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On the structure of divergence-free tensors

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Contravariant rank two tensors which are divergence-free on one index and which are constructed from the metric tensor, an auxiliary collection of arbitrary tensor fields, and the first and second partial derivatives of these quantities are classified. The results generalize existing mathematical arguments in support of the Einstein field equations.

1. INTRODUCTION

In gravitational field theories such as general relativity, contravariant rank two tensors A^{rs} which are divergence-free play a significant and well-known role. In this paper we shall classify those tensors which are locally of the form

$$A^{rs}=A^{rs}(g_{ij};g_{ij,h};g_{ij,hk};\rho_Q;\rho_{Q,h};\rho_{Q,hk}), \tag{1.1}$$

and which satisfy

$$A^{rs}{}_{|s}=0. \tag{1.2}$$

Here g_{ij} represents the components of a nondegenerate symmetric tensor defined on an n-dimensional orientable manifold M while ρ_Q symbolically denotes a collection of tensor fields of arbitrary rank and weight which are independent of g_{ij}. Covariant differentiation, as denoted in (1.2) by a vertical bar, is defined in terms of the Christoffel symbols Γ^h_{ij}. It should be emphasized that in classifying A^{rs} we do *not* assume it to be symmetric, divergence-free with respect to the index r or a polynomial in any of its variables. Nevertheless, as the following theorem indicates, a rather precise characterization of A^{rs} is obtained.

Theorem: If A^{rs} is a tensorial concomitant of the type (1.1) and satisfies (1.2), then

$$A^{rs}=B^{rs}+C^{rs}, \tag{1.3}$$

where B^{rs} is a symmetric divergence-free tensor of the type[1]

$$B^{rs}=B^{rs}(g_{ij};g_{ij,h};g_{ij,hk}), \tag{1.4}$$

and C^{rs} is a skew-symmetric divergence-free tensor of the type (1.1). Moreover, C^{rs} decomposes into the form

$$C^{rs}=D^{rs}+E^{rst}{}_{|t}, \tag{1.5}$$

where D^{rs} is a skew-symmetric divergence-free tensor of the type (1.4) and E^{rst} is an appropriate totally skew-symmetric tensor.

As immediate consequences of this theorem we have the following three corollaries.

Corollary 1: If A^{rs} is a tensorial concomitant of the type (1.1), then $A^{rs}{}_{|s}=0$ if, and only if $A^{rs}{}_{|r}=0$.

Corollary 2: If A^{rs} is a symmetric divergence-free concomitant of the type (1.1), than A^{rs} is independent of the fields ρ_Q and their derivatives.

Corollary 3: In a space of four dimensions, the most general divergence-free tensor of the type (1.1) is

$$A^{rs}=aG^{rs}+bg^{rs}+E^{rst}{}_{|t}, \tag{1.6}$$

where G^{rs} is the Einstein tensor, g^{rs} the inverse of g_{rs} and a and b are constants. If, in addition, A^{rs} is independent of $\rho_{Q,hk}$, then E^{rst} is explicitly given by

$$E^{rst}=\epsilon^{rstu}(V^Q\rho_{Q|u}+W_u),$$

where V^Q and W_u are arbitrary concomitants of g_{ij} and ρ_Q.

If the underlying manifold M is a spacetime, then we may replace the tensor g_{ij} by the generalized Pauli spin-matrices and assume that the fields defining ρ_Q are spin-tensorial in nature. Provided A^{rs} is now treated as a scalar under spinor (i.e., tetrad) transformations, the previous results can be reproduced verbatim and Eqs. (1.3)–(1.6) remain valid. The proof of these results depend upon several technical innovations and involves numerous lengthy calculations. Consequently in the next section we shall briefly describe the proof by outlining the main steps.[2]

These results lead us to a generalization of that derivation of general relativity in which the gravitational field equations are assumed to be of the form

$$V^{rs}=T^{rs}. \tag{1.7}$$

Here T^{rs} is the energy-momentum tensor for the external, i.e., nongravitational, source fields and V^{rs} is a tensor which is usually constructed from only those fields which characterize the gravitational field. To ensure that (1.7) implies

$$T^{rs}{}_{|s}=0$$

it is customary to restrict V^{rs} by the identity

$$V^{rs}{}_{|s}=0.$$

If, for example, V^{rs} is assumed to be a tensorial concomitant of the type

$$V^{rs}=V^{rs}(g_{ij};g_{ij,h};g_{ij,hk}), \tag{1.8}$$

then it is known[3] that V^{rs} is a linear combination of G^{rs} and g^{rs} in which case (1.7) gives rise to the usual Einstein field equations with cosmological term,

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$$aG^{rs}+bg^{rs}=T^{rs}. \tag{1.9}$$

Consequently if for some reason one wishes to modify the Einstein field equations then, as emphasized by Ehlers,[4] the assumption (1.8) would have to be changed. One possible alternative to (1.8) would be to assume that V^{rs} is of the type (1.1), where the fields ρ_Q represent either the external source fields or auxiliary gravitational variables. However, the above results clearly indicate that even under these very general circumstances, *the Einstein field equations must inevitably be retained.* Furthermore, our analysis shows that any attempt to modify T^{rs} by adding to it a symmetric divergence-free tensor of the type (1.1) cannot lead to different gravitational field equations.[5]

Finally, we would like to point out that a considerable effort has been made in recent years[6–14] to classify tensors S^{rs} of the type (1.1) which satisfy certain identities arising either from properties enjoyed by the energy–momentum tensors of general relativity or from the conservation laws derived from invariant variational principles. All of these identities may be expressed in the form

$$S^{rs}{}_{|s}=U^r, \tag{1.10}$$

where U^r is a vector which is constructed in a prescribed fashion. Since (1.10) is linear in S^{rs}, it follows that U^r uniquely determines S^{rs} up to a divergence-free tensor of the type (1.1). Therefore, in view of our present work, *a knowledge of U^r suffices to characterize S^{rs}*. Accordingly, it is anticipated that the results of this paper will simplify the existing solutions to classification problems of this kind and perhaps lead to new results in this area.

2. THE PROOF OF THE THEOREM

We denote the infinitesimal generators for the coordinate transformations of the tensor fields by[15] $F_i{}^j{}_Q{}^R$. Consequently, the covariant derivative of ρ_Q assumes the form

$$\rho_{Q|h}=\rho_{Q,h}-\Gamma^R_{Qh}\rho_R,$$

where $\Gamma^R_{Qh}=F_i{}^j{}_Q{}^R\Gamma^i_{jh}$, in which case

$$\rho_{Q|hk}-\rho_{Q|kh}=-R_Q{}^R{}_{hk}\rho_R,$$

where $R_Q{}^R{}_{hk}=F_i{}^j{}_Q{}^R R_j{}^i{}_{hk}$. If ξ_Q denotes another collection of tensor fields of the same type as ρ_Q, then the *principle form of the tensorial concomitant*

$$T^U=T^U(g_{ij};\ldots;g_{ij,i_1\ldots i_m};\rho_Q;\ldots;\rho_{Q,i_1\ldots i_m})$$

with respect to ξ_Q is the tensor[16,17]

$$P^U(\xi)=\sum_{l=0}^{m}(\partial^{Qi_1\cdots i_l}T^U)\xi_{Q,i_1\cdots i_l}, \tag{2.1}$$

where

$$\partial^{Qi_1\cdots i_l}T^U=\frac{\partial T^U}{\partial\rho_{Q,i_1\cdots i_l}}.$$

Since $\xi_{Q,i_1\cdots i_l}$ can be expressed as a unique linear combination of the symmetrized covariant derivatives $\xi_Q,\xi_{Q|i_1},\ldots,\xi_{Q|(i_1\cdots i_l)}$,

$$P^U(\xi)=\sum_{l=0}^{n}T^{U;Qi_1\cdots i_l}\xi_{Q|i_1\cdots i_l} \tag{2.2}$$

where $T^{U;Qi_1\cdots i_l}$ is the tensorial derivative[18] of T^U with respect to $\rho_{Qi_1\cdots i_l}$. It is not difficult to show[19] that for $m=2$,

$$T^{U;Qhk}=\partial^{Qhk}\,T^U$$

$$T^{U;\,Qhk}=\partial^{Qh}\,T^U+2\Gamma^Q_{Sk}\partial^{Shk}\,T^U+\Gamma^h_{lk}\partial^{Qlk}T^U$$

and

$$T^{U;Q}=\partial^Q T^U+\Gamma^Q_{Sh}(\partial^{Sh}T^U+\Gamma^S_{Tk}\partial^{Thk}T^U)+\Gamma^Q_{Sh,k}\,\partial^{Shk}T^U.$$

The transformation properties of T^U lead to certain differential *invariance identities*, one of these being (again for $m=2$),

$$2T^{U;l(j,hk)}g_{li}+T^{U;Q(hk}F_i{}^{j)}{}^R_Q\rho_R=0, \tag{2.3}$$

where

$$T^{U;lj,hk}=\frac{\partial T^U}{\partial g_{lj,hk}}.$$

The following lemma will be used to show that if A^{rs} is a divergence-free concomitant of the type (1.1), then it is a polynomial in the second derivatives of both g_{ij} and ρ_Q.

Lemma 2.1: Let S denote a set with n elements and for integers $p,q\geqslant 1$ let

$$f=f(s_1,\ldots,s_p,t_1,u_1,\ldots,t_q,u_q):S^{p+2q}\to\mathbb{R}$$

be any function with the following three properties:

(i) it is totally skew-symmetric in the arguments $s_1,s_2,\ldots,s_p$, e.g.,

$$f(s_1,s_2,\ldots s_p,t_1,\ldots,u_q)=-f(s_2,s_1,\ldots s_p,t_1,\ldots,u_q);$$

(ii) it is a symmetric in the arguments $t_l u_l$ for each $l=1,2,\ldots,q$, e.g.,

$$f(s_1,\ldots s_p,t_1,u_1,\ldots,u_q)=f(s_1,\ldots s_p,u_1,t_1,\ldots,u_q);$$

and

(iii) it satisfies the cyclic identity with respect to the arguments $s_k t_l u_l$ for each $k=1,\ldots,p$ and $l=1,\ldots,q$ e.g.,

$$f(s_1,\ldots s_p,t_1,u_1,\ldots,u_q)+f(u_1,\ldots s_p,s_1,t_1,\ldots,u_q)+f(t_1,\ldots,s_p,u_1,s_1,\ldots,u_q)=0.$$

Then f vanishes identically whenever $p+q>n$.

Proof: We introduce subsets A and $B_{\sigma,l}$ of S^{p+2q} with the definitions

$A=\{(s_1,\ldots,s_p,t_1,u_1,\ldots,t_q,u_q)\,|$
each $\sigma\in S$ occurs at least twice amongst the components $s_1,\ldots,s_p,t_1,u_1,\ldots,t_q,u_q\}$
for each $\sigma\in S$ and $l=1,2,\ldots,q$,

$$B_{\sigma,l} = \{(s_1,\ldots,s_p,t_1,u_1,\ldots,t_q,u_q) \mid \sigma \notin \{t_j,u_j\} \text{ for all } j \neq l\}.$$

The union of all the sets A and $B_{\sigma,l}$ equals S^{p+2q} and so it suffices to prove that the restrictions of f to A and to $B_{\sigma,l}$ vanish. The former may be established using arguments devised by Lovelock[20] while the latter follows by induction on n.

In the next lemma n' equals n or $n+1$ according to whether n is even or odd.

Lemma 2.2: Let $T^{rs,i_1j_1i_2j_2\cdots i_pj_p}$ be a tensorial concomitant of g_{ab} which enjoys the following symmetry properties:

(i) it is symmetric in the indices i_lj_l for all $l=1,2,\ldots,p$; and

(ii) it satisfies the cyclic identity with respect to the indices si_lj_l for all $l=1,2,\ldots,p$.

Then, provided $n>2$ and $1 \leqslant p \leqslant n'-2$, $T^{rs,i_1j_1\cdots i_pj_p}$ satisfies the cyclic identity with respect to the indices ri_lj_l for all $l=1,2,\ldots,p$. If $p<n'-2$, then $T^{rs,i_1j_1\cdots i_pj_p}$ is also symmetric in the indices rs.

Proof: We shall establish this lemma by induction on p. For $p=1$ and $n>2$, the result follows from the explicit construction of the most general tensorial concomitant which satisfies (i) and (ii).[21] Let us now suppose that the lemma is true for $p=q-1$, where $2 \leqslant q \leqslant n'-2$, and proceed to establish its validity when $p=q$. On multiplying the invariance identity

$$g^{tr}T^{us,i_1j_1\cdots i_qj_q} + g^{ts}T^{ru,i_1j_1\cdots i_qj_q} + \sum_{l=1}^{q}[g^{ti_l}T^{rs,i_1j_1\cdots uj_l\cdots i_qi_q} + g^{tj_l}T^{rs,i_1j_1\cdots i_lu\cdots i_qj_q}]$$

$$= g^{ur}T^{ts,i_1j_1\cdots i_qj_q} + g^{us}T^{rt,i_1j_1\cdots i_qj_q}$$

$$+ \sum_{l=1}^{q}[g^{ui_l}T^{rs,i_1j_1\cdots tj_l\cdots i_qj_q} + g^{uj_l}T^{rs,i_1j_1\cdots i_lt\cdots i_qj_q}]$$

by g_{ts}, on replacing u by s, and by repeatedly invoking properties (i) and (ii), it is found that

$$T^{sr,i_1j_1\cdots i_qj_q} + (n-q-1)\,T^{rs,i_1j_1\cdots i_qj_q} = g^{rs}U^{i_1j_1\cdots i_qj_q}$$

$$-\tfrac{1}{2}\sum_{l=1}^{q}\Big[g^{si_l}\underset{(l)}{V}^{rj_l,i_1j_1\cdots(i_lj_l)\hat{}\cdots i_qj_q} + g^{sj_l}\underset{(l)}{V}^{ri_l,i_1j_1\cdots(i_lj_l)\hat{}\cdots i_qj_q}\Big] \tag{2.4}$$

where

$$U^{i_1j_1\cdots i_qj_q} = T^{ts,i_1j_1\cdots i_qj_q}g_{ts},$$

$$\underset{(l)}{V}^{rs,i_1j_1\cdots(i_lj_l)\hat{}\cdots i_qj_q} = T^{rs,i_1j_1\cdots i_{l-1}j_{l-1}abi_{l+1}j_{l+1}\cdots i_qj_q}g_{ab},$$

and the circumflex ( )^ enclosing a pair of indices indicates that those indices are to be deleted.

Since the induction hypothesis is applicable to $\underset{(l)}{V}^{rs,i_1j_1\cdots i_{q-1}j_{q-1}}$ each $\underset{(l)}{V}^{rs,i_1j_1\cdots i_{q-1}j_{q-1}}$ is symmetric in the indices rs. It now follows that each of these tensors is also symmetric under interchange of the pairs of indices rs and i_hj_h for all $h=1,2,\ldots,q-1$. By multiplying (2.4) by g_{rs} we therefore obtain the relation

$$U^{i_1j_1\cdots i_qj_q} = \frac{1}{q}\sum_{l=1}^{q}\underset{(l)}{V}^{i_1j_1,i_2j_2\cdots i_qj_q},$$

which implies that $U^{i_1j_1\cdots i_qj_q}$ enjoys all the symmetry properties of $\underset{(l)}{V}^{i_1j_1,i_2j_2\cdots i_qj_q}$. Consequently if we successively cycle on the indices $ri_{2l-1}j_{2l-1}$ and $si_{2l}j_{2l}$ in (2.4) we find, after a simple analysis of the resulting equation, that

$$U^{i_1j_1i_2j_2\cdots i_qj_q} = \underset{(l)}{V}^{i_1j_1i_2j_2\cdots i_qj_q}$$

for each $l=1,2,\ldots,q$. In view of this result and the various symmetry properties of $U^{i_1j_1\cdots i_qj_q}$, (2.4) simplifies to

$$T^{sr,i_1j_1\cdots i_qj_q} + (n-q-1)\,T^{rs,i_1j_1\cdots i_qj_q}$$

$$= g^{rs}U^{i_1j_1\cdots i_qj_q} - \tfrac{1}{2}\sum_{l=1}^{q}[g^{si_l}U^{i_1j_1\cdots rj_l\cdots i_qj_q} + g^{sj_l i_1j_1\cdots i_lr\cdots i_qj_q}].$$

On account of the invariance identity for $U^{i_1j_1\cdots i_qj_q}$, the left-hand side of this equation is symmetric in rs and satisfies the cyclic identity on si_lj_l. Since the same must be true of the right-hand side, this proves the lemma for the case $p=q$, thereby completing our induction proof.

Theorem 2.3: Let A^{rs} be a tensorial concomitant of the type

$$A^{rs} = A^{rs}(g_{ij}; g_{ij,h}; g_{ij,hk}).$$

Then $A^{rs}{}_{|s}=0$ if, and only if $A^{rs}{}_{|r}=0$.

Proof: On account of the formula[22]

$$A^{rs}{}_{|t} = \tfrac{2}{3}A^{rs;ij,hk}R_{hijk|t},$$

it readily follows that the equations $A^{rs}{}_{|s}=0$ and $A^{rs}{}_{|r}=0$ are equivalent to

$$A^{rs;ij,hk} + A^{rk;ij,sh} + A^{rh;ij,ks} = 0, \tag{2.5}$$

and

$$A^{rs;ij,hk} + A^{ks;ij,rh} + A^{hs;ij,kr} = 0, \tag{2.6}$$

respectively. Consequently it suffices to show that (2.5) implies (2.6). From (2.3) it follows that

$$A^{rs;\,ij,hk} = A^{rs;\,hk,ij},$$

and so $A^{rs;\,ij,hk}$ also satisfies the cyclic identity with respect to the indices sij. We can apply Lemma 2.1 to the pth-order derivative of A^{rs} with respect to $g_{i_{2l-1}j_{2l-1},i_{2l}j_{2l}}$ ($l=1,2,\ldots,p$), viz.,

$$A^{rs;\,i_1j_1;\,i_2j_2;\,i_3j_3,i_4j_4;\,\cdots;i_{2p-1}j_{2p-1},i_{2p}j_{2p}},$$

to deduce that $A^{rs;\,i_1j_1,h_1k_1}$ is a polynomial in $g_{ab,cd}$ of degree no greater than $m-2$, where $m=n'/2$. This polynomial can be expressed in the form

$$A^{rs;\,i_1j_1,i_2j_2} = \sum_{l=0}^{m-2} T^{rsi_1j_1i_2j_2i_3j_3i_4j_4\cdots i_{2l}\;\;j_{2l}\;\;i_{2l}j_{2l}}$$

$$\times R_{i_3i_3i_4j_4}\cdots R_{i_{2l}\;\;i_{2l}j_{2l}\;\;j_{2l}},$$

where the coefficients $T^{rsij_1\cdots i_{2l}j_{2l}}$ are tensorial concomitants of g_{ab} alone and enjoy the symmetry properties enumerated in Lemma 2.2. By virtue of this lemma each coefficient satisfies the cyclic identity with respect to the indices ri_2j_2. Therefore, (2.6) holds and the theorem is proved.

Lemma 2.4: If A^{rs} satisfies (1.1) and (1.2), then

$$A^{rs;\,ij,hk} + A^{rk;\,ij,sh} + A^{rh;\,ij,ks} = 0, \tag{2.7}$$

$$A^{rl;\,Qhk} + A^{rk;\,Qlh} + A^{rh;\,Qkl} = 0, \tag{2.8}$$

$$\tfrac{1}{2}(A^{rh;\,Qk} + A^{rk;\,Qh}) + A^{rs;\,Qhk}{}_{|s} = 0, \tag{2.9}$$

$$A^{rh;\,Q} + A^{rs;\,Qh}{}_{|s} + A^{rl;\,Rhk} R_R{}^Q{}_{lk}$$
$$+ \tfrac{2}{3} A^{rl;\,Qmk} R_m{}^h{}_{lk} = 0, \tag{2.10}$$

and

$$\tfrac{1}{2} R_l{}^r{}_{hk}(A^{lh;\,Qk} - 2A^{lh;Qkm}{}_{|m}) - \tfrac{2}{3} R_l{}^r{}_{hk|m} A^{lh;\,Qkm} = 0. \tag{2.11}$$

Proof: Equation (2.7) is the consequence of differentiating (1.2) with respect to $g_{ij,hks}$. To derive (2.8)–(2.10) we first remark that the principal forms $P^r(\xi)$ and $P^{rs}(\xi)$ of $A^{rs}{}_{|s}$ and A^{rs} with respect to ξ are related by

$$P^r(\xi) = [P^{rs}(\xi)]_{|s}.$$

By substituting into this identity from (2.2) and by equating the coefficients of $\xi_{Q|(hkl)}$, $\xi_{Q|(hk)}$, and $\xi_{Q|h}$ and ξ_Q in the resulting equation, we can conclude that the tensorial derivatives $[A^{rs}{}_{|s}]^{;\,Qhkl}$, $[A^{rs}{}_{|s}]^{;\,Qhk}$ and $[A^{rs}{}_{|s}]^{;\,Qh}$ are given by the left-hand sides of (2.8)–(2.10) respectively while $[A^{rs}{}_{|s}]^{;\,Q} = A^{rs;\,Q}{}_{|s} + \frac{1}{2}A^{rh;\,Rk} R_R{}^Q{}_{hk} + \frac{1}{3} A^{rh;\,Rkl} R_R{}^Q{}_{hk}\,|\,l.$

Equations (2.8)–(2.10) now follow on account of (1.2). Finally, (2.11) arises via the simplification of the identity

$$-[A^{rs}{}_{|s}]^{;\,Q} + [A^{rs}{}_{|s}]^{;\,Qh}{}_{|h} - [A^{rs}{}_{|s}]^{;\,Qhk}{}_{|hk}$$
$$+ [A^{rs}{}_{|s}]^{;\,Qhkl}{}_{|hkl} = 0.$$

Lemma 2.5: If A^{rs} satisfies (1.1) and (1.2), then

$$D^{rs;\,Qhk} = 0, \tag{2.12a}$$

$$D^{rs;\,Qh} = 0, \tag{2.12b}$$

$$D^{rs;\,Q} = 0, \tag{2.12c}$$

where $D^{rs} = \frac{1}{2}(A^{rs} + A^{sr})$.

Proof: To derive the first of of (2.12), we begin by repeatedly differentiating (2.11), first with respect to $g_{ab,cde}$ and then with respect to $\rho_{R_1,t_1u_1}, \rho_{R_2,t_2u_2}, \ldots, \rho_{R_p,t_pu_p}$ and $g_{i_1j_1,h_1k_1}$, $g_{i_2j_2,h_2k_2}, \ldots, g_{i_qj_q,h_qk_q}$. On multiplying the resulting equation by a totally symmetric but otherwise arbitrary tensor ψ_{cde}, we find that

$$[g^{ra}E^{Qb} + g^{rb}E^{Qa} + \psi_{cde}g^{re}D^{ab;\,Qcd}]^{;\,\alpha(p);\,\beta(0,q)}$$

$$+ \sum_{l=1}^{q} [g^{ri_l}F^{h_lk_lQj_lab} + g^{rj_l}F^{h_lk_lQi_lab} + g^{rh_l}F^{i_lj_lQk_lab}$$

$$+ g^{rk_l}F^{i_lj_lQh_lab}]^{;\alpha(p);\,\beta(l,q)} - \tfrac{8}{3}$$
$$R_l{}^r{}_{hk}F^{lhQkab;\,\alpha(p);\,\beta(0,q)} = 0, \tag{2.13}$$

where

$$E^{Qb} = \psi_{cde}D^{cd;\,Qeb}, \quad F^{lhQkab} = \psi_{cde}D^{lh;\,Qke;ab,cd},$$

$$\alpha(p) = R_1t_1u_1;\, R_2t_2u_2;\cdots;\, R_pt_pu_p,$$

and

$$\beta(l,q)$$
$$= i_1j_1,h_1k_1;\, i_2j_2,h_2k_2;\cdots;\,(\;\;i_lj_l,h_lk_l)\hat{}\,;\cdots;\, i_qj_q,h_qk_q.$$

Moreover, on account of (2.8), it follows from Lemma 2.1 that

$$D^{rs;\,Qhk;\,\alpha(n-1)} = 0.$$

Hence, to establish the lemma by mathematical induction, it suffices to show that if

$$D^{rs;\,Qhk;\,\alpha(p+1)} = 0, \tag{2.14}$$

where $0 \leqslant p \leqslant n-2$, then

$$D^{rs;\,Qhk;\,\alpha(p)} = 0. \tag{2.15}$$

In order to derive (2.15) from (2.14) we shall use a second inductive argument. To start, we note that on account of (2.14) $D^{rs;\,Qhk;\,\alpha(p)}$ is independent of $\rho_{Q,ij}$ in which case (2.3) leads to

$$D^{rs;\,Qhk;\alpha(p);\,ab,cd} = D^{rs;\,Qhk;\alpha(p);\,cd,ab}\,.$$

By appealing once more to Lemma 2.1, we find that

$$E^{Qb;\,\alpha(p);\,\beta(0,q')} = 0 \quad \text{and} \quad F^{i_l j_l Qbh_l k_l;\,\alpha(p);\,\beta(l,q')} = 0$$

whenever $p+2q' \geqslant n$. Therefore, let us suppose that

$$E^{Qb;\,\alpha(p);\,\beta(0,q+1)} = 0 \quad \text{and} \quad F^{i_l j_l Qbh_l k_l;\,\alpha(p);\,\beta(l,q+1)} = 0 \tag{2.16}$$

for $p+2q<n$ and proceed to establish the validity of (2.16) with $q+1$ replaced by q. In view of the second of (2.16), multiplication of (2.13) by g_{ra} gives rise to

$$nE^{Qb;\,\alpha(p);\,\beta(0,q)} - \sum_{l=1}^{q} [F^{h_l k_l Qbi_l j_l} + F^{i_l j_l Qbh_l k_l}]^{;\alpha(p);\,\beta(l,q)} = 0. \tag{2.17}$$

Similarly multiplication of (2.13) by g_{rh_1} yields (on replacing the indices k_1ab by bh_1k_1)

$$\begin{aligned} &-E^{Qb;\,\alpha(p);\,\beta(0,q)} + nF^{i_1j_1Qbh_1k_1;\,\alpha(p);\,\beta(1,q)} \\ &\quad - F^{h_1k_1Qbi_1j_1;\,\alpha(p);\,\beta(1,q)} - \sum_{l=2}^{q} [F^{h_l k_l Qbh_1k_1;\,i_1j_1,\,i_l j_l} \\ &\quad + F^{i_l j_l Qbh_1k_1;\,i_1j_1,\,h_l k_l}]^{;\,\alpha(p);\,i_2j_2,h_2k_2;\cdots;\,(i_l j_l,h_l k_l)\hat{};\,\cdots;\,i_q j_q,h_q k_q} \\ &\quad = 0. \end{aligned} \tag{2.18}$$

By repeatedly permuting all pairs of indices $i_l j_l$ and $h_l k_l$ for $l=1,\ldots,q$ in (2.17) and (2.18) we obtain a homogeneous system of linear equations whose coefficient matrix is nonsingular.[23] This implies that (2.16) remains valid with $q+1$ replaced by q and thus, by induction on q,

$$E^{Qb;\,\alpha(p)} = 0 \quad \text{and} \quad F^{ijQbhk;\,\alpha(p)} = 0.$$

It is now an elementary matter to obtain (2.15) from (2.13) (with $q=0$). This completes our original induction argument and establishes (2.12a).

Due to (2.12a), (2.8), (2.9), and Bianchi identities, (2.11) reduces to

$$R_l{}^r{}_{hk} D^{lh;\,Qk} = 0. \tag{2.19}$$

By cycling on the indices rhk in (2.9) and by covariantly differentiating (2.9) with respect to x^r it is found that

$$D^{rh;\,Qk} + D^{kr;\,Qh} + D^{hk;\,Qr} = 0 \tag{2.20}$$

and

$$\tfrac{1}{2}(A^{rh;\,Qk} + A^{rk;\,Qh})_{|r} - \tfrac{1}{2}A^{rs;\,Rhk}R_R{}^Q{}_{rs} - \tfrac{1}{3}A^{kr;Qms}R_m{}^h{}_{rs} = 0. \tag{2.21}$$

Differentiation of (2.21) with respect to $g_{ab,cds}$ leads to

$$D^{rh;\,Qs;\,ab,cd} + D^{rh;\,Qd;\,ab,sc} + D^{rh;\,Qc;\,ab,ds} = 0. \tag{2.22}$$

On account of the symmetry properties (2.20) and (2.22), it is possible to derive (2.12b) from (2.19) by an induction argument similar to that used to obtain (2.12a).

Finally, on noting (2.21), (2.12c) follows directly from the result of symmetrizing (2.10) on the indices r and h.

Lemma 2.6: If C^{rs} is a skew-symmetric divergence-free concomitant of the type (1.1), then

$$P^{rs}(\xi) = V^{rst}{}_{|t}, \tag{2.23}$$

where $P^{rs}(\xi)$ is the principal form of C^{rs} and[24]

$$V^{rst}(\rho;\xi) = \tfrac{3}{2}C^{[rs;\,Qt]u}\xi_{Q|u} + [C^{[rs;\,Qt]} - \tfrac{1}{2}C^{[rs;\,Qt]u}{}_{|u}]\xi_Q.$$

Proof: Let $C^r = C^{rs}{}_{|s}$. Then (2.23) follows immediately from the formula

$$\begin{aligned} P^{rs}(\xi) = {} & V^{rst}{}_{|t} + \tfrac{3}{2}C^{[r;\,Qs]tu}\xi_{Q|tu} \\ & + [\tfrac{4}{3}C^{[r;\,Qs]t} - C^{[r;\,Qs]tu}{}_{|u}]\xi_{Q|t} \\ & + [C^{[r;\,Qs]} - \tfrac{2}{3}C^{[r;\,Qs]t}{}_{|t} + \tfrac{1}{2}C^{[r;Qs]tu}{}_{|tu}]\xi_Q \end{aligned}$$

which may be verified by direct calculation.

It is now a simple matter to prove the theorem stated in the introduction. Indeed, in view of Lemma 2.5, the tensor[25]

$$C_1^{rs} = \int_0^1 [\widetilde{A}^{rs;\,Q}\rho_Q + \widetilde{A}^{rs;\,Qh}\rho_{Q|h}] + \widetilde{A}^{rs;\,Qhk}\rho_{Q|hk}\;dt,$$

where

$$\widetilde{A}^{rs} = A^{rs}(g_{ij};\,g_{ij,h};\,g_{ij,hk};\,t\rho_Q;\,t\rho_{Q,h};\,t\rho_{Q,hk})$$

is skew-symmetric in the indices rs. Moreover, on recalling (2.1) and (2.2) it is easily seen that

$$A^{rs} = A_0^{rs} + C_1^{rs}, \tag{2.24}$$

where $A_0^{rs} = A^{rs}(g_{ij};\,g_{ij,h};\,g_{ij,hk};\,0;\,0;\,0)$. Since (1.2) implies that A_0^{rs} is divergence-free with respect to the index s we may refer to Theorem 2.3 to conclude that A_0^{rs} is divergence-free with respect to the index r and can therefore be expressed in the form

$$A_0^{rs} = B^{rs} + C_2^{rs},$$

where B^{rs} and C_2^{rs} are respectively symmetric and skew-symmetric divergence-free concomitants of the type (1.4). Because A^{rs} and A_0^{rs} are divergence-free on the index s, it follows from (2.24) that the same must be true of C_1^{rs}. However, C_1^{rs} is skew-symmetric in its indices and therefore by setting

$$C^{rs} = C_1^{rs} + C_2^{rs}$$

we arrive at (1.3). Finally, (1.5) follows from Lemma 2.6 upon setting

$$E^{rst} = \int_0^1 V^{rst}(\rho; t\rho)\, dt.$$

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[22]See Ref. 16.
[23]See Ref. 2, Lemmas 2.2 and 4.2.
[24]Square brackets indicate skew-symmetrization over the enclosed lower case indices only.
[25]It is assumed that A^{rs} is well defined at $\rho_Q = 0$ and that the field variables ρ_Q form a star-shaped set with respect to $\rho_Q = 0$.