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A Classroom Experiment: Implementing a Math-Talk Environment in a University Setting

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A Classroom Experiment:

Implementing a Math-Talk Environment in a University Setting

By

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A report submitted in partial fulfillment
of the requirements for the degree

of

MASTER OF MATHEMATICS

in

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Abstract

A need for reform in teaching mathematics has long been recognized. The traditional classroom with the sage on the stage lacks a higher level of engagement for the students and therefore, produces a lower level of student satisfaction. The math-talk classroom is one attempt to engage students and to raise the level of interaction and discussion, thus enabling students to increase their level of comprehension. In an effort to create a math-talk experience in my Calculus II course, I applied the methodology of Thinking Through a Lesson Protocol (TTLP) to test its effectiveness in assisting the instructor in creating an atmosphere conducive to student discussions (or math-talk).

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Introduction

For decades the mathematics education community has denounced the more traditional solely transmission-based instruction model and called for reform that would engage students in the learning conversation. "In traditional mathematics instruction, the role of the teacher is essentially to transmit knowledge to, and validate answers for, students, who are expected to learn alone and in silence" (Silver & Smith, 1996). When mathematics is presented in this way, the topics discussed in a math course may be generally regarded by the students as having little to no relevance outside of the classroom. Hufferd-Ackles, Fuson, and Sherin via the National Council of the Teachers of Mathematics insist that "the successful implementation of mathematics education reform requires that teachers change traditional teaching practices significantly, and develop a discourse community in their classroom (2004)."

In order to put my classroom more in line with the educational reform the mathematics education community is calling for, I began a journey to create an environment that fosters more math-talk (conversations about mathematics) in the classroom. The idea for the project was inspired as I worked on another project examining the questioning strategies of a professor in a project based math-biology course (Powell, 2012). My research on questioning strategies and classifications led me to what had been written about discourse in the classroom and math-talk classrooms. I became interested in finding a systematic, practical way to help a typical college classroom incorporate some math-talk into it.

Throughout the journey I discovered what other researchers had to say about discourse in the classroom, the role questions play in promoting discourse (math-talk), and what other researchers have learned about the progression teachers and students pass through as math-talk was practiced and developed in the classroom. In this report, I summarize those findings. I also

discuss how I used an instrument for lesson development called Thinking Through a Lesson Protocol (TTLP) as a tool to encourage math-talk in my classroom. I describe my experience with TTLP as I prepared three 50-minute final exam review sessions for my Calculus II class. I invited colleagues to observe the review sessions and I share their observations as well as my own reflections of what I saw in my university classroom after implementing this lesson planning procedure. The reform methods I practiced during this project represent the beginning of my pursuit to invite and encourage more math-talk in my classroom.

Problem Statement

I believe that the behaviors we see from teachers and students in a traditional classroom (where the teacher lectures while students are expected to listen and learn in silence, etc…) were created, practiced, and perfected because it created an environment that was predictable and easier to control than other environments where student ideas and discussions were included. Teachers have gravitated towards the traditional classroom setup because they did not like the unpredictability of where the conversation would go if students were given input and feared losing their status of the almighty knower of all things mathematical in their classroom. Student attitudes may also affect the success of discussions in the classrooms. Students may be afraid of the effect on their image if they offer an idea that is not generally accepted, especially if the atmosphere in the classroom is not such that a discussion and defense of ideas is practiced. I also believe that the pressure of achieving all points of the required curriculum has put teachers in a time crunch. What if the students bring up issues that the teacher cannot answer or does not have the time to deal with? Many teachers perceive it more time efficient to direct the lecture,

providing clear expositions of course content through polished presentations, and expecting little or no input from the students.

I currently teach in a university classroom where there is a mountain of curriculum to cover. I believe that generally my students expect (and want) me to prepare an entertaining lecture that helps them understand the mathematical topic of the day with minimal effort on their part. I do not blame my students for this attitude, but view it as a consequence of the educational world they have been brought up in. I also have a difficult time creating a nontraditional classroom experience because I was brought up in the same traditional educational world. I wanted to incorporate more mathematical discussions (or math-talk) in my classroom where there exists a mountain of curriculum to get through, not a lot of time to work with, and students who have been trained to listen and learn in silence and tend to resist being asked to participate in discussions, preferring to wait until I explain. I asked myself these questions as I pondered on how to proceed to create more math-talk in my classroom.

- If a gigantic curriculum and limited time is an issue, is it possible to implement discourse without totally compromising the material or time you have to work with?
- What steps do we take to create a discourse community in the classroom?
- How do we create a classroom atmosphere conducive to student explorations and discussions about mathematics?
- How can controversy serve as an aid to help the students think more deeply about their own assumptions and encourage them to revisit and refine what they originally thought about a mathematical idea?
- How can I create opportunities for my students to think about, explain, and perhaps defend their understanding of a particular piece of mathematical content?

These questions drove my review of literature in mathematics education and the design of the implementation project defined in this report.

Review of Literature

The differences between a traditional mathematics classroom and the math-talk classroom are in the responsibility of knowledge sharing. In a traditional mathematics classroom, the teacher has the responsibility to decide what knowledge is important for the students to know, transmit that knowledge, and validate the answers for the students (Silver & Smith, 1996). In a traditional mathematics classroom the students sit quietly and pay attention to what the teacher is saying and ask questions if they are brave enough, but many of them choose to learn in silence (Silver & Smith, 1996).

The math-talk classroom is "a classroom community in which the teacher and students use discourse to support the mathematical learning of all participants. A primary goal [of a math-talk community] is to understand and extend one's own thinking as well as the thinking of others in the classroom" (Hufferd-Ackles, Fuson, Sherin, 2004). The math-talk classroom is one in which discourse about mathematics is encouraged. The teacher and the students share the responsibility of transmitting and sharing knowledge in an open discourse. Students feel empowered to share their insights, ideas, and questions about the mathematics being presented. All have the opportunity at one time or another to present and defend their mathematical ideas. Students are prepared to answer questions about their theories. The classroom no longer has the same structure as a traditional classroom where the teacher solely directs the knowledge and ideas that are presented (i.e., sage on the stage). Traditional classrooms are the norm and an integral part of my personal experience, both as student and teacher. Although unfamiliar, the math-talk classroom holds intrigue and curiosity as a pedagogical application to the current teaching environment.

Encouraging mathematical discourse is one of my goals in the classroom. On occasion I have enjoyed watching my students spontaneously beginning to discuss why the mathematics works, the nuts and bolts of proofs and concepts. This occurs most commonly when there is a controversy, spurring interest and excitement. Teaching is enjoyable in this paradigm, but in my experience it does not happen often. How can we encourage this communication without a controversy, and still elicit opinions and points of view? Or how can we generate productive controversies to get student discussions going?

Some teachers may shy away from the math-talk classroom because "the learning environment becomes complex and less predictable as teachers attempt to interpret and understand [student] responses. To do this effectively requires principled knowledge of mathematical concepts and an understanding of how students think and reason mathematically" (Moyer & Molewicz, 2002). The teachers' tendency to shy away from the math-talk environment can be overcome in part by understanding how mathematical knowledge is developed. Lampert (1990) described Lakatos' ideas about the development of mathematical knowledge as recorded in his book *Proofs and Refutations*. Lakatos suggests that mathematical knowledge develops by consciously guessing about relationships between quantities and shapes. Proofs are then shaped and developed by following a path of thought that meanders through a maze of conjectures, assumptions, and counterexamples, until the mathematical truth is found. Of course, these mathematical truths are based on assumptions that can be questioned, scrutinized, reexamined or even changed, which allows mathematics to grow and develop. The art of performing mathematics, therefore, requires some intellectual risk for those participating. Sharing a conjecture carries with it a willingness to have others analyze, scrutinize, and perhaps challenge the idea. A person therefore needs to be willing to revise or change a belief when a reason for the

change is presented. On the other hand, a person should not be too quick to abandon their beliefs without serious examination. These qualities allow people to do mathematics.

In order to effectively help students through this process, teachers should examine the particular mathematical topic they would like the students to understand, make note of the assumptions made, the misconceptions that may arise, and be prepared to ask vital questions that will help get at what the students are thinking about the content. In other words, "in order to get students to become more capable at math-talk, teachers need to ask probing questions to try to understand the students' thinking and to get student to articulate their thinking" (Roach, 2010). If teachers are to be successful at understanding student thinking and helping students' articulate their reasoning, they need to be good at asking questions. Moyer and Milewicz (2002) stated that "effective questioning in mathematics actually requires well-developed oral-questioning skills [including] preparing important questions ahead of time, delivering questions clearly and concisely, posing questions [that] stimulate thought, and giving [enough] time to think about and prepare an answer." (Moyer, Milewicz, 2002). They used the following scheme to classify questions pre-service teachers asked during one-on-one diagnostic mathematics interviews with children. The scheme is useful when thinking about effective questions to ask.

- Questions that helped [students] make sense of the mathematics. (e.g., Can you explain to me why that makes sense?)
- Questions that helped [students] rely more on themselves to determine whether something was mathematically correct. (e.g., How did you reach that conclusion?)
- Questions that helped [students] learn to reason mathematically. (e.g., How could you prove that to me?)
- Questions that helped [students] conjecture, invent and solve problems. (e.g., What would happen if…?)
- Questions that helped [students] to connect mathematics, its ideas, and its applications. (e.g., Have we solved any problems like this one before?)

Gall (1970) suggested that "it is important [to] identify the types of questions which students should be encouraged to ask". For example teachers may want to consider asking students what interests them about an unfamiliar topic before presenting students with any information about the topic, or elicit questions about the understanding of a newly presented topic (implying that students also need to be trained in question-asking skills). To encourage math-talk in the classroom, teachers must learn skills for good questioning and also skills to help their students to pose questions.

Research supports the idea that students need to be equally responsible for asking questions in the classroom. "According to a number of models in cognitive science, question generation is a fundamental component in cognitive processes that operate at deep conceptual levels, such as the comprehension of text and social action, the learning of complex material, problem solving, and creativity. There is also empirical evidence that improvements in the comprehension, learning, and memory of technical material can be achieved by training students to ask good questions" (Graesser & Person, 1994).

"Students are [capable] of engaging in active inquiry, but the classroom environment does not foster it" (Graesser & Person, 1994). In an effort to encourage more student questions in the classroom, Michael Shodell (1995) has developed some activities for students to participate in to help them take an active role in class by asking questions. The first day exercise begins by stating that "the essence of 'thinking' is really question asking. The best thinking comes from the best asking." He then guides the students around the ideas of active questioning, what it means to be an active participant in class by asking questions, and then offering up an example of what types of questions a student could ask when presented with a new topic. For example, suppose you were a student in a trigonometry course presented with the following

information: "*Thus far we have been using degrees as our unit of measurement for angles. However, there is another way of measuring angles that is often more convenient. The idea for the new unit of measure is simple: associate a central angle of a circle with the arc that it intercepts. The central angle of a circle that intercepts an arc that has the same length as the radius of the circle will be called a radian, our new unit of measure.*" As the student, can you think of two or three questions you would like to have answered after considering what was said. After the student has an opportunity to think of some questions they have, the teacher provides them with some other possible questions and the scheme Shodell (1995) used for classifying them.

- Clarification (e.g. Just what is a radian? What is a central angle?)
- Interpretation (e.g. How big is a radian?)
- Extension (e.g. When is it convenient to use radians?)
- Critical (e.g. Why can't degrees be used?)
- Associative (e.g. Can I use radians to determine the linear distance a wheel has traveled?)

In another exercise Shodell (1995) displayed an excerpt from a text used in schools in 1851. The book was written in a question and answer style. He then asked the students to "assume you have time-traveled back to a classroom of the 1850s where this was being taught. Describe in detail how you could explain to those earlier students (and their teacher!) why much of this material was, in fact, in error and what the real bases were for the concepts and processes being considered" (Shodell, 1995). He then instructed them to consider going forward in time and speculate about the misconceptions we may have now and how the future scientific understanding may be different. "Knowing the answer to a question may or may not indicate an

understanding of the subject matter. However, being able to formulate a good question is always contingent upon such understanding." (Shodell, 1995)

A second method for building students' question asking skills is presented by Maskill & Pedrosa de Jesus (1997). They ask students to write their questions down, as they have them throughout the course of a class, then the teacher collects the questions and uses that information to help in planning the next lesson. In doing this they found that students do have questions and are capable of formulating those questions even if they do not always do so verbally. Students may still be hesitant to fully participate in the math-talk discussions even if they have been instructed on how to ask good questions. There may be social reasons for their reservations. Graesser & Person (1994) stated, "The low frequency and sophistication of student questions can be attributed to barriers at three different levels."

- Students have difficulty identifying their own knowledge deficits (i.e., Students have difficulty detecting contradictory information, in identifying missing data that are necessary for a solution, and in discriminating superfluous from necessary information).
- Social editing (i.e.,the student reveals ignorance and loses status when a bad question is asked).
- A deficit in acquiring good question-asking skills (teachers not acting as good role models).

Hence, methods are needed to help teachers build environments in which questioning strategies can be developed.

As teachers and students work to improve their ability to ask good questions and overcome their inhibitions about participating, the level of math-talk in the classroom will increase. There are many different roles that the teacher and students will take on as they engage in math-talk. Denise Jarrett (1997) described those roles more specifically. "Teachers will

- Create a rich learning environment;

- Identify important concepts students will investigate;
- Plan the inquiry;
- Solicit student input to narrow the focus of the inquiry;
- Initiate and orchestrate discussion;
- Ask prompting and probing questions; pursue students' divergent comments and questions, when appropriate;
- Guide students' learning in order to get at the core of the content; and
- Provide opportunities for all students to demonstrate their learning by presenting a product or making a public presentation.

Students will

- Contribute to the planning of an inquiry investigation;
- Observe and explore;
- Experiment and solve problems;
- Work both as a team member and alone;
- Reason logically, pose questions;
- Confer and debate with peers and the teacher;
- Discuss their own ideas, as well as develop ideas and knowledge collaboratively;
- Make logical arguments and construct explanations;
- Test their own hypotheses;
- Communicate findings;
- Reflect on feedback from peers and the teacher;
- Consider alternative explanations; and
- Retry experiments, problems, and projects"

When teachers and students engage in math-talk they will be participating in "activities [and acquiring] skills that focus on the active search for knowledge to satisfy a curiosity" (Jarrett, 1997).

Hufferd-Ackles, Fuson, and Sherin studied the development of a discourse [or math-talk] community during mathematics lessons taught in an Elementary School. They said a math-talk learning community is "a classroom community in which the teacher and students use discourse to support the mathematical learning of all participants" (Hufferd-Ackles, Fuson, Sherin, 2004). Teachers and students accept the challenge of knowing and explaining mathematical concepts together. It is no longer the sole duty of the teacher to convey the knowledge, but the duty of all participants to explain their thinking and contribute to the lessons learned in the course. During the study they followed the progress of four teachers, teaching from first to fourth graders, each working to create a math-talk learning community in their classroom. The result of the study was a table that outlined a continuum of the different levels of math-talk observed as the teacher and students attempted to develop a discourse community in their classroom (Appendix A).

Level	Paradigm	Descriptors
Level 0	Traditional Classroom	Teacher directs discourse with brief answers or responses required from the students. The teacher is the only one who asks questions and the questions mostly require a yes or no response.
Level 1	Talk Show Classroom	Teacher attends to limited students' mathematical thinking and focuses less on correct answers; however, the teacher is still the center through which communication occurs. The teacher is the only one who asks questions and there are more follow up questions about procedures.
Level 2	Co-Teaching Classroom	Teacher expects and supports students to build new, inquiry rich roles as the students may even be "co-teaching." In this sense, a teacher is modeling "math talk." The teacher asks probing questions and facilitates the students talking to each other by asking the students to explain to each other their reasoning.
Level 3	Math-Talk Classroom	Teacher is co-teacher and co-learner. While the teacher observes and monitors everything that is going on, students are expected to ask each other about their work and explain their thinking to one another without prompting.

Table 1: Levels of Teaching Progression towards a Math-Talk Classroom adopted from Hufferd-Ackles, et. al., 2004; summarized by Roach, 2010

In a 2010 study, Roach, Robertson, et. al., used the continuum that Hufferd-Ackles, Fuson, and Sherin developed. Roach, et. al., summarized the math-talk continuum table as shown in the descriptors in Table 1.

"At Level 3, many questions are "Why?" questions that requires justification (in addition to the kind of explanation seen at Level 2)" (Roach, et.al., 2010). The Hufferd-Ackles, et. al, framework as summarized by Roach presents a view of the levels of math-talk in questioning, explaining mathematical thinking, source of mathematical ideas, and responsibility for learning that was observed by Hufferd-Ackles, et. al., during their study of teachers in the elementary school that attempted to shift their class from the more traditional teacher-led discussions to a math-talk classroom where the students as well as the teacher contributed ideas to the lesson and the student ideas were used to influence the learning of all in the classroom.

Silver and Smith discussed how the National Council of Teachers of Mathematics is calling for a change in the way that mathematics is taught in the schools. They believe that discourse should be an integral part of the mathematics classroom. Discourse requires communication and many teachers struggle to effectively implement communication strategies into the classroom. They presented a few examples of discourse challenges in the mathematics classroom. Firstly, motivating students to participate in the discussion presents common challenges. Secondly, students may be resistant to share their ideas in school because they fear how they will be viewed by their peers. Thirdly, students may not be able or willing to discuss the mathematics that is being presented due to lack of interest. Specifically, Silver and Smith (1996) discussed the experience of a sixth grade mathematics teacher who asked her students to gather data following the theme "What is your favorite _____" , present the data in a graph, and lastly assigned one member of the group to present the data to the class. She found

that students didn't generally question or discuss the mathematical ideas in the presentation, but rather questions like "How did you decide which TV shows to include?", "How did you divide the work?" or "How long did it take to design the graph?", rather than why a particular graphical form was selected for the data being presented or matters of scaling, etc… Even though the teacher tried to redirect the discussion into more mathematically relevant topics, the students did not follow through and as such the teacher allowed many mathematical issues to go unexplored. What happens when students don't know or don't want to discuss the mathematics? Do we just ignore it, or continue to try to teach them how to think critically?

Another issue brought out of this paper was that of choosing a task that would help bring out discourse on worthwhile mathematical ideas. The example used was from a seventh grade class assignment on ratios. The students were asked to express ratios in various forms in simplest terms. The students were given time to work together, the teacher walking among them answering questions and keeping them on task. A large group discussion began after most of the students completed the assignment. The majority of the discussion focused on how the simplest form of each ratio was obtained by applying a particular procedure individually and generalizations about ratios and proportions were not considered because they were not considered as a whole. The teachers' focus plays an important role in this process including what the teacher wants the students to learn (i.e., bigger more general ideas or specific procedures).

The researchers then described ways that we can support teachers in the process of creating a discourse community. "Teachers need a broad, deep, flexible knowledge of content and pedagogical alternatives. Moreover, they need to be capable of modeling reasonably good mathematical thinking and reasoning as they engage in "deciding what to pursue in depth" and

"when to provide information, when to clarify an issue, when to model, when to lead" (Silver, 1996). Teachers need safe supportive environments where they receive encouragement from colleagues and administration. "[M]ost teacher education programs do not furnish prospective teacher with extensive experience with mathematical discourse, nor do most graduate-degree programs for teachers. Most teachers have learned the mathematics they know in traditional classrooms, they are being asked to create instructional environments with which they have had little direct experience either as teachers or as learners" (Silver, 1996). The goal of a math-talk classroom is to create a classroom where:

- Posing questions and tasks that elicit, engage, and challenge each student's thinking;
- Listening carefully to students' ideas;
- Asking students to clarify and justify their ideas orally and in writing;
- Deciding what to pursue in depth from among the ideas that students bring up during a discussion;
- Deciding when and how to attach mathematical notation and language to students' ideas;
- Deciding when to provide information, when to clarify an issue, when to model, when to lead, and when to let a student struggle with a difficulty; and
- Monitoring students' participation in discussions and deciding when and how to encourage each student to participate.

Creating a classroom atmosphere where math-talk is nurtured presents a great goal, but little is said about the various paths that one could take to get there and the various problems that one may encounter along the way. Smith, Bill, and Hughes (2008) developed a procedure called Thinking Through Lesson Protocol (TTLP), originally designed to help teachers implement tasks that help students engage in high-level thinking. TTLP, briefly outlined in Table 2 (shown in Appendix B), "provides a framework for developing lessons that use students' mathematical thinking as the critical ingredient in developing their understanding of key disciplinary ideas"

Section	Title	Descriptor
Part 1	Selecting and Setting Up a Mathematical Task	Lays the groundwork for subsequent planning by asking the teacher to identify the mathematical goals for the lesson and set expectations regarding how students will work. The mathematical ideas to be learned provide the direction for all decision making during the lesson.
Part 2	Supporting Students' Exploration of the Task	Monitoring students as they explore the task (individually or in small groups). Students are asked questions based on the solution method used to assess what they currently understand so as to move them toward the mathematical goal of the lesson.
Part 3	Sharing and Discussing the Task	Orchestrating a whole-group discussion of the task that uses the different solution strategies produced by students to highlight the mathematical ideas that are the focus of the lesson.

Table 2: Thinking Through a Lesson Protocol (TTLP), adopted from Smith, Bill, & Hughes, 2008

(Smith, Bill, & Hughes, 2008). TTLP is intended to help "teachers anticipate what students will do and generate questions teachers can ask that will promote student learning prior to the lesson being taught." (Smith, Bill, & Hughes, 2008)

The purpose of my project was to learn strategies for increasing the level of math-talk in my own university classroom. From the literature reviewed here, I gained specific ideas for fostering the growth of questioning strategies among my students and as an instructor. I also learned ways for categorizing questions that are helpful in my reflections on classroom practice. Further literature has illuminated the development of math-talk (Hufferd-Ackles, et. al., 2004, Roach, 2010) in classrooms as teachers make changes to their practice. While this guideline is valuable for tracking progress and assessing classrooms, I found it lacking in its ability to provide specific moves I could make to implement my own lesson improvement. The TTLP tool by Stein and Smith filled that purpose, and consequently the use of that tool guided my lesson design and implementation.

Methodology

To help me achieve my goal of fostering more math-talk in my classroom, I decided to use the question based guidelines of TTLP to design a final exam review for the Calculus II course that I taught at Utah State University in the fall of 2011. In this section I provide some background about the university and course, and also describe my methods for designing and evaluating my review sessions.

Utah State University (USU) is an R1 university located in northern Utah that serves more than 14,000 students. This section of Calculus II was one of nine sections taught on campus. The class consisted of 38 undergraduate students, most of whom were engineering majors required to take the course. Throughout the course we studied some applications of integration (work, centers of mass), integration techniques (integration by parts, trigonometric substitution, partial fractions, integration tables, numeric approximation, indefinite integrals), convergence and divergence of infinite sequence and series, polar coordinates, conic sections, three dimensional space, vectors, applications. (For a complete list of course objectives see Appendix C.)

The review for the final exam took place over three separate 50 minute class periods. Because the students wanted a comprehensive review of all topics to help them prepare for the final, I decided to devote each day to a different section of the course: Day 1 – Integration Applications (solving separable differential equations, work, center of mass), Integration Techniques (parts, trigonometric substitution, partial fractions, improper), and the Convergence or Divergence of infinite Sequences or Series; Day 2 – Power Series, Taylor's Theorem, Binomial Series, Polar Coordinates (converting from rectangular coordinates, sketching polar curves, area under polar curves, arc length), Conic Sections (in rectangular and polar form),

Vectors in Space (addition, scalar multiplication, dot product, cross product and applications); Day 3 – Vector Equations in space (definition of a vector equation, differentiating and integrating vector equations, finding the unit tangent, normal, and binormal vectors, curvature, torsion) and applications.

Identifying the objectives for the review and dividing those objectives into three separate review sessions was the essential first step of using the TTLP. The next step was creating or finding appropriate tasks that would address the objectives. I began the hunt for the tasks by referencing the textbook we were currently using in the course (Hass, Weir, & Thomas, 2007), a different calculus textbook (Stewart, 2005), and the teaching resource guide associated with the second text (Shaw, et. al., 2005). I devoted a couple days to gathering tasks I thought addressed the objectives of the course. My thought was that once I gathered a number of tasks for each day, I would then analyze the tasks following the method of TTLP and then decide which tasks would be most beneficial to use for the review. After I gathered the tasks I sat down and answered the questions posed by the TTLP process about the tasks, then I chose the six or seven tasks for each day that I felt would be beneficial for my students to study in preparation for their final exam.

As the review days approached, I asked some of my colleagues to observe the class and take notes about their observations of the class, paying particular attention to the discussion that occurred. I provided the observers with the Hufferd-Ackles math-talk continuum table and asked them to identify the level of math-talk they observed in the classroom. During the first two days of review there was one observer, one teacher, and approximately 38 students present in the classroom (there was no observer on the third day). I (the teacher) took time after each class to write down my reflections and impressions about what went on in the course. I also made note

of where I thought the class fell along the Hufferd-Ackles math-talk continuum table. What follows are the details of my plans for the review that resulted from addressing questions in the TTLP, a summary of observations and reflections of the implementation of my review, and in the conclusion section, I revisit the driving questions that motivated this project.

Findings

In this section I describe how I planned the review days using the method of Thinking Through Learning Protocol (TTLP). It also contains a summary of the observations and reflections of each day, along with an analysis of the level of math-talk based on my reflections and evaluation from my colleagues. Throughout the session I describe my work using TTLP to plan each day of the review sessions. Day 1 and day 2 are presented completely, but day 3 is not described in detail because I did not get observation data from that day to more fully analyze the discourse in the classroom.

Day 1

The first day was devoted to reviewing the topics we studied during the first part of the course. I developed the activity while considering the mathematical objectives in these chapters (see Appendix C). With these objectives in mind, I chose or created six different prompts (see Table 3) that would serve to help the students recall what they had learned in those chapters. I included as many topics as I could from the list of objectives while informing my students that the prompts should not be considered an all-inclusive, comprehensive review.

1. Archaeologists have determined that the great statue of Aruba was really a giant magnet placed on top of an iron table. At a height of x feet above the table, the magnetic force exerted on the statue was given by

$$f(x) = \frac{1600}{(2x+1)^2}\ lbs$$

When the mighty Hercules lifted the statue 3 feet before hurling it at Ares, how much work did he do? (Note: Don't forget gravity! The statue weighed 1200 lbs.)
(Adapted from Shaw, et. al., 2005, p426 #1)

2. Suppose A & B are constants. Verify that $y(x) = Ax^{-1} + B + \ln x$ is a solution of the differential equation $y'' + 2y' = \frac{1}{x}$. (Adapted from Shaw, et. al., 2005, p458 #1)

3. Here is a copious list of sequences and series. Determine whether each converges or diverges. Justify your answer.

(a) $a_n = \dfrac{e^n}{3^n}$ (Shaw, et. al., 2005, p531 #1)

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2-10}$

(c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$

(d) $a_n = \dfrac{(-1)^n + n}{(-1)^n - n}$ (Shaw, et. al., 2005, p531 #5)

(e) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sin\left(\pi + \frac{\pi}{n}\right)$

(f) $a_n = (-1)^n \sqrt{n}$ (Shaw, et. al., 2005, p531 #2)

(g) $a_n = \dfrac{\ln\left(\left(e^4\right)^n\right)}{3^n}$ (Shaw, et. al., 2005, p531 #6)

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{-2/3}}{n^{7/5}}$

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{3n}{e^{5n}}$

(j) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-3}{n^3+5}$

(k) $a_n = (-1)^n \dfrac{1}{\sqrt{n}}$ (Shaw, et. al., 2005, p531 #3)

(l) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n + 2^n}{6^n}$

(m) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3n}{9n-2}$

(n) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{4n}{n^2-32}$

(o) $a_n = (-1)^n \cos\left(\dfrac{\pi}{2}(n+1)\right)$ (Shaw, et. al., 2005, p531 #7)

(p) $a_n = (-1)^{2n+1}$ (Shaw, et. al., 2005, p531 #4)

(q) $\displaystyle\sum_{n=1}^{\infty} \frac{5n^2}{3n^2+1}$

(r) $\displaystyle\sum_{n=1}^{\infty} \frac{5-2\sqrt{n}}{n^3}$

(s) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$

(t) $a_n = \left(1 - \dfrac{1}{n}\right)^n$

(Hass, et. al., 2007, p512 #50)

4. Can you find a sequence $\{a_n\}$ such that $\{a_n\}$ converges to zero and the series $\sum_{k=1}^{\infty} a_k$ diverges? Justify your answer. (Adapted from Shaw, et. al., 2005, p563 #1)

5. Can you find a sequence $\{a_n\}$ such that $\{a_n\}$ diverges and the series $\sum_{k=1}^{\infty} a_k$ converges? Justify your answer. (Adapted from Shaw, et. al., 2005, p.563 #3)

6. Evaluate each integral by using u-substitution, integration by parts, trigonometric substitution, trigonometric identities, or techniques of integrating with improper integrals.

(a) $\int \frac{x^3+2}{4-x^2}\, dx$ (Hass, et. al., 2007, p499 #70)	(b) $\int \frac{9}{81-v^2}\, dv$ (Hass, et. al., 2007, p499 #75)	(c) $\int \frac{dx}{x(x^2+1)^2}$ (Hass, et. al., 2007, p499 #71)
(d) $\int \frac{\sqrt{1-x^2}}{x^2}\, dx$ (Hass, et. al., 2007, p499 #82)	(e) $\int \frac{\sin^2 x}{\cos^2 x}\, dx$ (Hass, et. al., 2007, p499 #74)	(f) $\int \tan^4 x\ \sec^2 x\ dx$ (Hass, et. al., 2007, p498 #39)
(g) $\int_2^{\infty} \frac{dx}{(x-1)^2}$ (Hass, et. al., 2007, p499 #76)	(h) $\int_0^3 \frac{dx}{x-1}$ (Adapted from Hass, et. al., 2007)	(i) $\int \theta \cos(2\theta+1)\ d\theta$ (Hass, et. al., 2007, p499 #77)
(j) $\int \ln(x+1)\ dx$ (Hass, et. al., 2007, p497 #1)	(k) $\int_{-1}^{1} \frac{dy}{y^{2/3}}$ (Hass, et. al., 2007, p499 #55)	(l) $\int_{-\infty}^{0} xe^{3x}\, dx$ (Hass, et. al., 2007, p499 #60)

Table 3: Student Prompts for Day 1 Review

My desire was that they while they worked through the prompts (see Table 3), they would be reminded of the important ideas and skills that they obtained during this course. Due to the lengthy list of objectives and the number of prompts selected, this presentation focuses on my experience and thought process with prompt #6 for the first day of review (see Table 4). Prompt #6 not only served to remind the students of the integration techniques we discussed during class, but to help them decide when to apply a particular technique. I presented a list of integration problems for the students to solve. The problems I chose to include on the list have varying solution methods, and for some of the problems, more than one method can be applied. Table 4 includes all problems from prompt #6 and my attempt at peering into the minds of my students to predict how they will attempt to solve each problem.

Instructions: Evaluate each integral by using u-substitution, integration by parts, trigonometric substitution, trigonometric identities, or techniques of integrating with improper integrals. (Objective Ch.7B)	
Problem	*Possible Solution Method(s)*
a. $\int \frac{x^3+2}{4-x^2}\, dx$ (Objective Ch.7Aiv)	Partial Fractions
b. $\int \frac{9}{81-v^2}\, dv$ (Objective Ch. 7Aiii/iv)	Trigonometric Substitution or Partial Fractions
c. $\int \frac{dx}{x(x^2+1)^2}$ (Objective Ch.7Aiv)	U-Substitution or Partial Fractions
d. $\int \frac{\sqrt{1-x^2}}{x^2}\, dx$ (Objective Ch.7Aiii)	Trigonometric Substitution
e. $\int \frac{\sin^2 x}{\cos^2 x}\, dx$ (Objective Ch.7Aii)	Rewriting Integrand using Trigonometric Identities
f. $\int \tan^4 x\, \sec^2 x\, dx$ (Objective Ch.7Aii)	Rewriting Integrand using Trigonometric Identities then a U-Substitution
g. $\int_2^\infty \frac{dx}{(x-1)^2}$ (Objective Ch.7Aiv/E)	U-Substitution or Partial Fractions then evaluating Improper Integral with a limit.
h. $\int_0^3 \frac{dx}{x-1}$ (Objective Ch.7E)	U-Substitution and evaluating Improper Integral with a limit.
i. $\int \theta \cos(2\theta+1)\, d\theta$ (Objective Ch.7Ai)	Integration by Parts and a U-Substitution
j. $\int \ln(x+1)\, dx$ (Objective Ch.7Ai)	Integration by Parts and a U-Substitution
k. $\int_{-1}^1 \frac{dy}{y^{2/3}}$ (Objective Ch.7E)	Evaluating Improper Integrals with a limit.
l. $\int_{-\infty}^0 xe^{3x}\, dx$ (Objective Ch.7Ai/E)	Integration by Parts and evaluating Improper Integral with a limit.

Table 4: Day 1 Review - Prompt #6

Prompt #6 is especially useful for helping the students achieve Objective Ch.7B. The directions of the prompt give a list of possible solution methods, but it is up to the students to determine which would be most helpful to them in solving the integral. Students could possibly try to classify the integrals according to the solution method they think will work, before actually trying to solve each individual integral. Students may have difficulty remembering to use long division before the method of partial fractions in part (a). I expected that the students would have some difficulty recognizing that part (h) and (k) are improper (because the asymptote occurs within the bound of the integral) and therefore may attempt to solve the integral without taking a limit. I also expected that the students would not choose trigonometric substitutions

over partial fractions. Any method students chose to attempt to solve the integral would be useful to help them achieve the learning goal (Objective Ch.7B). Through the process of decision-making they will be able to develop their skill of choosing the appropriate solution method.

I gave the students a worksheet with the list of prompts. As a class, we decided what order we would work on the prompts. After the decision was made, the students were given 5-15 minutes to work out a solution, returned to the discussion, and finally volunteered to share their solution method with the class and answer any questions that may arose. I then asked if any other solution methods were used. Pros and cons of each method were discussed. Students did have access to their textbooks, calculators, pencils, and were free to ask questions. I encouraged the students to work in small groups. Students informally recorded their work on the worksheet (it was not collected) or in their notes, and reported on their work during the class discussion.

If students had questions, comments, or pleas for mercy while working on the task, I addressed those individually. Sometimes I publicly offered additional instruction, if it appeared that quite a few students were at a loss for how to start. I knew that students understood if they went to work right away. If there were murmurings, a general feeling of confusion, or off-task behavior, then I knew that more direction was needed.

Prior to the class meeting and in response to TTLP part 2 and 3, I prepared the following list of questions to help me support students while they were working on prompt #6:

- What is the formula for integration by parts? Where does it come from? Why might you want to use it?
- When applying the formula for integration by parts, how do you choose u and dv? How can you apply integration by parts to an integral of the form $\int f(x)\, dx$?

- What is the goal of the method of partial fractions?
- When the degree of a polynomial $f(x)$ is less than the degree of a polynomial $g(x)$, how do you write $f(x)/g(x)$ as a sum of partial fractions if $g(x)$
    - is a product of distinct linear factors?
    - consists of a repeated linear factor?
    - contains an irreducible quadratic factor?
    - What do you do if the degree of f is not less than the degree of g?

Using TTLP to prepare helped me to successfully coach the students as they worked on the prompts. I felt that the questions and prompts that were prepared for this day of review helped to encourage math-talk in the classroom. My students were engaged in conversations about the content. When confronted with difficult questions they worked collaboratively to solve the problems. Although they had engaged in math-talk, they continued to see me as the math guru to whom they must receive validation. This process was further guided towards a greater feeling of independence which was built upon on Day 2.

Day 2

I tackled the preparations for day two much like I did for day one. The mathematical goals for day two came from the topics we covered during the middle third of the course. We reviewed the topics we learned about in Chapter 8 (Part 2: Infinite Sequences and Series), Chapter 9 (Polar Coordinates and Conics), and Chapter 10 (Vectors and the Geometry of Space). The students previously participated in a series of lectures and completed homework assignments relevant to those objectives and I selected and designed prompts that would remind the students about what I deemed to be the big ideas of these chapters (see Table 5). The student responses were

1. Find the Taylor series for the function $g(x) = \sin(x - \pi)$ centered at $a = 0$. Determine the radius and interval of convergence of the series you found. (Adapted from Hass, et. al., 2007)

2. Examples, examples, examples.
   (a) Find a sequence $\{a_n\}$ such that $\lim_{n\to\infty} a_n = 0$ and $\sum a_n$ diverges. (Adapted from Stewart, 2005)
   (b) Find a sequence $\{a_n\}$ such that $\lim_{n\to\infty} a_n$ does not exist. (Adapted from Stewart, 2005)
   (c) Find a polar equation whose graph is a circle.
   (d) Find a vector equation that describes a circle.
   (e) Find a polar equation for a parabola with a vertical directrix.
   (f) Find polar coordinates of the origin.
   (g) Find an equation of a hyperbola centered at the Cartesian point $(-3, 6)$.
   (h) Find two orthogonal vectors.
   (i) Find two unit vectors parallel to $\boldsymbol{w} = \langle w_1,\ w_2,\ w_3 \rangle$. (Adapted from Hass, et. al., 2007)
   (j) Find the equation of a line in space.

3. Find the area inside one leaf of the four-leaved rose $r = \cos 2\theta$. (Hass, et. al., 2007, p589 #3)

4. Find the length of the curve described by the polar equation $r = 1 + \cos\theta$. (Hass, et. al., 2007, p590 #19)

5. Given $\vec{u} = \langle 1, 1, 2 \rangle$ and $\vec{v} = \langle -1, -1, 0 \rangle$ find $-2\vec{u} + \vec{v}, |\vec{v}|, \vec{u} \cdot \vec{v}, \vec{v} \times \vec{u}$, the angle between $\vec{u}$ and $\vec{v}$, and the vector projection of $\vec{u}$ onto $\vec{v}$. (Adapted from Hass, et. al., 2007)

6. The planes $3x + 6z = 1$ and $2x + 2y - z = 3$ intersect in a line.
   (a) Show that the planes are orthogonal.
   (b) Find parametric equations for the line of intersection. (Hass, et. al., 2007, p659 #45)

7. For what value or values of a will the vectors $\boldsymbol{u} = 2\boldsymbol{i} + 4\boldsymbol{j} - 5\boldsymbol{k}$ and $\boldsymbol{v} = -4\boldsymbol{i} - 8\boldsymbol{j} + a\boldsymbol{k}$ be parallel? (Adapted from Hass, et. al., 2007)

Table 5: Student Prompts for Day 2 Review

influenced by their previous experiences in class. I attempted to peer into the minds of the students in my class and predicted the possible ways this task was attempted. These solution methods are listed in Table 6.

Instructions: Examples, examples, examples. Find an example of each statement.	
Problem	*Possible Solution Method(s)*
A. Find a sequence $\{a_n\}$ such that $\lim_{n\to\infty} a_n = 0$ and $\sum a_n$ diverges. *(Objective Ch.8B)*	I think that the students will recall the harmonic series, the most famous example of a series whose terms approach 0, but whose sum diverges. Also, the students could attempt to construct a random sequence and then use one of the series tests for convergence/divergence to determine whether the series converges or diverges. Some may attempt to look for such a series in the textbook or from their neighbor. Students may also find a divergent sequence, misinterpreting the notation.
B. Find a sequence $\{a_n\}$ such that $\lim_{n\to\infty} a_n$ does not exist. *(Objective Ch.8A)*	I predict that students will discuss the meaning of a limit not existing (whether or not it includes infinite limits or just sequences that bounce around, never settling somewhere) before creating the sequence. Students may just create an unbounded sequence. Students may submit a convergent sequence whose series would diverge as a solution.
C. Find a polar equation whose graph is a circle. *(Objective Ch.9A)*	Cartesian equation of circle submitted. Students begin with Cartesian equation then convert it directly to polar coordinates. Construct polar equations of form $r = a$, where $a\epsilon\mathbb{R}$. Or look up in textbook (or neighbors notes) polar equations of the form $a^2 = r_o^2 + r^2 - 2r_o r\cos(\theta - \theta_o)$, where a is the radius and (r_o, θ_o) is the center. Perhaps modifications of equations for ellipse would be considered. Random made up equations with r's and theta's.
D. Find a polar equation for a parabola with a vertical directrix. *(Objective Ch.9G)*	Cartesian equation of ellipse submitted. Refer to the textbook to find general form of polar equations of parabolas, ellipses, and hyperbolas. Select eccentricity either correctly or incorrectly for a parabola.
E. Find polar coordinates of the origin. *(Objective Ch.9A)*	$(0,0)$ submitted with no further explanation or some justification for including any number as second component for ordered pair. No solution possible.
F. Find an equation of a hyperbola centered at the Cartesian point $(-3, 6)$. *(Objective Ch.9D,E)*	Standard equation of hyperbola (centered at origin) submitted. Attempt made to transform the standard equation, but incorrectly executed. Incorrectly submitting equation for ellipse or circle. Equation of form $\frac{(x+3)^2}{a^2} - \frac{(y-6)^2}{b^2} = 1$ or $\frac{(y-6)^2}{a^2} - \frac{(x+3)^2}{b^2} = 1$ where $a, b\epsilon\mathbb{R}$.
G. Find two orthogonal vectors. *(Objective Ch.10C)*	Sketching two orthogonal vectors. Displaying component form (or linear combination of standard unit vectors) of two orthogonal unit vectors. Demonstrating that dot product equals zero. Incorrectly demonstrating that the cross product equals zero.

H. Find two unit vectors parallel to $\boldsymbol{w} = \langle w_1,\ w_2,\ w_3 \rangle$. *(Objective Ch.10A)*	Finding any two unit vectors. Displaying two vectors parallel to $\boldsymbol{w}$ but not of length one. Picking specific values for w_1, w_2, and w_3, then finding the corresponding unit vectors. Displaying the vectors $\frac{\boldsymbol{w}}{\lvert\boldsymbol{w}\rvert}$ and $-\frac{\boldsymbol{w}}{\lvert\boldsymbol{w}\rvert}$. Discussion on what a unit vector is before decisions about solution are made.
I. Find the equation of a line in space. *(Objective Ch.10E)*	Submitting equation of form $y = mx + b$. Vector equation of form $\langle x, y, z \rangle = \langle x_o, y_o, z_o \rangle + t\langle a, b, c \rangle$ where (x_o, y_o, z_o) is a known point on the line and $\langle a, b, c \rangle$ is a vector parallel to the line. Parametric equations of the form $x = x_o + ta$, $y = y_o + tb$, $z = z_o + tc$ where (x_o, y_o, z_o) is a known point on the line and $\langle a, b, c \rangle$ is a vector parallel to the line. Incorrectly submitting equation of plane. Submitting polar equation of line $r\cos(\theta - \theta_o) = r_o$ where (r_o, θ_o) is the foot of the perpendicular from the origin to the line or polar equation of the form $\theta = \theta_o$, $\theta_o \epsilon \mathbb{R}$.

Table 6: Day 2 Review - Prompt #2

During class each student was given a list of the prompts. The class was given time to peruse through the prompts and decide which to work on. The students were given the liberty to work individually or in groups (encouraged). After 10 minutes of deliberation, the groups (or individuals) presented and discussed different solution options. Students were not required to formally record their work, but most used the worksheet and recorded their attempts to determine the solution to the prompts.

Students were given advance notice of the review and came to class expecting to discuss a certain subset of topics we had previously spent some time on in the course. I assumed that some students would be better prepared to successfully navigate through the prompts, than others. The classroom atmosphere was an open, comfortable place for students to express their opinions. I expected that the student concerns would come out immediately if they did not understand. I tried to clarify the prompt, without immediately offering a solution. I again, expected to hear the quiet sounds of students' productivity if they understood what they were

doing. I prepared the following questions to help me help the students as they worked on the prompts.

- What does the notation $\{a_n\}$ mean? What is it that you are looking for?
- What does it mean for $\lim_{n\to\infty} a_n = 0$?
- What does it mean for $\sum a_n$ to diverge (or converge)?
- What does it mean when $\lim_{n\to\infty} a_n$ does not exist? If a sequence increases (or decreases) without bound, do we say the limit does not exist? If a sequence oscillates, do we say the limit does not exist?
- What is an infinite sequence? What does it mean for such a sequence to converge? To diverge?
- What is an infinite series? What does it mean for such a series to converge? To diverge?
- What is a power series? How do you test a power series for convergence? What are the possible outcomes?
- What is the Taylor series generated by a function $f(x)$ at a point $x = a$? What information do you need about f to construct the series?
- What is a Maclaurin series?
- What are polar coordinates? What equations relate polar coordinates to Cartesian coordinates? Why might you want to change from one coordinate system to the other?
- How do you find the area of a region $0 \leq r_1(\theta) \leq r \leq r_2(\theta)$, $\alpha \leq \theta \leq \beta$, in the polar coordinate plane?
- Under what conditions can you find the length of a curve $r = f(\theta),\ \alpha \leq \theta \leq \beta$, in the polar coordinate plane?
- What is a parabola? What is an ellipse? What is a hyperbola? What are the Cartesian equations of each?
- What is eccentricity of a conic section?
- What are standard equations for lines and conic sections in polar coordinates?
- How do you find a vector's magnitude and direction?
- Define the dot product (scalar product) of two vectors. When is the dot product of two vectors equal to zero? What geometric interpretation does the dot product have?

- Define the cross product (vector product) of two vectors. When is the cross product of two vectors equal to zero? What geometric or physical interpretations do cross products have?
- How do you find equations for lines, line segments, planes and spheres in space? Give examples.

Helping the students navigate through the prompts and orchestrating an environment conducive to math-talk was easier during the second day review. The prompts chosen seemed to generate more controversy than the prompts during day #1, naturally spurring more discussion. Only after one math-talk class meeting, the students were more centrally engaged within their groups and less focused on the dissemination of my knowledge. Students became more confident during the Day 2 review and were more willing to defend their position with their peers. I served as a facilitator during this process instead of (or rather than) the provider of information.

Day 3

The third day of review consisted of topics from Chapter 11, the only chapter that was untested up to that point. The students were aware that about half of the Final Exam would be taken from topics in Chapter 11. With that in mind, I created a list of prompts to help assist the students in preparing for the exam (see Table 7). In Table 8, I show the first prompt and possible solution methods for that prompt.

1. Find the length of the curve $\vec{r}(t) = \langle 2t^{3/2},\ \cos 2t,\ \sin 2t\rangle$ for $0 \le t \le 1$. (Adapted from Hass, et. al., 2007)

2. A particle moves in space with parametric equations $x = t,\ y = t^2,\ z = \frac{4}{3}t^{3/2}$. Find each of the following when $t = 1$.
   a. The unit tangent vector $\vec{T}$.
   b. The unit normal vector $\vec{N}$.
   c. The binormal vector $\vec{B}$.
   d. The curvature κ of its trajectory.
   e. Find an equation for the osculating plane. (Adapted from Hass, et. al., 2007)

3. A particle starts at the origin with initial velocity $\vec{\imath} - \vec{\jmath} + 3\vec{k}$. Its acceleration is $\vec{a}(t) = 6t\vec{\imath} + 12t^2\vec{\jmath} - 6t\vec{k}$. Find its position function. (Adapted from Hass, et. al., 2007)

4. Find the point on the curve $f(x) = x^2$ where curvature is the greatest. Justify your answer. Hint: Use the curvature formula $(x) = \frac{|f''(x)|}{\left[1+(f'(x))^2\right]^{3/2}}$. (Adapted from Hass, et. al., 2007)

5. $\vec{r}(t) = \langle \sec t, \tan t, \frac{4}{3}t \rangle$ is the position of a particle in space at time t.
   (a) Find the particle's velocity and acceleration vectors.
   (b) Find the equation of the line tangent to the curve at $t = \frac{\pi}{6}$.
   (c) Are the particle's velocity and accelerating vectors orthogonal at $t = 0$? (Adapted from Hass, et. al., 2007)

6. Solve the initial value problem where $\frac{d^2\vec{r}}{dt^2} = \langle -1, -1, -1 \rangle$ and $\vec{r}(0) = \langle 10, 10, 10 \rangle$ and $\left.\frac{d\vec{r}}{dt}\right|_{t=0} = \langle 0, 0, 0 \rangle$. (Adapted from Hass, et. al., 2007)

Table 7: Student Prompts for Day 3 Review

Problem	*Possible Solution Method(s)*
Find the length of the curve $\vec{r}(t) = \langle 2t^{3/2}, \cos 2t, \sin 2t \rangle$ for $0 \le t \le 1$. *(Objective Ch.11D)*	It can be solved using the arc length formula $L = \int_a^b \lvert\vec{v}\rvert\, dt$. I predict that some students may try and use the arc length formula given in Chapter 6, they would then need to find a function y in terms of x and it is a much messier way of obtaining the solution.

Table 8: Day 3 Review - Prompt #1

The overall atmosphere of the class during this review session was much like the first two days. Students were given time to peruse through the prompts, choose which to work on, given time to work, and then we discussed it as a class. I observed that students were equally comfortable with the math-talk classroom as Day 2, although they spent a great deal of time asking questions on the semantics and specifics of the final due to its proximity.

Reflections and Observations on the Implementation of the Review

Day 1

On December 6, 2011 the students from my Calculus II course filed into our 9:30 am class as usual. There was some buzz from their individual conversations as they entered. Most of those conversations ended when I began class, while one group persisted quietly for another minute or so. This casual conversation indicates that a comfortable classroom atmosphere has already been established. I distributed the list of prompts and instructed my students to take some time to peruse the prompts and then choose one to focus on first. The first prompt the students decided to work on was prompt #6. In this prompt the students given a list to integrals and were asked to determine and execute an appropriate method of integration for each (see Table 3). Initially I guided the students to look at the overall set of problems and try to decide which integration method to use before attempting to actually integrate. Those words prompted some discussion from the students as they tried to organize their integrals according to integration method. After most felt satisfied with their organization we briefly discussed as a class which method they chose for each integral. I led the discussion, asking for input from the students. The largest difference of opinion among the students was in choosing a method for integrals a, d, h, and j. As a result, these integrals spurred the most discussion in the class. In some cases the students began to try one method and got stuck, so were then unsure about the method they had chosen. In other cases, students did not want to continue with the method they initially thought was right and tried other ones, or just could not determine a method at all. As a class we settled on a method for those problems and then time was given for the student to work out the integral. This spurred discussions among the groups of students as they tried to remember the steps for each method or the algebra or trigonometry required for each problem.

After a while, the students asked me to help them with integral j, because none of the groups had successfully worked through that integral. I walked the students through my solution by projecting it for the whole class to see and question.

The students decided to move onto prompt #1. In this prompt the students were asked to determine how much work Hercules performed lifting and throwing a statue at Ares (see Table 3). As it turned out, this prompt did not spur much discussion, although the students liked the story told in the prompt.

Prompt #3 was the last prompt we had time for. This prompt gave a long list of sequences and series and required students to determine whether each converges or diverges, justifying their answers. The students were initially shocked by the number of prompts. I guided them to look through the prompts and determine which would diverge and which could converge before trying the tests to determine convergence. This prompted some discussion about how we would know if a series or sequence diverges and some careful consideration of the difference between sequences and series. The students asked for help with the sequences that included trigonometric functions because they did not remember the Squeeze theorem. We ran out of time before this prompt was completed.

I felt that this activity did promote more discussions among the students. Because it was a review and they had been exposed to the mathematical concepts and ideas beforehand, the students had previous experience to draw upon as they attempted to classify and solve the problems that were presented. One group of students impressed me by determining another method of integration that was not explicitly listed in the directions of the prompt. My

impression was that the class was between a level 1 and level 2 in the Hufferd-Ackles math-talk continuum table (see Appendix A).

Observer 1, a fellow graduate student, observed this class. His impressions were that the students were comfortable sharing their ideas and that they worked together well to solve the problems. He noted that the students asked many questions directed to the teacher and the teacher was usually the one to respond. He also noted that in one instance the students came up with an integration method not listed by the teacher. Overall he felt the class fell between level 2 and 3 in questioning, and between level 1 and 2 in explaining mathematical thinking, source of mathematical ideas, and responsibility for learning on the Hufferd-Ackles math-talk continuum table (see Appendix A). Overall, I felt that the level of math-talk in my classroom was higher using this activity than it had been in the past.

Day 2

Day 2 began much like day 1. I stood at the door as the students came into the classroom. As I greeted them I handed them the prompts I had prepared for the day. As class started, I gave the students time to peruse the prompts and decide as a class where they would like to begin. They decided on prompt #1 which directed the students to generate a Taylor series for a function (see Table 5). As they worked on solving the problem lots of questions about the mechanics of generating a Taylor series were asked. There were not many questions about the relationship between the function and the Taylor series. We then moved onto prompt #2, which directed the students to find or create examples of various mathematical concepts (see Table 5). This prompt generated more questions than I anticipated. I believed that the statements were straight forward, but the students had many questions about what exactly they were asking for and seemed to have difficulty coming up with their own example of the different mathematical

ideas. The rest of the class time was spent discussing these ideas. My impression was that prompt #2 generated the best discussion among the class. There was a lot of discourse between me and the students and between students. I believe that the class registered between a level 2 and level 3 on day 2.

Observer 2, a professor at USU, noted that the students began to work on prompt #1 individually. After realizing that they had questions about the mechanics of generating a Taylor Series the discussions amongst the teacher and the students began. The observer noted that the students began to look through their notes to find the formula for the Taylor Series, while another began to take derivatives. Some students waited for verification from the teacher. The teacher gave time for the students to finish their work with the Taylor series, asking for a volunteer to come share her work with the class. One student was brave enough to share. After presenting his work the student asked for verification from the teacher. Other students congratulated him. The teacher asked for other methods of generating the series and right away one student said no and the class moved on.

This observer also noted that prompt #2 generated lots of questions by the students. About 70% chose to work in small groups, 30% alone. The questions that the groups could not figure out were brought to the attention of the teacher. Questions were answered or a follow up question was asked to help the students think about it a different way. As time ran out, the student asked the teacher for solutions to the prompts and for guidance about what to focus their study on.

Overall, the observer thought that questioning was at a level 2, explaining mathematical thinking was at a level 1.5, source of mathematical ideas at a level 2.5, and responsibility for

learning at a level 2 for math-talk in the classroom, as outlined in the Hufferd-Ackles math-talk table. This analysis was comparable to my own analysis of the discussion during this day of review.

Day 3

Day 3 felt and ran much like the other two days. My impression was that the students achieved a level 2 in all categories on this day. The person I arranged to observe Day 3 was unable to make it at the last minute.

Over the course of the 3 days, my observations and reflections generally were affirmed by the observers that came to class. I tended to be more critical of my own level of questioning, but in some instances I had more insight into student thinking.

Conclusion

Throughout the course of this project I aimed to find answers for the following questions. I discovered insights about each question at the end of my journey. The literature review and the classroom experiment I conducted provided me with a greater understanding of these issues. In this section, I present a brief synopsis of my current understanding of each question.

- If a gigantic curriculum and limited time is an issue, is it possible to implement discourse without compromising the material or time you have to work with?

Throughout the process of this paper I learned that students need some knowledge about a topic before a productive discussion is possible. It therefore makes sense for direct instruction to come before the discussions one hopes to incorporate into the class. Posing questions or problems for the students after a new topic is presented is good for generating discussions.

Review days are ideal for these inquiry style discussions, because students have been exposed to the material and the discussions give them an opportunity to clear up any misconceptions and solidify the concepts. An easy-to-implement change in the right direction would be for instructors to allow students time to ponder new knowledge at class closing.

- What steps do we take to create a discourse community in the classroom?

First and foremost, instructors must allow time for such discussions to take place. Discussions rarely spontaneously occur, and sometimes enduring a few moments of silence will help to give students time to think about how they could contribute to the discussion. Instructors should prepare a list of possible questions that get at what the student is thinking and will help the students clarify their thoughts as this preparation helped me in my own experiment conducting discussions. These prompts should be prepared ahead of time and are best used if the instructor is acutely aware of the mathematical objectives of the discussion.

- How do we create a classroom atmosphere conducive to student explorations and discussions about mathematics?

Classrooms conducive to math-talk are ones in which the instructor is open with the students. Teachers must not be afraid to make mistakes in front of the class or follow erroneous thinking. When this occurs, students can see the results and hopefully be better equipped to navigate their way though mathematical thinking in the future. Teachers should also allow students time to navigate the mathematical waters themselves, asking follow up or guiding questions rather than just offering solutions. It is good practice for teachers to listen to the students and try to understand student ideas and point of view. Teachers should also be willing to follow the mathematical path the students choose. Remember, the teacher is there to help them along the journey.

- How can controversy serve as an aid to help the students think more deeply about their own assumptions and encourage them to revisit and refine what they originally thought about a mathematical idea?

Choosing prompts that have controversy built into them can be a great tool for enticing students into participating in discussions. These types of prompts can assist you in helping the students really understand the finer points of certain theorems or topics that otherwise might be over generalized or misunderstood. These prompts can also help the students really think about under what conditions a particular theorem applies.

- How can I create opportunities for my students to think about, explain, and perhaps defend their understanding of a particular piece of mathematical content?

I created this opportunity by using the methods of TTLP to identify prompts that would promote discussions during a review session. In the future I would like to experiment with using discussions facilitated through an online medium in an effort to save time in the classroom.

Another goal of this project was to see if the methods of TTLP would assist in facilitating math-talk in the classroom. My impression was that more math-talk occurred in the classroom when the students were presented with the prompts that I chose using the TTLP than occurred before. I found that the TTLP gives a thorough method of looking at what the mathematical goals of the lesson are, and a way of helping the instructor keep on task throughout the lesson. However, it is time consuming and cumbersome and would not be a practical method for everyday use. I spent approximately 10 hours selecting the prompts and preparing supporting questions for each day. Determining the mathematical goals and considering accommodations for students all required time on top of what I had already spent. I felt that TTLP was most helpful in preparing for a course review. It was difficult at times to predict the possible student responses, but that skill may become easier as the teacher uses such prompts more frequently and

sees how the students respond. I would use the method of TTLP again, especially when designing lessons around big ideas of the course curriculum.

In the future I hope to refine my ability to choose prompts that generate controversy among students and thus spur discussions. I have enjoyed seeing mathematical discussions emerge and blossom in my classroom through this project. I learned that discussions do indeed occur in a classroom setting, with lots of curriculum to cover, through careful planning and prompt selection. Given the time and opportunity, students are willing to share their ideas with others in the classroom and will learn to listen to each other. It is most rewarding to create an open and communicative atmosphere where students feel comfortable sharing their ideas and questioning one another and their instructor. The application of the math-talk classroom will benefit the students through the exchange of ideas with each other and with their instructors.

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Appendices

Appendix A

Levels of the Math-Talk Learning Community: Action Trajectories for Teacher and Student (Hufferd-Ackles, Fuson, & Sherin, 2009)

Overview of Shift over Levels 0-3: The classroom community grows to support students acting in central or leading roles and shifts from a focus on answers to a focus on mathematical thinking.			
A. **Questioning**	B. **Explaining mathematical thinking**	C. **Source of mathematical ideas**	D. **Responsibility for learning**
Shift from teacher as questioner to students and teacher as questioners.	Students increasingly explain and articulate their math ideas.	Shift from teacher as the source of all math ideas to students' ideas also influencing direction of lesson.	Students increasingly take responsibility for learning and evaluation of others and self. Math sense becomes the criterion for evaluation.
Level 0: Traditional teacher-directed classroom with brief answer responses from students.			
A. **Questioning**	B. **Explaining mathematical thinking**	C. **Source of mathematical ideas**	D. **Responsibility for learning**
Teacher is the only questioner. Short frequent questions function to keep students listening and paying attention to the teacher. Students give short answers and respond to the teacher only. No student-to-student math talk.	*No or minimal teacher elicitation of student thinking, strategies, or explanations; teacher expects answer-focused responses. Teacher may tell answers.* No student thinking or strategy-focused explanation of work. Only answers are given.	*Teacher is physically at the board, usually chalk in hand, telling and showing students how to do math.* Students respond to math presented by the teacher. They do not offer their own math ideas.	*Teacher repeats student responses (originally directed to her) for the class. Teacher responds to students' answers by verifying the correct answer of showing the correct method.* Students are passive listeners; they attempt to imitate the teacher and do not take responsibility for the learning of their peers or themselves.

Level 1: Teacher beginning to pursue student mathematical thinking. Teacher plays central role in the math-talk community.

A. **Questioning**	B. **Explaining mathematical thinking**	C. **Source of mathematical ideas**	D. **Responsibility for learning**
Teacher questions begin to focus on student thinking and focus less on answers. Teacher begins to ask follow-up questions about student methods and answers. Teacher is still the only questioner. As a student answers a question, other students listen passively or wait for their turn.	*Teacher probes student thinking somewhat. One or two strategies may be elicited. Teacher may fill in explanations herself.* Students give information about their math thinking usually as it is probed by the teacher (minimal volunteering of thoughts). They provide *brief descriptions* of their thinking.	*Teacher is still the main source of ideas, though she elicits some student ideas. Teacher does some probing to access student ideas.* Some student ideas are raised in discussions, but are not explored.	*Teacher begins to set up structures to facilitate students listening to and helping other students. The teacher alone gives feedback.* Students become more engaged by repeating what other students say or by helping another student at the teacher's request. This helping mostly involves students showing how *they* solved a problem.

Level 2: Teacher modeling and helping students build new roles. Some co-teaching and co-learning begins as student-to-student talk increases. Teacher physically begins to move to side or back of the room.

A. **Questioning**	B. **Explaining mathematical thinking**	C. **Source of mathematical ideas**	D. **Responsibility for learning**
Teacher continues to ask probing questions and also asks more open questions. She also facilitates student-to-student talk, e.g., by asking students to be prepared to ask questions about other students' work. Students ask questions of one another's work on the board, often at the prompting of the teacher. Students list to one another so they do not repeat questions.	*Teacher probes more deeply to learn about student thinking and supports detailed descriptions from students. Teacher open to and elicits multiple strategies.* Students usually give information as it is probed by the teacher with some volunteering of thoughts. They begin to stake a position and articulate more information in response to probes. They explain steps in their thinking by proving *fuller descriptions* and *begin to defend* their answers and methods. Other students listen supportively.	*Teacher follows up on explanations and builds on them by asking students to compare and contrast them. Teacher is comfortable using student errors as opportunities for learning.* Students exhibit confidence about their ideas and share their own thinking and strategies even if they are different from others. Student ideas sometimes guide the direction of the math lesson.	*Teacher encourages student responsibility for understanding the mathematical ideas of others. Teacher asks other students questions about student work and whether they agree or disagree and why.* Students begin to listen to understand one another. When the teacher requests, they explain other students' ideas in their own words. Helping involves clarifying *other* students' ideas for themselves and others. Students imitate and model teacher's probing in pair work and in whole-class discussions.

Level 3: Teacher as co-teacher and co-learner. Teacher monitors all that occurs, still fully engaged. Teacher is ready to assist, but now in more peripheral and monitoring role (coach and assister).

A. **Questioning**	B. **Explaining mathematical thinking**	C. **Source of mathematical ideas**	D. **Responsibility for learning**
Teacher expects students to ask one another questions about their work. The teacher's questions still may guide the discourse. Student-to-student talk is student-initiated, not dependent on the teacher. Students ask questions and listen to responses. Many questions are "Why?" questions that require justification from the person answering. Students repeat their own or other's questions until satisfied with answers.	*Teacher follows along closely to student descriptions of their thinking, encouraging students to make their explanations more complete; may ask probing questions to make explanations more complete. Teacher simulates students to think more deeply about strategies.* Students describe more complete strategies; they *defend* and *justify* their answers with little prompting from the teacher. Students realize that they will be asked questions from other students when they finish, so they are motivated and careful to be thorough. Other students support with active listening.	*Teacher allows for interruptions from students during her explanations; she lets students explain and "own" new strategies. (Teacher is still engaged and deciding what is important to continue exploring.) Teacher uses student ideas and methods as the basis for lessons or miniextensions.* Students interject their ideas as the teacher or other students are teaching, confident that their ideas are valued. Students spontaneously compare and contrast and build on ideas. Student ideas form part of the content of many math lessons.	*The teacher expects students to be responsible for co-evaluation of everyone's work and thinking. She supports students as they help one another sort out misconceptions. She helps and/or follows up when needed.* Students listen to understand, then initiate clarifying other students' work and ideas for themselves and for others during whole-class discussions as well as in small group and pair work. Students assist each other in understanding and correcting errors.

Appendix B

TTLP – Thinking Through a Lesson Protocol (Slightly modified from Smith, Bill, & Hughes, 2008)

Part 1 – Selecting and setting up the mathematical task.

1. Ask, what is the mathematical goal for the lesson? Using the selected task, discuss what you are trying to accomplish through the use of the task. CHALLENGE: Be clear about what mathematical ideas the students are to learn and understand from their work on the task, not just what they will do.

    a. In what ways does the task build on students' previous knowledge, life experiences, and culture?

    b. What definitions, concepts, or ideas do students need to know to begin work on the task?

    c. What questions will you ask to help students access their prior knowledge and relevant life and cultural experiences?

2. Select a task that is presented in such a way that the solution path is not predictable or explicitly suggested.

3. Identify all the ways that the task can be solved. Consider both correct and incorrect approaches that students are likely to use. Identify a subset of solution methods that would be useful in reaching the mathematical goals.

4. Consider the challenges for struggling students or English Language Learners and how you will address those challenges.

5. Decide on the expectations for students as they work on and complete the task.

    a. What resources or tools will students have to use in their work that will give them entry into, and help them reason through, the task.

    b. How will the students work – independently, in small groups, or in pairs – to explore the task? How long will they work individually or in small groups or pairs? Will students be partnered in a specific way? If so, in what way?

    c. How will students record and report their work?

6. How will you introduce all students to the activity so as to provide access to all students while maintaining the cognitive demands of the task? How will you ensure that students understand the context of the problem? What will you hear that lets you know students understand what the task is asking them to do?

Part 2 – Supporting students' exploration of the task

1. Create questions to ask students that will help them focus on the mathematical ideas that are at the heart of the lesson as they explore the task. Ask questions that

    a. Help students get started or make progress on the task.

    b. Focus student thinking on the key mathematical ideas in the task.

    c. Clarify what the student has done and what the student understands. (Use the possible solution methods to help with this.)

    d. Help students advance toward the mathematical goals of the lesson.

    e. Encourage all students to share their thinking with others or to assess their understanding of their peers ideas.

    f. While exploring the possible solution paths, develop 'what if' questions

    g. What are the misconceptions

    Once you have a clear sense of how the student is thinking about the task, you are better positioned to ask questions that will advance his or her understanding and help the student build a sound argument based on the mathematical work.

2. Consider what you will do to ensure that students remain engaged in the task.

    a. What assistance will you give or what questions will you ask a student (or group) who becomes quickly frustrated and requests more direction and guidance in solving the task?

    b. What will you do if a student (or group) finishes the task almost immediately? How will you extend the task so as to provide additional challenge?

    c. What will you do if a student (or group) focuses on non-mathematical aspects of the activity (e.g., spends most of his or her (or their) time making a poster of their work)?

Part 3 – Sharing and discussing the task.

1. Decide which solution paths you want to have shared during the class discussion. Which order?

2. In what ways will the order in which solutions are presented help develop students' understanding of the mathematical ideas that are the focus of your lesson?

3. What specific questions will you ask so that students will –

    a. Make sense of the mathematical ideas you want them to learn?

   b. Expand on, debate, and question the solutions being shared?
   c. Make connections among the different strategies that are presented?
   d. Look for patterns?
   e. Begin to form generalizations?

4. How will you ensure that, over time, each student has the opportunity to share his or her thinking and reasoning with their peers?
5. What will you see or hear that lets you know that all students in the class understand the mathematical ideas that you intended for them to learn?
6. What will you do tomorrow that will build on this lesson?

Appendix C

Calculus II Mathematical Goals (Course Objectives)
[Chapters refer to the course textbook (Hass, Weir, & Thomas, 2007)]

Chapter 6 – Applications of Definite Integrals

A. Students will be able to determine if a given equation is a solution to a differential equation.

B. Students will be able to find the general solution or a particular solution (given initial conditions) of a separable first-order differential equation.

C. Students will be able to find the work required to

   i. stretch or compress a spring x length units from its natural (or unstressed) length.

   ii. lift an object (i.e. leaking bucket, sandbag, rope).

   iii. pump all or part of the liquid from a container.

D. Students will be able to locate the center of mass of a thin flat plate of material.

Chapter 7 – Techniques of Integration

A. Students will be able to use each of the following integration techniques:

   i. Integration by Parts

   ii. use Trigonometric Identities to rewrite the integrand

   iii. Trigonometric Substitution

   iv. Partial Fractions

   v. Referencing the Table of Integrals

B. Students will be able to determine which of the integration techniques listed above will be most helpful in solving an integral.

C. Student will be able to approximate the value of an integral using

   i. The Trapezoidal Rule

   ii. Simpson's Rule

D. Students will be able to determine an upper bound for the magnitude of the error obtained by using The Trapezoidal Rule or Simpson's Rule. Students will be able to determine the

minimum number of subintervals needed to approximate the integrals within a given error of magnitude.

E. Students will be able to evaluate improper integrals. Students will be able to determine the convergence or divergence of improper integrals.

F. Students will be able to use the Direct Comparison Test or Limit Comparison test to help determine the convergence or divergence of improper integrals.

Chapter 8 – Infinite Sequences and Series

A. Students will be able to determine if an infinite sequence converges or diverges and give examples of such sequences.

B. Students will be able to distinguish between an infinite sequence and series.

C. Students will be able to determine if an infinite series converges or diverges and give examples of such series.

D. Students will be able to determine if / when an infinite sequence converges or diverges and give examples of such sequences.

E. Students will be able to distinguish between an infinite sequence and series.

F. Students will be able to determine if / when an infinite series converges or diverges and give examples of such series.

G. Students will be able to identify a power series and determine when it converges.

H. Students will be able to generate a Taylor series and determine when it converges.

I. Students will understand that Taylor polynomials give polynomial approximations of functions.

Chapter 9 – Polar Coordinates and Conics

A. Students will be able to relate polar coordinates to Cartesian coordinates.

B. Students will be able to find the area of a polar region.

C. Students will be able to find the length of a polar curve.

D. Students will be able to identify conic sections (parabola, ellipse, and hyperbola) and find the Cartesian standard-form equations of each.

E. Students will be able to transform standard-form conic section equations.

F. Students will understand eccentricity of a conic section.

G. Students will be able to find the standard equation conic sections in polar coordinates.

Chapter 10 – Vectors and the Geometry of Space

A. Students will be able to identify distinguishing characteristics of vectors (i.e., magnitude, length).

B. Students will be able to perform vector operations (addition, scalar multiplication).

C. Students will be able to define and find the dot product of two vectors and describe the geometric interpretation of the dot product.

D. Students will be able to define and find the cross product of two vectors and describe the geometric interpretation of the cross product.

E. Students will be able to determine equations for lines, line segments, planes, and spheres in space.

Chapter 11 – Vector-Valued Functions and Motion in Space

A. Students will be able to identify a vector valued function and find limits and derivatives of vector valued functions.

B. Students will be able to find velocity and acceleration vectors, speed and direction of motion of vector-valued functions.

C. Students will be able to integrate vector valued functions.

D. Students will be able to find the arc length of a space curve.

E. Students will be able to identify the unit tangent vector.

F. Students will be able to determine the curvature of a smooth space curve.

G. Student will be able to find the principal unit normal vector.

H. Students will be able to find the binormal vector and the tangential and normal scalar components of acceleration.