $\dfrac{T(A'D')}{T(BC)} \approx \left(\dfrac{\lambda f_1}{\pi^2 R_d R_L}\right)\left(\dfrac{\lambda d_{BB}}{\pi^2 R_{ph} R_{BB}}\right)$

Conclusions

- One can itemize diffraction effects in a way that separates effects order by order.
  - *This can give physical/mathematical insight into the reason for various behaviors.*

- Useful approximations can be found that work to varying degrees.
  - *Further work remains to fully assess their utility.*

- In the few cases studied to date, it appears that, for systems other than the source-aperture-detector cases, a “clean” expansion of effects versus wavelength or temperature is not assured.