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EDUCATION ARTICLE

Carrying BioMath Education in a Leaky Bucket

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Abstract In this paper, we describe a project-based mathematical lab implemented in our Applied Mathematics in Biology course. The Leaky Bucket Lab allows students to parameterize and test Torricelli's law and develop and compare their own alternative models to describe the dynamics of water draining from perforated containers. In the context of this lab students build facility in a variety of applied biomathematical tools and gain confidence in applying these tools in data-driven environments. We survey analytic approaches developed by students to illustrate the creativity this encourages as well as prepare other instructors to scaffold the student learning experience. Pedagogical results based on classroom videography support the notion that the Biology-Applied Math Instructional Model, the teaching framework encompassing the lab, is effective in encouraging and maintaining high-level cognition among students. Research-based pedagogical approaches that support the lab are discussed.

Keywords Project-based education · Pedagogy · Torricelli's law · Leaky bucket

1 Introduction

Mathematical modeling is at the heart of interdisciplinary work in mathematical biology. Unlike engineering and physics, in which governing equations are generally well-established and their adaptation to particular circumstances, simplification and integration is what requires mathematical insight, in most biological situations it is not at all clear to what model equations mathematics should be applied. Modeling,

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model evaluation and model competition in an arena determined by available data are where the rubber meets the road on the interface between mathematics and biology (Hilborn and Mangel 1997). Sadly, the educational process at almost all levels does not adequately address the need for students to develop modeling skills, either on the quantitative or biological sides of the curriculum.

To prepare future biologists, interdisciplinary laboratory work should be part of students' educational experiences. Among recommendations in the National Research Council's report, BIO 2010 (2003), are the following:

- Laboratory courses should be as interdisciplinary as possible, since laboratory experiments confront students with real-world observations that do not separate well into conventional disciplines.
- Efforts must be made on individual campuses and nationally to provide faculty the time necessary to refine their own understanding of how the integrative relationships of biology, mathematics, and the physical sciences can be best melded into either existing courses or new courses in the particular areas of science in which they teach.

In the mathematical world, the explosion of new theory and methods in computational and applied mathematics has resulted in similar calls to integrate modeling and applications in the mathematics curriculum at all levels. A recent study of modeling in the science education literature shows a progression in learning modeling that we can expect of our students (Schwartz et al. 2009). By the 1990s there was broad general agreement that modeling had an important role in the learning of mathematics for the following purposes (quoting Usiskin 1991):

- to foster creative and problem-solving attitudes, activities, and competencies;
- to generate a critical potential towards the use and misuse of mathematics in applied contexts;
- to provide the opportunity for students to practice applying mathematics that they will need as individuals, citizens, or professionals;
- to contribute to a balanced picture of mathematics;
- to assist in acquiring and understanding mathematical concepts.

This has influenced instruction at colleges through the incorporation of more application and interdisciplinary projects and with the addition of writing experiences to foster student development of critical thinking skills (Bressoud 2001). Small-group learning experiences have also been widely studied and incorporated into science and mathematics instruction (Springer et al. 1999).

However, integration of modeling projects into collegiate coursework has a long way to go. While projects and problem-based learning have become more prevalent (Hmelo-Silver 2004; Smith et al. 2005; DebBurman 2002; Barak and Dori 2004), encouraging students to work out not only the solution but also the problem is time-consuming and raises challenges for professors and students (Chin and Chia 2005). More thorough documentation of projects would help the advancement of this style of instruction (Helle et al. 2006), which motivates this paper.

Reform requires changing not only what we teach but also how we teach it. Lecture-based instruction often fails to engage students in higher levels of cognition, but traditional collegiate math education is lecture-based. However, critiques of 'traditional' instructional strategies have not necessarily been helpful. As stated by Chazan and Ball (1999), "An exhortation to avoid telling is about what not to do. It contributes nothing toward examining what teachers should or could do. While it is intended to allow students a larger role in classroom discussions, it oversimplifies the teacher's role, leaving educators with no framework for the kinds of specific, constructive pedagogical moves that teachers might make." Thus, this paper will aim to communicate both the technical details and a specific guide for what teachers *can* do to implement the project, as opposed to what *not* to do.

Effective teachers conduct classroom discourse in a way that enables students to engage in higher order cognition and become part of the generation of their own knowledge. Quoting the National Council of Teachers of Mathematics (NCTM) professional standards (1991):

> Discourse entails fundamental issues about knowledge: What makes something true or reasonable in mathematics? How can we figure out whether or not something makes sense? That something is true because the teacher or the book says so is the basis for much traditional classroom discourse. Another view, the one put forth here, centers on mathematical reasoning and evidence as the basis for the discourse. In order for students to develop the ability to formulate problems, to explore, conjecture, and reason logically, to evaluate whether something makes sense, classroom discourse must be founded on mathematical evidence… The teacher's role is to initiate and orchestrate this kind of discourse and to use it skillfully to foster student learning.

We should expect, then, that discourse is a critical component in helping students develop mathematical modeling skills. A central part of scientific discourse is writing. Particularly on the modeling side of mathematical biology the development of writing skills is critical; successful models must be conveyed to non-mathematical audiences to become scientifically useful. From a teacher's perspective, writing strongly encourages higher cognitive function among students (Keyes 1999; Tynjälä 1998). Written discussion encourages students to consider mechanism and justification, to compare and contrast models and their behavior, to determine who the audience is and what is important to tell them. Simultaneously written discourse offers the teacher a chance to see the student's thought process, to question and comment on it in a formative way and thereby encourage student growth and development. Moreover, the self-conscious nature of writing helps students to synthesize their understanding.

Integrating these notions of discourse and writing into a pedagogical system to teach modeling in mathematical biology, we proposed the Biology/Applied Mathematics Instructional Model (BAMIM; Powell et al. 1998)—which draws from best practices research in mathematics and science education and requires students to work at high levels of cognition. BAMIM holds that the optimal mental state for a student to learn modeling and the mathematics associated with model application is precisely the mental state in which a professional applied mathematician would approach a novel biological problem. Our own experiences and research have led us to believe in the importance of engaging students in the collecting and organizing of their own data for modeling (Kohler et al. 2009; Powell et al. 1998). We have developed a class, Applied Mathematics in Biology (AMB), built around BAMIM. The

AMB class proceeds by confronting mixed educational teams of students with real-world experiments, which they design and perform. Teams develop and parameterize multiple models and ultimately convey their results in written papers approaching scientific standard.

Here we survey a specific laboratory project designed for the AMB course, the Leaky Bucket (LB) Lab. Students attempt to explain and predict the time trajectory of fluid exiting a container through a small aperture. The LB Lab requires students to comprehend and parameterize the classic Torricelli model as well as formulate an alternate model of their own to explain drainage dynamics. The full derivation is provided below, but the essential equations are the rate equation that relates fluid speed at the exit point, gravity and the height of the fluid above the exit: $u = \sqrt{2gh}$. The second equation relates the change in height (drainage) to speed and the ratio of aperture to cross-sectional area of the bucket: $\frac{dh}{dt} = -\frac{a}{A}u$. With these concepts of fluid speed and container dimensions, students experiment, collect data, justify model components mechanistically, fit parameters, display their results and predictions, analyze the accuracy of the models and synthesize their work in written reports. Although the LB project was designed for an interdisciplinary mathematical biology course, we have incorporated the lab into a broader spectrum of mathematics courses, including freshman calculus, differential equations and an advanced course in methods of applied mathematics.

Despite its apparent basis in physics, the LB lab provides fertile ground for germination of common concepts in mathematical biology, and has become a mainstay in the AMB course. Invariably, more than half of the students are not mathematicians or physicists, have either never seen a differential equation, and come to their first experience with continuous modeling in fear and awe. These students have actually found the physical context, with concrete grounding in observation of a clearly continuous process, a painless way to begin understanding differential equations. Additionally, many modern biological applications require some knowledge of fluid mechanics. Examples include individual-based flight or swimming models (e.g. Lemasson et al. 2008; Haefner and Bowen 2002; Powell and Engelhardt 2000), microbes in a chemostat (Smith and Waltman 1995), nutrient cycling dynamics in mountain lakes (Wurtsbaugh et al. 2005; Rigley 2009), mathematical physiology (Keener and Sneyd 1998), to name only a few. We have placed the LB lab directly following experiences in discrete modeling, and before more obviously biological labs (growth of yeast, maturation of larvae, diffusion of shrimp), where it serves to pave the way to continuous models. Finally, the LB lab provides a good introduction to many mathematical tools that students will need for other biological applications.

The LB project shares with many biological systems the need to consider continuous time, spatial heterogeneity, real number measurements, statistical error and lack of observer control over the rules that the system obeys. Additional intersections include (but are not limited to) the virtue of multiple working hypotheses, techniques for parameterization and validation, interaction between uncertainty and model selection, trade-off between model complexity and process understanding and the ambivalent role of variability. From the standpoint of specific quantitative tools, students learn modeling techniques, (re-)discover some portion of ODE and simple calculus, learn a variety of linear and nonlinear fitting techniques, and develop an appreciation for and skill with dimensional analysis. Most importantly, students cross the threshold of constructing and criticizing models on their own; the LB exercise gets them quickly over the notion that there is a single 'correct' model. The fact that the LB lab is not directly biological is actually an equalizer and facilitator in the AMB class. Neither mathematics nor biology students have an advantage in terms of their background; students from different backgrounds (assigned non-randomly to lab groups to increase diversity) learn to appreciate and use different points of view as they attack the problem in teams.

In this paper we will survey the materials used to launch the LB lab and common analytic approaches generated by students. The aim is to give instructors a background with which to support or 'scaffold' student discovery and learning. We will also discuss teaching techniques used to manage the LB lab, including research-based pedagogical strategies for maintaining a high level of cognition as well as effective approaches to sustaining the BAMIM environment. Pedagogical results, both observed dialogues and distribution of thought-level questions as the lab progress, provide evidence that the pedagogical framework is both appropriate and effective. We hope to provide readers the wherewithal to incorporate the LB lab into a BAMIM experience in their own mathematics, statistics or biology classes.

2 Launching the Leaky Bucket Lab

We have placed the LB lab second in the class, following the Disease Game Lab (Powell et al. 1998) but well before our lab investigating diffusion of brine shrimp (Kohler et al. 2009). From a mathematical and modeling perspective the Disease Game serves to review/introduce discrete modeling, steady states and stability, probability and mass action, and basic computational concepts. To pave the way for the rest of the class students need a similar review of/introduction to differential and compartment models, simple numerical and analytic solution of differential equations, regression, calculation of correlation coefficients and dimensional analysis. Consequently a brief lecture and derivation of Torricelli's law (see below) is used to set up the LB lab, contemporaneous with a computational lab covering regression and numerical solution of differential equations.

Introducing the Torricelli model to launch the lab not only provides opportunities to review and introduce necessary math and philosophy of science but also 'hooks' students. The mathematical and physical development tends to grip them (and also exemplifies mechanistic model justification). When they make observations to test the model in a 'normal' scientific way they expect the model to perform much better than it does, and often assume that they have done something wrong. Students then seek to perfect their data collection, aiming not to falsify but to validate the Torricelli model. This is the normal fate of scientific labs in undergraduate experience, and the Achilles heel of 'normal' science (Kuhn 1962). However, the general failure to validate the Torricelli model leaves students in a fertile position with respect to class goals. Clearly there is room for improvement in modeling, and the students are in complete control of the experiment. At this point they can see the point of alternate models (multiple working hypotheses!), are anxious to do better than Torricelli, and are ripe to develop ways to compare and evaluate multiple working models in an arena mediated by their own data. We stress the importance and relevance of strong validation (mimicking Platt's 1964 "Strong Inference"). Students create and parameterize several models on data from containers of their own creation, then test them on independent containers to distinguish among competing models.

2.1 Student Expectations and Lab Agenda

The general objectives for students are:

1. Accurately predict the rate of drainage of fluid from a leaky bucket, given knowledge of the bucket's geometry and the size/shape of drainage aperture.
2. Create two models (one of which may be the Torricelli model or a close relative) which will predict the emptying time of a leaky bucket *which can only be measured, not tested* in advance. The models must be "significantly different" from each other.
3. Calibrate models (i.e., estimate parameters) using data collected from buckets teams construct and test.
4. Develop protocol by which team models can be applied to similar, but independent, containers which can only be measured before validation begins.

We ask students to generate a brief paper (with Introduction, Methods, Results, Discussion and Conclusion sections). Specific **tasks** they must complete (and elements to appear in the report) are:

1. Define and justify the models (**Methods**);
2. Define the experimental protocol used to estimate the parameters (**Methods**);
3. Perform measurements and estimate the parameters (**Results**);
4. Verify that the models perform "acceptably well" (as justified and defined by the modelers) on the original containers (**Results**);
5. Apply the models (with parameters determined by calibration and measurement of validation bucket geometry) to the new containers supplied for strong validation (**Results**); and
6. Answer the questions: "Which model did best? Why?" (**Discussion and Conclusion**).

In the AMB class, students submit rough drafts upon which we comment but score only for participation. This allows us to initiate a written scientific and mathematical discourse with students. A (loose) script for how the LB lab proceeds as follows:

(**Lecture**) Introduction to LB Lab, initial data collection [15 minutes]
(**Lecture**) Derivation of Torricelli model [20 minutes]
(**Data Collection/Model Construction**) Group Time: Design and creation of initial buckets and protocol, drainage observations and initial comparison with Torricelli predictions [60 minutes]
(**Data Collection/Model Construction**) Class Discussion: Groups sketch data, comparison w/Torricelli model, share ideas on what is wrong [20 minutes]
(**Data Collection/Model Construction**) Group Time: Discussion and development of alternate models. Collection of additional calibration data [120 minutes]

(**Model Presentation**) Class Discussion: Groups present alternate models, calibration strategy, scheme for addressing validation [45 minutes]
(**Validation**) Validation Buckets Revealed: Groups measure relevant geometry from new buckets [15 minutes]
(**Validation**) Validation Challenge: Each group does one or two validation runs, contributes to public data pool [15 minutes]

We have divided these activities into the four categories (**in parentheses**) for purposes of question assessment (see Pedagogical Results below). This agenda is covered over a few lab/lecture days, with the expectation that student groups should be meeting, discussing their models, parameterizing and comparing with data. In classes which have less scheduling freedom, many details can be streamlined; e.g. buckets with holes and benchmarks can simply be provided to students, or the data collection done as a demo in front of the class. In classes where the point is more that applications exist (e.g. of non-polynomial integration in calculus, or separation of variables in ODEs) the class can be provided with one of the models discussed below and allowed to work with it and class-collected data. Working with tangible data the students have collected will enliven any associated mathematical material, and the instructor can expand upon the investment in data collection through the rest of the class (e.g. working through more advanced models or developing simple numerical integration approaches later in an ODE class). In the AMB class evaluation is primarily via written report; a draft report on the LB lab is due 1–2 weeks after the final validation data is taken and distributed among groups. Now we turn to some of the technical details surrounding presentation of the Torricelli model and scaffolding possible student approaches to alternative models.

2.2 Lab Materials and Methods

The following materials are needed (for each group):

- 1–2 quart translucent or clear plastic jugs such as those containing milk, soda or juice for use as leaky buckets.
- Scalpels or X-Acto knives for cutting apertures and removing burrs (a drill with bits is useful for circular holes, but not necessary).
- Waterproof marker.
- Stopwatch.
- Duct tape (just on general principles).
- Ruler with at least millimeter scale.
- Graduated cylinders or kitchen measuring cups for measuring metric volumes.
- Access to tap water.
- Plastic dish-washing tub to capture drained water if a large sink is not available.

The experience is most effective when students develop their own laboratory procedures, based on the need to evaluate the Torricelli model and develop/parameterize their own models. Students must plan to measure all parameters needed in their models, either directly or using fitting techniques. This may involve different levels of ingenuity, flexibility, and special equipment from the instructors, depending on the models used. Containers may have complicated shapes (as with plastic milk jugs), so there are non-trivial issues (e.g., where to place holes and the initial water level) that students must struggle with. It is helpful for them to keep in mind that they will validate with a container of unknown geometry, an unknown initial volume of fluid and unknown shape of aperture. If validation buckets have more than one hole, holes will be cut at the same vertical level. Holes will not be larger than 1 cm^2 or so small as to preclude a free-flowing stream, but otherwise shape, size, and number of holes are unconstrained. All groups evaluate all the validation containers, so it is possible to create friendly competition among groups according to the relative merits of their alternative models.

When time is not available for groups of students to develop and refine their own procedures, or if instructors wish to offer a starting point to get things rolling, we provide the following procedure (based on using a 1/2-gallon milk jug):

1. Divide into groups of 3–4. Each group will need at least one person to manage the stopwatch (*Timer*), spot fluid levels (*Spotter*) and record data (*Recorder*).
2. Set up the bucket. Where the jug begins to have regular horizontal cross sections (2–4 cm above base for a standard US plastic half-gallon milk jug), cut a horizontal slit 1–2 mm tall and 1–2 cm wide, being careful that the top and bottom of the slit are parallel to the base of the jug. Every centimeter vertically from the *bottom* of the slit make a horizontal mark, up to between 10 and 15 cm above the bottom of the slit (depending on how far the jug maintains a relatively consistent cross section).
3. Measure the bucket. At a minimum, students need to estimate the cross-sectional area of the bucket and the area of the aperture. Students may wish to measure the cross section volumetrically, adding a known volume to the bucket and dividing by a measured vertical height.
4. Observe drainage trajectories.
   (a) Fill the bucket to the desired initial height (12 or 13 cm are used in this paper), as measured by the bottom of the fluid meniscus. The aperture will need to be covered either with a piece of duct tape or a convenient finger. If using a finger be careful not to press hard enough to deform the container.
   (b) Position the bucket so that it can drain into a sink or basin.
   (c) Spotter removes tape and says "Start!" Timer starts stopwatch.
   (d) As fluid passes each vertical mark, Spotter calls "Mark!" and Timer gives the time of the split, which Recorder records next to the appropriate vertical level.
   (e) Continue until the bottom of the fluid meniscus is level with the top of the slit. Timer records final emptying time. For a 1/2-gallon container with aperture of 0.4 cm^2 filled to 12 cm above the slit, this will be between 30 and 60 seconds.
5. Repeat the observation sequence at least three times for the same initial height of fluid to assess variability.

One of the biggest issues is determining when to stop; depending on the size and shape of both bucket and aperture the flow may transition from a free stream to an attached dribble to periodic drips. Ideally students should discover and address this on their own; if time is tight, instructors can experiment with the bucket in advance to determine a stopping rule for the observation sequence.

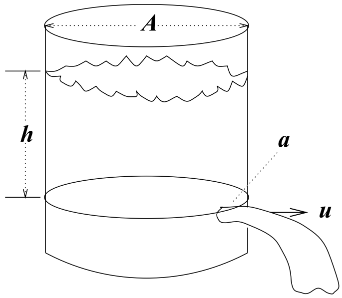

Fig. 1 *A Leaky Bucket.* The leaky bucket has cross-sectional area A. The height of the fluid above the hole is h, a is the cross-sectional area of the hole, and u is the velocity of the fluid passing through the hole

2.3 Torricelli's Model

2.3.1 Derivation

Galileo's student Evangelista Torricelli deduced a square root relationship between the speed at which fluid flows from a vessel and the height of fluid above the hole, which was later generalized into Bernoulli's Principle. From a modern perspective, the Torricelli model for draining fluid is based on two physical concepts:

1. The Bernoulli relationship between pressure (P), density (ρ), and speed (u) for fluid along a streamline:

$$\Delta P = \frac{1}{2}\rho u^2;$$

2. The hydrostatic relationship between changes in pressure over the height of a fluid column and gravity (acceleration g), fluid density ρ and the height h of the column above the hole (see Fig. 1)

$$\Delta P = \rho g h.$$

Equating these two gives a relationship between height of the fluid above the hole and speed at the aperture:

$$u = \sqrt{2gh}. \tag{1}$$

Since the volume of fluid lost from the bucket must equal the flux of fluid through the bucket's hole (with area a),

$$\frac{dV}{dt} = -ua,$$

and using the fact that (for a bucket with regular sides and constant cross section in height) the volumes above and below the hole (V and V_0) are related, $V = Ah + V_0$, we get

$$-ua = \frac{dV}{dt} = \frac{d}{dt}[Ah + V_0] = A\frac{dh}{dt}.$$

Substituting into the quadratic speed/height relationship, (1), gives a differential equation:

$$\frac{dh}{dt} = -\frac{a\sqrt{2g}}{A}\sqrt{h}, \quad h(t=0) = h_0. \tag{2}$$

2.3.2 Solution

To solve Eq. (2), separate variables,

$$\frac{dh}{\sqrt{h}} = -\frac{a\sqrt{2g}}{A}\,dt,$$

and integrate. Then:

$$\begin{aligned}
\int_{h_0}^{h} h^{-\frac{1}{2}}\,dh &= -\int_0^t \frac{a\sqrt{2g}}{A}\,dt,\\
\frac{1}{\frac{1}{2}}\left(h^{\frac{1}{2}}\right)\Big|_{h_0}^{h} &= -\frac{a\sqrt{2g}}{A}(t-0),\\
\sqrt{h}-\sqrt{h_0} &= -\frac{a\sqrt{2g}}{2A}t,\\
h &= \left(\sqrt{h_0}-\frac{a\sqrt{2g}}{2A}t\right)^2.
\end{aligned}$$

One can calculate the predicted time for the bucket to empty by setting $h = 0$, which gives

$$\sqrt{h_0} - \frac{a\sqrt{2g}}{2A}t = 0,$$

or

$$t_{\text{empty}} = \frac{2A\sqrt{h_0}}{a\sqrt{2g}} = \frac{A\sqrt{2h_0}}{a\sqrt{g}}. \tag{3}$$

While Torricelli's model is elegant and mechanistic, it does a poor job of predicting drainage trajectories (see Fig. 2). From a pedagogical standpoint this is a nice setup for student creativity; students expect such an elegant model to do a great job predicting emptying time. Students often spend a fair amount of time reviewing their experimental procedure, certain that the model should do better. But it will not, and the lab proceeds with students discussing alternate models and collecting data to determine what variables are important.

3 Mathematical Analysis and Alternative Approaches

In order to facilitate this lab, instructors must anticipate student responses; some idea of the variety of approaches students conceive is provided to help instructors scaffold the student learning experience. We provide a description of the data and analyze a variety of alternate model approaches generated by former students. This will help instructors forecast resource requirements and also indicate the breadth of mathematical discoveries made by students in the context of the lab.

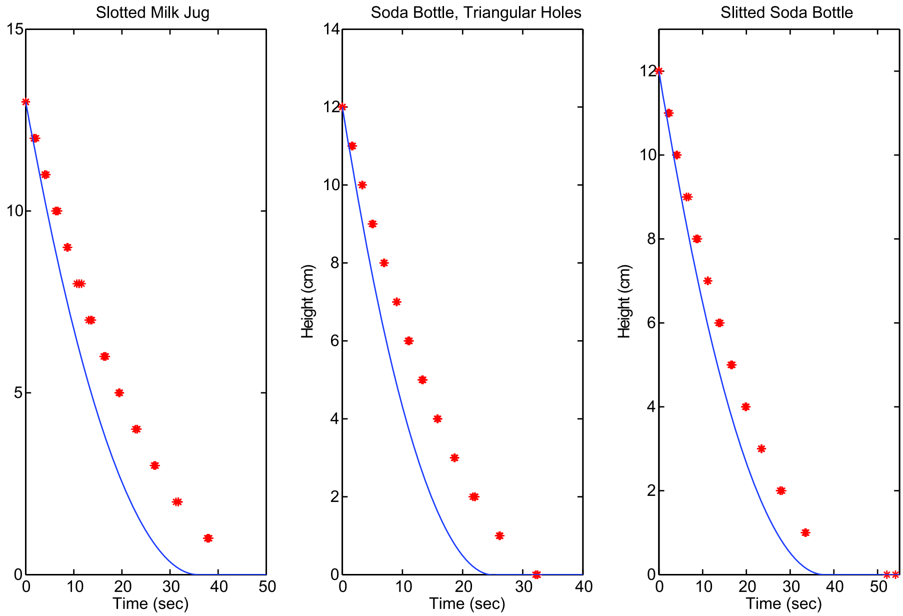

Fig. 2 Calibration data (*) for three buckets: a half-gallon milk jug with a 0.39 cm^2 slit (four replicates, *left*), a two-liter soda bottle with two triangular apertures of net area 0.575 cm^2 (four replicates, *center*) and a second soda bottle with a rectangular slit of area 0.38 cm^2 (three replicates, *right*). Predictions from Torricelli's law are plotted as *solid curves* for each bucket. Numerical data is presented in Table 1

3.1 Geometry and Data

Our 'buckets' have traditionally been formed from empty half-gallon plastic milk jugs or two-liter soda bottles, which are common, recyclable, robust, see-through, simple in shape and easy to cut. We provide a selection of rulers, permanent markers, razor knives, simple augers, rasps and hand drills with which students manufacture holes. Occasionally we save 'buckets' from previous classes as examples; these can be used for whole-class demos so that lab groups have some initial data to compare with the Torricelli model. Typical apertures have linear dimensions on the order of 2–5 mm and may be circular, rectangular, triangular and occasionally star-shaped. Holes are located 2–4 cm above the bottom (generally at the lowest part of the clearly vertical portion of the jug); water levels are set initially 10–15 m above the aperture.

For purposes of illustration we pool observations made by students in Spring, 2010 and test various models against validation data generated in 2006. Calibration was done using three separate containers: two cylindrical soda containers with diameters of approximately 10.8 cm (students estimated areas of 91.95 and 92 cm^2) and one half-gallon milk container of vaguely octagonal cross section (with area estimated at 86 cm^2). The milk jug had a single rectangular slit for drainage, of height 2 mm and width 1.95 cm, and was filled 13 cm above hole centerline. One soda jug had a rectangular hole of height 2 mm and width 1.9 cm while the other had two triangular holes at the same horizontal level, one with area 0.315 cm^2 and the other with area 0.26 cm^2 (giving total aperture area 0.575 cm^2). The two soda containers were filled

Table 1 Calibration data from three buckets with differing apertures, collected by the AMB 2010 class. The 'buckets' are two two-liter soda bottles and a 1/2-gallon milk jug. One milk jug and one soda bottle were drained through a rectangular slit (with areas, a, indicated above) while the remaining soda bottle was drained through two triangular holes (bases horizontal to ground level) with total area $a = 0.575$. Student estimates for the cross-sectional area, A, of the container are also given above

h (cm)	Slitted milk jug $A = 86$, $a = 0.39$ cm^2				Soda jug, $\Delta + \Delta$ $A = 92$, $a = 0.575$ cm^2				Slitted soda jug $A = 91.95$, $a = 0.38$ cm^2		
13	0	0	0	0							
12	1.72	2.17	1.87	1.72	0	0	0	0	0	0	0
11	3.99	4.31	3.99	3.99	1.62	1.55	1.61	1.74	2.17	2.43	2.1
10	6.7	6.4	6.17	6.7	3.37	3.27	3.36	3.3	3.98	4.21	4.13
9	8.63	8.8	8.58	8.63	4.93	5.08	5.12	5.12	6.28	6.78	6.32
8	11.1	11.6	10.63	11.1	7.02	6.89	6.99	6.93	8.7	8.91	8.5
7	13.54	13.75	13.09	13.54	9.12	8.99	9.05	9.05	11.12	11.26	11.14
6	16.47	16.64	16.2	16.47	11.18	10.92	11.08	11.08	13.65	14.05	13.76
5	19.52	19.56	19.33	19.52	13.43	13.33	13.42	13.21	16.45	16.78	16.59
4	23.19	23.12	22.78	23.19	15.93	15.8	15.83	15.77	19.74	20.00	19.82
3	26.97	26.96	26.67	26.97	18.74	18.58	18.64	18.68	23.38	23.52	23.44
2	31.73	31.68	31.24	31.73	22.08	21.96	22.11	21.68	27.7	28.17	27.82
1	38.11	37.73	37.85	38.11	26.21	26.08	26.17	26.24	33.59	33.66	33.39
0	92.2	81.84	62.14	92.2	32.12	32.39	32.36	32.49	54.08	55.83	52.12

Table 2 Validation data collected in 2006 using a 1/2-gallon milk jug with circular aperture of diameter 0.556 cm filled to 14 cm. Unlike the 2010 calibration data, students in 2006 chose to measure heights at specified times instead of the other way around

t (s)	0	10	20	30	40	50	60	70	80	90
h (cm)	14	10.6	8.5	6.7	4.9	3.5	2.2	1.0	0.5	0.1

to 12 cm above aperture centerline. Data for all replicates are presented in Table 1. Validation data, from a 1/2-gallon milk jug with circular aperture of diameter 0.556 cm, filled to 14 cm, appears in Table 2.

Given freedom, students generally start by measuring emptying time directly with a stopwatch—making the implicit assumption that the Torricelli model will provide a good approximation to the drainage time (which it does only rarely). They then repeat the experiment with increasing desperation, trying to remove sources of perceived variability. Issues that often come up include: how to start the experiment (plugging the aperture with a finger deforms the sides of the container, altering initial height of the fluid and geometry), how to measure the initial height (fluid level on the side of the container or bottom of the meniscus), when does the drainage really stop (there is often a transition from a clear jet to a trickle attached to the container exterior and occasionally a further transition to discrete drops). Students find that the concept of 'emptying time' is not clearly defined. Often students choose to call a bucket 'empty' when the meniscus comes level with the top of the hole, or when the flow from the aperture transitions to dribbles. Eventually students are convinced that the Torricelli model is fairly far off no matter what (see Fig. 2). This is the doorway to BAMIM.

After group discussion students will subdivide the height into intervals (1–2 cm) and measure the height as a function of time to investigate how the Torricelli approach deviates from observation (i.e. does variance grow progressively, is the model accurate until the flow transitions to a dribble or is there some clear trend suggesting an explanation?). Students may try different starting heights, aperture shapes and sizes, or containers (bigger milk jugs, cardboard cartons, empty cans of Red Bull) as they explore. Some groups measure velocities emerging from the aperture (using the distance the jet squirts as a function of height and deducing velocity from the parabola) or try different fluids (oil or soapy water) to test sensitivity to density, viscosity or surface tension.

Given the size of errors in measuring physical parameters, could the Torricelli model be correct but measurement errors are polluting the observations? This leads (as in the transcript of classroom discourse below) to a sensitivity analysis of emptying time (3). This begins with calculation of the partial derivatives of predicting emptying time:

$$\frac{\partial}{\partial A} t_{\text{empty}} = \frac{2\sqrt{h_0}}{a\sqrt{2g}}, \qquad \frac{\partial}{\partial a} t_{\text{empty}} = -\frac{2A\sqrt{h_0}}{a^2\sqrt{2g}}, \quad \text{and} \quad \frac{\partial}{\partial h_0} t_{\text{empty}} = 2\frac{A}{a\sqrt{gh_0}},$$

which can then be used to calculate proportional sensitivity,

$$\begin{aligned} \frac{\Delta t_{\text{empty}}}{t_{\text{empty}}} \approx \frac{dt_{\text{empty}}}{t_{\text{empty}}} &= \frac{dA\frac{2\sqrt{h_0}}{a\sqrt{2g}} - \frac{da}{a^2}\frac{2A\sqrt{h_0}}{\sqrt{2g}} + \frac{dh_0}{2\sqrt{h_0}}\frac{A}{a\sqrt{g}}}{\frac{A\sqrt{2h_0}}{a\sqrt{g}}} \\ &\approx \frac{\Delta A}{A} - \frac{\Delta a}{a} + \frac{1}{2}\frac{\Delta h_0}{h_0}. \end{aligned} \tag{4}$$

Thus a ten-percent error in measuring either a or A gives a comparable percent error in predicted emptying time, whereas a ten-percent error in observing h_0 generates five-percent error. The deviation in emptying time is 10–50 %, and most students do not feel that their measurement errors are nearly that large.

3.2 Alternate Model Approaches

As an introduction to the virtue of multiple working hypotheses students are required to describe, parameterize and test at least one alternate model. The alternate must embrace observed phenomena, be completely described and well-justified. Working in groups students come up with novel and surprising approaches. Below are surveyed some of the more common alternatives and their pros and cons as revealed by validation data.

3.2.1 Phenomenological Models

A purely phenomenological approach makes no attempt to respect underlying mechanisms, although it should reflect observed dependencies among parameters and variables (e.g. emptying time increases as aperture size decreases). Students, particularly from biological and/or statistical backgrounds, are often inclined to fit decreasing, concave functions of time to observed height trajectories. The most popular candidates are exponential models,

$$h = h_0 e^{-\lambda t}, \tag{5}$$

and power-law models,

$$h = h_0\left(1 - \frac{t}{\tau}\right)^{\beta} \quad \text{or} \quad h = h_0\left(\frac{\tau}{t+\tau}\right)^{\beta}. \tag{6}$$

In both cases some discussion is often required within groups to come up with a form that satisfies both initial conditions and qualitative behavior in time.

Technically there are two broad challenges for the students, one tactical and one strategic. The tactical problem is simply fitting the models to observations. At this point in the class we have generally introduced linear regression (see Kohler et al. 2009 for details). Fitting a polynomial model directly to observed height–time trajectories is straightforward, but the fitted models may have no predicted emptying time (often with quadratic models) or a completely unnatural emptying behavior (as with cubic models). Moreover, the extension of fitted polynomials to an unknown validation bucket is not straightforward; this is the strategic issue. Extrapolating a polynomial model to a container of unknown dimensions, starting conditions and aperture size often dissuades students from the direct polynomial approach.

The power law and exponential models have the additional tactical difficulty that the unknown parameters (β and λ) appear nonlinearly, so that direct linear regression is not possible. Many students have experience with 'logarithmic regression' (fitting the log of the observation to a line) or 'power law regression' (fitting the log of the observations to a linear function of the log of the times). Either approach is educational; since logarithmic transformations accentuate observed variability at lower fluid levels, the fits in the physical domain are unsatisfying.

This provides an excellent context in which to introduce nonlinear least squares or (equivalently) maximum likelihood estimation (MLE). If students choose to head down this path, we offer them an MLE tutorial. For MLE one assumes that differences (d_j) between the jth predicted and observed heights are distributed normally:

$$d_j = \text{Pred}_j - \text{Obs}_j \sim \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{d^2}{2\sigma^2}}. \tag{7}$$

The likelihood of the observations, given h_0, λ and σ, is the product of the various observation probabilities:

$$\begin{aligned} \mathcal{L} &= \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(\text{Pred}_j - \text{Obs}_j)^2}{2\sigma^2}\right] \\ &= \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(h_0 e^{\lambda t_j} - h_j)^2}{2\sigma^2}\right]. \end{aligned} \tag{8}$$

The problem now becomes choosing λ, σ and possibly h_0 (if students believe there is some uncertainty measuring an initial height) to make the observations as likely as

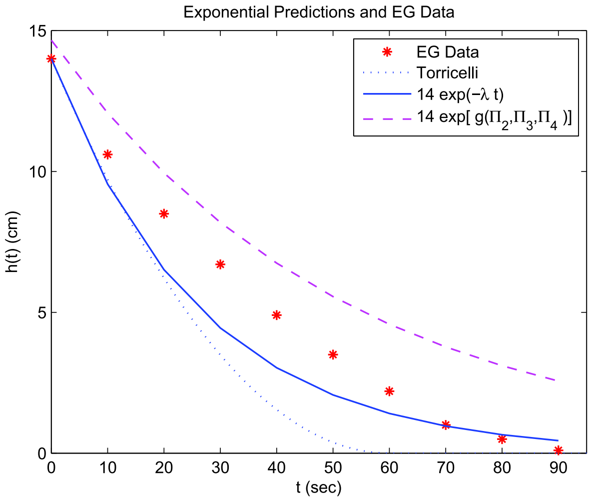

Fig. 3 Comparison of two exponential fits and validation data (*). Torricelli predictions appear for reference (*dotted curve*). The *solid curve* depicts exponential predictions generated by fitting exponentials to calibration data individually, then using linear regression to extrapolate to a and A values needed for the validation bucket. The dashed curve depicts the use of a Pi Theorem approach to generating exponential predictions for the validation data; in this case the Pi Theorem approach is vastly inferior

possible, given the exponential model. This is done by minimizing the negative log likelihood:

$$L(h_0,\lambda,\sigma) = -\log(\mathcal{L}) = \sum_{j=1}^{n}\left[\log\left(\sqrt{2\pi\sigma^2}\right) + \frac{(h_0 e^{\lambda t_j} - h_j)^2}{2\sigma^2}\right]. \tag{9}$$

Computationally can use any number of prepackaged optimization codes to minimize L; the class standard is `fminsearch` in MATLAB. Within this whole process there are numerous discovery opportunities for students, from writing computational optimization codes for advanced students to manipulating probabilities and logarithms for novices.

The strategic issue faced by all phenomenological approaches is extending predictions from a fitted, non-mechanistic model to a novel container, as required by strong validation. What students are most likely to do is to fit unknown parameters within treatments but across replicates and then fit a secondary model which extrapolates parameters across the range of feasible physical conditions. For example, using an exponential model (5) and MLE students generate a selection of λs for varying A and a (see Table 3) then fit the linear model

$$\lambda = \lambda_0 + \lambda_a a + \lambda_A A.$$

For linear regression using data in Table 3 this gives

$$\lambda = -0.0617 + 0.0768a + 0.000946A$$

and for the validation bucket $\lambda_{\text{val}} = 0.0383$, which is plotted in Fig. 3. Results from this approach can be very mixed, depending entirely upon how student experiments bracket validation parameters.

An alternate approach is to use the Buckingham Pi Theorem. Any mathematical truth about the system must be phrased as a dimensionally consistent equation, and therefore admit representation using the Pi Theorem. That is, any natural law $f(q_1, q_2, \dots, q_m) = 0$ involving m measurable dimensional quantities, q_i, can be

Table 3 Exponential decay parameters (λ) for calibration data generated using nonlinear maximum likelihood. Intercepts are set to hit initial conditions so that only the exponential decay parameter is fitted

	Slitted milk jug				Soda jug, $\Delta + \Delta$				Slitted soda jug		
a (cm^2)	0.390	0.390	0.390	0.390	0.575	0.575	0.575	0.575	0.380	0.380	0.380
A (cm^2)	86.0	86.0	86.0	86.0	92.0	92.0	92.0	92.0	91.95	91.95	91.95
λ (s^{-1})	0.0494	0.0490	0.0506	0.0494	0.0692	0.0698	0.0693	0.0695	0.0548	0.0537	0.0548

written as an equivalent law $F(\Pi_1, \Pi_2, \dots, \Pi_{m-r}) = 0$ with only $m - r$ (fewer!) dimensionless 'Pi' variables $\Pi_1, \Pi_2, \dots, \Pi_{m-r}$, where r is the rank of the associated dimensional matrix (see Logan 2006 for a complete discussion).

If required, we offer a discussion of dimensional consistency and the Pi Theorem, and then students determine which physical properties seem important. The quantities h, t, h_0, a, A and g are critical, and a brief foray into dimensional analysis (or just canceling units) gives 'natural' dimensionless quantities

$$\Pi_1 = \frac{h}{h_0}, \qquad \Pi_2 = \frac{a}{A}, \quad \text{and} \quad \Pi_3 = \sqrt{\frac{g}{h_0}}t.$$

In fact, the Torricelli solution

$$h = h_0\left(1 - \frac{\sqrt{2}}{2}\frac{a}{A}\sqrt{\frac{g}{h_0}}t\right)^2$$

can be written dimensionlessly as

$$F(\Pi_1, \Pi_2, \Pi_3) = \Pi_1 - \left(1 - \frac{\sqrt{2}}{2}\Pi_2\Pi_3\right)^2 = 0.$$

The Pi Theorem can be used to build a phenomenological model introducing effects *not* included in Torricelli's law (e.g. fluid viscosity). Keeping the above natural Π variables but introducing the kinematic viscosity, ν (with units length squared/time), a reasonable new dimensionless Π becomes

$$\Pi_4 = \frac{\nu}{a}t,$$

measuring viscous effects at the aperture. Formulated via the Pi Theorem, the truth about the leaky bucket can be written as

$$H(\Pi_1, \Pi_2, \Pi_3, \Pi_4) = 0$$

and w.l.o.g. as

$$\frac{h}{h_0} = \Pi_1 = G(\Pi_2, \Pi_3, \Pi_4) = e^{g(\Pi_2, \Pi_3, \Pi_4)},$$

using the Implicit Function Theorem. The choice of an exponential on the right-hand side is justified by $\Pi_1 >= 0$ always. Using a linear approximation for the exponent,

$$g \approx g_0 + g_2\Pi_2 + g_3\Pi_3 + g_4\Pi_4,$$

gives a rational phenomenological model for predicting fluid heights in the bucket:

$$h = h_0 \exp\left(g_0 + g_2\frac{a}{A} + g_3\sqrt{\frac{g}{h_0}}t + g_4\frac{\nu}{a}t\right). \tag{10}$$

Using MLE and the calibration data we found

$$g_0 = 0.045879, \qquad g_2 = -0.0512, \qquad g_3 = -0.0119, \qquad g_4 = 0.0193;$$

results are plotted in Fig. 3 against the Torricelli and the previous exponential models. In this case the exponential prediction generated using the Pi Theorem does not predict the validation data nearly as well as the direct exponential approach, since the calibration data, converted to Pi variables, do not encompass the validation data.

3.2.2 Semi-mechanistic Models

A semi-mechanistic model is based on a physical model but adds empirical or phenomenological elements to incorporate additional effects. The most common student correction to the Torricelli model is to include a term reflecting fluid friction at the aperture, generally assuming that the amount of fluid leaving is a fraction, α, of the volumetric flow predicted by Torricelli's law. This could occur for a variety of reasons: the velocity field at the aperture could be uniform, so that the amount of fluid leaving is less than the peak velocity times the area of the hole; flow could be impeded by the edges of the aperture, so that the effective area is smaller than measured, or the peak fluid velocity itself could be lower than expected. Each of these could lower the total flow rate at the aperture by some fraction, α. There are physical approaches to determining α, which will be discussed under **Additional Mechanistic Models** below. The semi-mechanistic approach is to assume that some reduction exists and attempt to determine the size of the reduction empirically.

The governing Torricelli equation with an empirical α is

$$A\frac{dh}{dt} = -\alpha a\sqrt{2gh}, \quad h(0) = h_0. \tag{11}$$

The solution follows directly:

$$h = h_0\left(1 - \alpha\frac{\sqrt{2}}{2}\frac{a}{A}\sqrt{\frac{g}{h_0}}t\right)^2. \tag{12}$$

Students may use (12) and regression techniques to determine α or use the data to approximate $\frac{dh}{dt}$ and then estimate α using (11).

Using (12), students often attempt to fit the quadratic

$$h = h_0\left(1 - c_1 t + c_2 t^2\right)$$

using linear regression techniques (or multi-linear techniques, if the data is structured in a and A). A good learning experience emerges as they attempt to deduce α from the two (generally incommensurate) equations

$$c_1 = \alpha\frac{\sqrt{2}}{2}\frac{a}{A}\sqrt{\frac{g}{h_0}} \quad \text{and} \quad c_2 = \alpha^2\frac{a^2}{A^2}\frac{g}{2h_0}.$$

This difficulty can be resolved using nonlinear estimation techniques, as discussed above, or by performing a linear regression using

$$1 - \sqrt{\frac{h}{h_0}} = \alpha\frac{\sqrt{2}}{2}\frac{a}{A}\sqrt{\frac{g}{h_0}}t.$$

The square root transformation alters the error structure, accentuating variance around small h disproportionately. However, this is not generally too big an issue; the transformed data looks quite linear, as can be seen in Fig. 4(a). Using this method we estimate $\alpha_{\text{root}} = 0.715$ for the calibration data; a comparison with validation data appears in Fig. 5. Students may also attempt to estimate different α for different trials without accounting explicitly for the dependence on a and A, then average the results (which generally works poorly).

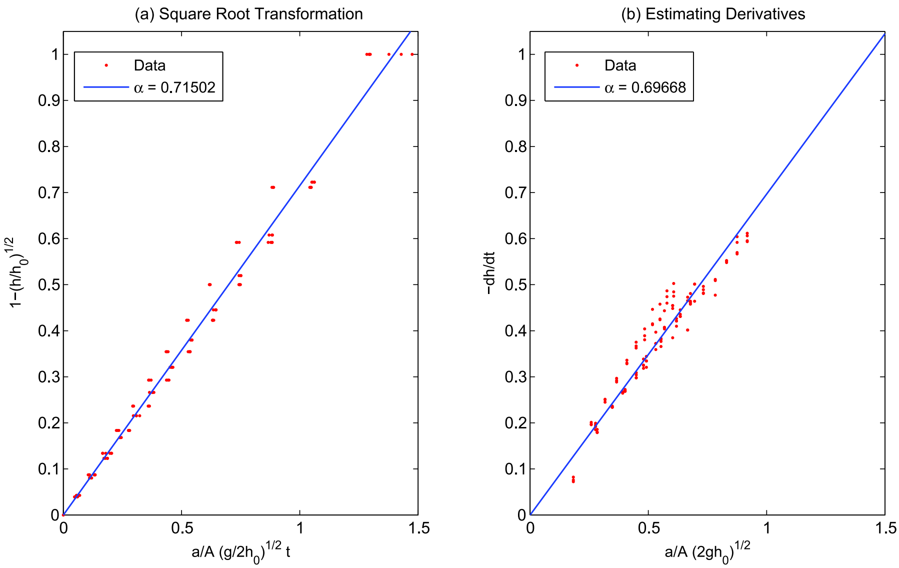

Fig. 4 Empirical estimates for α, the degree of aperture occlusion, using (**a**) root transformed data and (**b**) centered differences to estimate $\frac{dh}{dt}$. For the calibration data these two methods generate comparable results

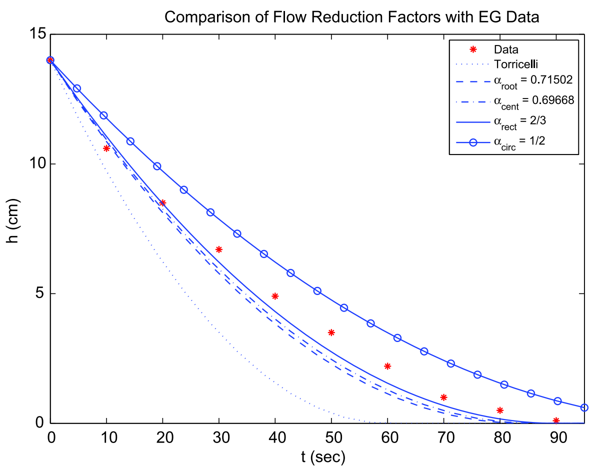

Fig. 5 Comparison of flow contraction factors with validation data (*). Torricelli's model with no correction appears as *small dotted line*. Regressions generated by square root transformation ($\alpha_{\text{root}} = 0.71502$) and centered differences ($\alpha_{\text{cent}} = 0.69668$) appear as *dashed* and *dash-dotted lines*, respectively. Analytic calculation using a parabolic model for rectangular flow ($\alpha_{\text{rect}} = 2/3$, not well justified for the circular aperture) appears as a *solid curve*; this is the best approximation of the lot. The appropriate flow reduction calculated using axisymmetric Poiseuille flow ($\alpha_{\text{circ}} = 0.5$) appears as a *solid line* with *circles*. A flow reduction of $\alpha = 0.62$ (not depicted) reproduces validation data almost perfectly

An alternate approach to determining α uses differences in subsequent observed heights to approximate $\frac{dh}{dt}$, then linear regression techniques to fit estimates, $\hat{h}'_j$, to the linear model

$$\hat{h}'_j = -\alpha \frac{a}{A}\sqrt{2gh_j}. \tag{13}$$

This gives students a good opportunity to think about how to approximate derivatives from data, using either 'forward', 'backward' or 'centered' differences. Forward and backward differences,

$$\frac{h_{j+1} - h_j}{\Delta t_j} \approx h'_j \approx \frac{h_j - h_{j-1}}{\Delta t_{j-1}},$$

where Δt_j is the duration of the time interval between the j and $(j+1)$st observations, both lead to a bias (below and above actual values, respectively), given the consistent concavity of the data. A centered difference,

$$h'_j \approx \frac{h_{j+1} - h_{j-1}}{\Delta t_j + \Delta t_{j-1}},$$

is both more accurate and relatively unbiased, as students can prove to themselves graphically or analytically. Data generated by centered differences, and the linear fit to estimate α, appear in Fig. 4(b); we estimated $\alpha_{\text{cent}} = 0.697$ using the calibration data and centered differences. Forward differences (not depicted) gave an estimate of $\alpha_{\text{fwd}} = 0.675$; backward differences gave an estimate of $\alpha_{\text{bwd}} = 0.730$. Comparisons of these reduction factors, Torricelli's law and validation data are depicted in Fig. 5. Generally speaking, using a centered difference and (13) with linear regression gives good estimates for α (in terms of reproducing observations), although for the data used here a forward difference (generating a smaller α) would have been more accurate.

3.2.3 Additional Mechanistic Models

Some physicists and engineers take AMB to learn about modeling and using data for model comparison and discrimination. These students may have some experience in fluid mechanics and use it to improve predictions from Torricelli's law. The 'correct' model would be the Navier–Stokes equations, including free boundary conditions for the jet as well as gravity, possibly needing surface tension at the aperture to describe transition to dripping flow. This is a research enterprise far beyond the scope of the AMB class.

However, a full Navier–Stokes approach may not be necessary; predicting the flow from apertures has a long history in fluid mechanics, harking back to luminaries like Kirchhoff, Helmholtz and Planck. As pointed out in G.K. Batchelor's *Introduction to Fluid Mechanics* (1985, p. 388), the essence of the classic problem is determining the degree of contraction of the jet (α above) or (equivalently, though not obviously) determining the structure of the fluid velocity field at the aperture. Batchelor surveys an approach to the two-dimensional problem (i.e. a vertical cross section through the aperture) using conformal mapping of container interior and the jet to flow between two planes. The resulting complex velocity potential has two free streamlines representing the upper and lower boundaries of the jet; these free streamlines are concave parametric curves which describe how much the jet contracts as it leaves the hole. Analytically this gives $\alpha \approx 0.61$ for a horizontal (approximately two-dimensional) slit, which compares well with experimental estimates in the range of 0.61–0.64 for horizontal slits.

The 'free streamline' approach is lovely, but generally far beyond the resources of students in AMB, and its accuracy questionable for holes which cannot be treated as vertical sections. Students have come up with alternate, physically-based ways to describe the flow at the aperture and calculate corrections to the Torricelli model. One approach commonly suggested by students is to include the effects of surface tension at the aperture; another is to approximate velocities at the aperture using Poiseuille flow and then calculate α analytically. Since surface tension would create a back-pressure at the aperture, reducing peak flow velocity, while the Poiseuille model describes transverse velocity structure at the aperture, the approaches can be used separately or together (or with empirical approaches described above).

Surface tension has units of force per length, reflecting the cohesion among fluid molecules along a material line at the interface between two fluids (water and air, for the leaky bucket). If p is the perimeter of the aperture, the net force, T, contracting a closed fluid line around the jet is $T = \gamma p$, where $\gamma = 72.8$ dyn/cm for a water/air interface at room temperature (Batchelor 1985). Assuming that this force exerts a back-pressure across the entire aperture area, Bernoulli's equation for the speed of fluid leaving the aperture becomes

$$\frac{1}{2}\rho u^2 = \rho g h - \gamma \frac{p}{a}.$$

Solving for the velocity

$$u = \sqrt{2\left(gh - \gamma \frac{p}{\rho a}\right)}$$

gives an adjustment to Torricelli's law:

$$A\frac{dh}{dt} = -ua = -a\sqrt{2\left(gh - \gamma \frac{p}{\rho a}\right)}, \quad h(0) = h_0. \tag{14}$$

Note that this equation is only valid when $h \geq \frac{p}{\rho a g}$. Equation (14) can be solved using the change of variables

$$\hat{h} = h - \gamma \frac{p}{\rho a g},$$

giving

$$h = \gamma \frac{p}{\rho a g} + \left(h_0 - \gamma \frac{p}{\rho a g}\right)\left(1 - t\frac{ag}{2A}\sqrt{\frac{2\rho a}{\rho a g h_0 - \gamma p}}\right)^2. \tag{15}$$

It should be noted that this solution, quadratic in t, is only valid while it is decreasing to $h = \gamma \frac{p}{\rho a g}$, at which point the velocity of outflow is zero.

Solution (15) never reaches zero (i.e. the bucket never drains), which is realistic for small enough holes. The solution predicts an asymptotic height of

$$\bar{h} = \gamma \frac{p}{\rho a g}.$$

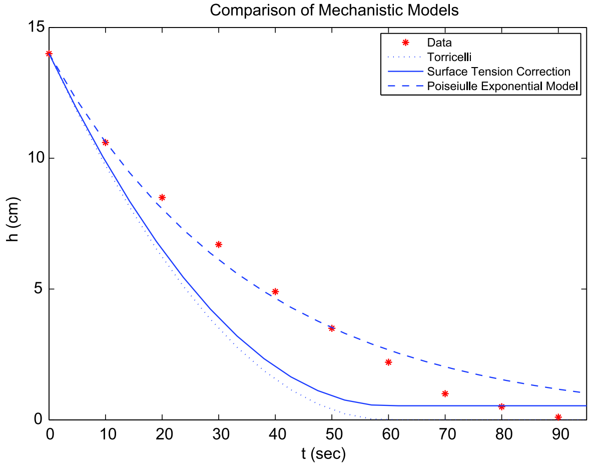

Fig. 6 Comparison of mechanistic models to validation data (*) for a milk carton with cross-sectional area $A = 86\ \text{cm}^2$ and circular hole of diameter 0.556 cm. Aperture size was $a = 0.243\ \text{cm}^2$, surface tension was $\gamma = 72.8$ dyn/cm and perimeter $p = 1.75$ cm. Original Torricelli model predictions appear as a *dotted line*. The surface tension correction to Torricelli's model appears as a *solid curve*, while the exponential model generated by assuming Poiseuille flow with Torricelli velocity appears as a *dashed line*

For the circular validation aperture of diameter 5.56 mm this gives $\bar{h} \approx 0.534$ cm or 5.34 mm, which is unrealistically high (see Fig. 6). The discrepancy occurs because surface tension is expressed through material lines stretching across the aperture as $u \to 0$, as opposed to compressing the neck of the emerging jet. These distances are shorter (i.e. smaller p), leading to a decrease in $\bar{h}$. However, between the stopping of the jet and a stable balance between surface tension and residual pressure there is an analytically difficult state of transitioning from a flowing jet to a stable, stationary bubble. Opposite sides of the jet must reconnect, requiring the fluid equations to pass through a singularity. Singularities and re-connections are objects of research in fluid mechanics, certainly beyond the scope of our lab. Depending on the student definition of 'empty' (some students cannily define transition to dribbling flow as the time at which the bucket becomes 'empty') and the size of the aperture the surface tension correction can perform quite well. For the 2006 validation data, however, this model does not do so well, as can be seen in Fig. 6.

An alternate model includes viscous effects on aperture flow using the Poiseuille simplification of Navier–Stokes equations for the horizontal velocity field, $u(y, z)$:

$$\nu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \frac{1}{\rho}\frac{\partial P}{\partial x}, \tag{16}$$

where P is the pressure field, x is in the direction of the flow from the aperture and y, z are horizontal and vertical variables parallel to the aperture, respectively. The Poiseuille model is analytically justified for steady flow through a pipe with $u = 0$ at the perimeter; its use at an aperture of negligible extent is not wholly appropriate, but it is within the realm of Navier–Stokes-related complications that students contemplate. In principle the pressure field, P, should be determined by a solution to a further, coupled elliptic problem, but in this case we may take $P = \rho g h$ at the container wall opposite the hole, a distance w away, and approximate $\frac{1}{\rho}P_x = -\frac{\rho g h}{\rho w} = -\frac{gh}{w}$. Equation (16) now becomes an exercise in solving either an ordinary or partial differential equation, depending on aperture geometry and student sophistication.

For a hole which is much wider (c in the y direction) than its height ($-b < z < b \ll c$), the transverse derivatives may be neglected, giving the Poiseuille solution

$$u = \frac{gh}{2\nu w}\left(b^2 - z^2\right). \tag{17}$$

Integrating across the aperture gives a new mass-balance equation

$$A\frac{dh}{dt} = -c\frac{gh}{2\nu w}\int_{-b}^{b}\left(b^2 - z^2\right)dz = -\frac{2}{3}\frac{cgh}{\nu w}b^3 \tag{18}$$

and generates an exponential model for drainage

$$h_{\text{rect}} = h_0 \exp\left(-\frac{2}{3}\frac{gab^2}{\nu w A}t\right), \tag{19}$$

where we have written aperture area $a = bc$.

The Poiseuille solution for a circular aperture of radius $r < b$ is quite similar:

$$u = \frac{gh}{4\nu w}\left(b^2 - r^2\right), \tag{20}$$

where the factor of 1/4 results from writing $\nabla^2 u = \frac{1}{r}\partial_r(r\partial_r u)$ in radial coordinates and solving (16). The mass balance equation becomes

$$A\frac{dh}{dt} = -2\pi\frac{gh}{4\nu w}\int_{0}^{b}\left(b^2 - r^2\right)r\,dr = -\frac{\pi}{2}\frac{gh}{4\nu w}b^4, \tag{21}$$

with solution

$$h_{\text{circ}} = h_0 \exp\left(-\frac{gab^2}{8\nu w A}t\right), \tag{22}$$

using $a = \pi b^2$. This Poiseuille exponential model performs reasonably well for an exponential model (see Fig. 6) but clearly deviates from observation as the bucket empties. The problem is that the linear approximation for the pressure gradient is not accurate as the fluid becomes shallow. However, w may be taken to be unknown, providing justification for the exponential phenomenological model (and a dimensionally consistent way to accommodate different geometries during validation).

A different Poiseuille approach hit upon by students is to assume that the transverse structure of the flow is parabolic but the peak flow velocity should be equal to the Torricelli velocity $u = \sqrt{2gh}$ (or the velocity given by the surface tension correction above). This gives, in the case of the rectangular slit,

$$u = \sqrt{2gh}\left(1 - \frac{z^2}{b^2}\right),$$

and generates a volumetric flow

$$\text{Flow}_{\text{rect}} = \sqrt{2gh}c\int_{-b}^{b}\left(1 - \frac{z^2}{b^2}\right)dz = \frac{2}{3}a\sqrt{2gh}, \tag{23}$$

which is equivalent to $\alpha_{\text{rect}} = \frac{2}{3}$ in (12) and generally fits the data quite well (see Fig. 5). However, for a circular hole,

$$\text{Flow}_{\text{circ}} = \sqrt{2gh}2\pi\int_{0}^{b}\left(1 - \frac{r^2}{b^2}\right)r\,dr = \frac{1}{2}a\sqrt{2gh}, \tag{24}$$

equivalent to $\alpha_{\text{circ}} = \frac{1}{2}$, which is unrealistically low, as depicted in Fig. 5. Students hitting on this approach (and biased toward circular holes) may find that it does poorly, while students biased rectangularly may use this approach and feel that they have completely solved the problem.

4 Pedagogical Results

The Leaky Bucket Lab, with all the exciting mathematical discovery potential outlined above, has been a successful component of the AMB course for several reasons. Students are lead to analyze, parameterize, and pose mathematical models based on data they collect themselves. On an affective level, students realize that slick theory is not always "correct" since the model of Torricelli, although appealing, fails to capture straightforward measurements. The LB project requires students to compare and evaluate multiple models and deal with strong validation using independent data. Above we have provided a survey of modeling approaches to give instructors some idea of technical issues that arise. Now we turn to some of the pedagogical issues involved in lab implementation.

Instructional methodology was analyzed using video recording, and below we communicate specific teaching moves made to promote a high level of cognitive demand and student performance. Recordings of each lab component (Lecture—Data Collection and Model Creation—Model Presentation—Validation) were transcribed, and questions posed by course instructors and students coded according to the cognitive level evidenced or promoted. Overall, we found clear evidence that the specific teaching factors associated with high levels of cognition from Stein et al. (2009) were present in our implementation of the LB lab (see Table 4). We also analyzed student reports along with instructor comments and scores in draft and final stages; these findings will be published separately. Below the teaching factors of Stein et al. (2009) are discussed, and we abstract vignettes of observed classroom discussion to illustrate pedagogical strategies in the LB lab. In the following subsections we report results from question scoring, cross-referenced with lab activity.

4.1 Pedagogical Strategies

A good project does not teach students by itself; teachers are essential for orchestrating the classroom discourse in such a way that students learn. For success, instructors must not only choose challenging tasks for students, but ask questions and foster classroom discussion to maintain high cognitive performance. There are many ways in which a project can flop in the classroom and fail to engage students in higher order thinking and learning. Education researchers Stein et al. (2009) articulate specific moves on the part of instructors that are associated with the maintenance and decline of high-level tasks in classrooms. While their research is centered around middle and secondary mathematics education, similar principles apply in college teaching. Table 4 summarizes various factors under the teacher's control associated with the maintenance/decline of high levels of cognition. The more closely an instructor can cleave to these guidelines, even in traditional classes, the higher the level of cognitive function and intellectual development of the students.

Table 4 Factors associated with the maintenance/decline of high cognitive demand. The entries are abstracted from Stein et al. (2009); the left-hand entries are characteristic of classrooms promoting high cognitive demand. Entries on the right are characteristic of classrooms with lower demand

Factors associated with the maintenance of high levels of cognitive demand	Factors associated with the decline of cognitive demand
Instructional methods include scaffolding of student thinking	Difficult aspects of problems become routinized and hence the work is done for the students
The task builds on students' prior knowledge and the teacher draws frequent conceptual connections	Students are stymied with an inappropriate task preventing their meaningful engagement in activities
There is a sustained press for justifications from instructors and students	The focus of work is on answer correctness or completeness
Instructor and competent students provide models of high-level performance	There is no accountability for high-quality products or adherence to a scientific and analytical process
Student have sufficient time to explore, create, and test their own ideas	Insufficient time is allotted for the task, or too much time is given allowing students to drift attention away from intended learning goals
Students have the means to monitor their own progress	Task expectations are not clear

4.2 Observations of Classroom Discourse

During most of the LB lab instructors do not lecture but rather circulate through the room, kibbutzing on group discussions, responding to queries, facilitating experiments, and monitoring student progress. When groups get stuck instructors may interject questions to aid progress. Instructors provide scaffolding of student thinking, i.e. use leading questions to guide the reasoning and direction of the conversation, yet allow students to make connections themselves. For example, in the following discussion students are struggling with understanding predictions of the Torricelli model, which consistently underestimates the time for emptying the vessel, but are confused about what to make of this discrepancy between model and result. The students are sitting in a circle with their notes in the middle, and mulling their findings. Instructor A joins the circle and asks the group to describe sources of errors in their own measurements. The instructor does not focus the questioning on correct or incorrect answers, but rather presses students to connect their measurements with the model. This sets up a situation in which students critically evaluate the data they have collected and are receptive to an exposition of the procedures for sensitivity analysis.

Instructor A: So, a basic question to ask is, what are the natural sources of error in what you're doing? Where do you think you make base errors?

Student M: Well, in measuring volumes—graduated cylinders are not very precise. And just hitting the stopwatch at the right time.

Instructor A: This model [pointing to the equations in the group's notes] doesn't actually have volume in it, so how does the volume measurement translate into something that is in the model?

Student M: Hmm… we use the volume and our measurement of the water's height to determine the cross-sectional area of the bucket. The area is based on two measurements that could have error in them.
Student T: That do have error in them, there's no could.
Student M: That do have errors in them, yeah.
Instructor A: So the cross-sectional area is a variable. What other measurements have error in them?
Student M: The hole was not exactly forty-two millimeters square. It's not a perfect square. So the area of the hole is an approximation.
Student J: Also the ending time. It is really hard to say for sure. I'm just eyeballing when the water hits the top of the hole. Could it be that our measurement errors throw the model off?
Instructor A: Well, that's the question. You've got this model to predict emptying time. Let's say you want to understand the sensitivity of the errors in measuring the cross-sectional area. You can say that a change in the emptying time would be like a partial derivative…

Instructor A then continues with a brief review of/introduction to sensitivity analysis, drawing connections between the current situation and other knowledge. This provides a tool for the group to use in subsequent work if they choose, but they are not required to apply or practice this particular strategy. Students must make their own decisions about what is relevant and what to include in their exploration and reporting. Hence, the instructor provides the scaffold, but students are doing their own construction.

Later on, the conversation in this same group turns to the topic of formulating an alternative model. The instructor has a direct role in guiding the conversation by suggesting that the Torricelli model, rather than measurement errors, might be a problem. He prods students for ideas and validates their thoughts, encouraging them to explore further and even to make appropriate experiments to test their ideas.

Student M: Our data from the different runs seem fairly consistent; just one run was far off. The data on the five second and ten second marks are lining up pretty well. The problem seems to be with the finishing time, but we might be marking it done too soon. I think our data is looking good, consistent with reasonable error. The problem has got to be our measurement of either the height or the volume.
Instructor B: Or the model …
Student M: Yeah or the model. It might not account for everything, like the water sticking to the inside.
Instructor B: Yeah, for example. It's worth thinking about things like that, though. There are some things you've already observed that are different from the model, right? This water sticking to the inside thing, it's clearly not something that the model addresses. So is that significant? Is it something that should be put in the model somehow?
Student T: I think so, the drag of the water along the side could slow it down.
Instructor B: You should think about that. How would you adjust the model? What kind of term would have an effect of deceleration or drag of the water?

Student M: Throw another factor into the denominator of the equation, some positive number, greater than one, to slow it down. We need something that would account for the drag along the side, or . . .

Student J: Or somehow take better account of the forces acting on the system. [Pause.] Gravity, for example, seems to be going in the wrong direction because the water is shooting out the side, not straight down. [Student J gestures with the bottle in her hand, pointing in the directions of the stream of water and in the direction of gravity.]

Instructor B: So you're saying that according to the model, water should be squirting straight out, but gravity is actually pulling down so it's like the size of the hole is decreasing somehow. Like rain falling on your windshield—the drops hit your windshield at a different frequency, depending on how fast you're driving. That's an interesting idea.

Student M: Yeah, that could slow it down, maybe explain the error.

Instructor B: So once velocity gets low, gravity does not have as much of an effect, really making the size of the hole get . . .

Student J: Get smaller.

Student T: I think I see what you're talking about.

Instructor B: So think some about it. You've got some time now, while you are together. There may be some measurement or observation you could experiment with, while you've got this in front of you.

Student M: Yep.

Student J: Thank you.

Here it is clear that students are involved in conjecturing, and connecting their understanding of the observed system to the equations that model it. Instructors do not allow difficult aspects of the student work to become routinized by dictating precise cookbook style laboratory or mathematical procedures. Rather students are encouraged to come up with ideas and justifications, then given time to explore options and test possibilities.

Another way in which these conceptual connections are emphasized for students is through tying the physical situation students are modeling to their experiences and prior knowledge. Instructors continually gauge students comprehension of important ideas through questioning and use analogies to help students make those connections. (This was illustrated by the reference to rain falling on a windshield in the previous example.)

During the lecture portion of this lab exercise, the class discourse has a different nature. More questions address direct recall of factual information. Still, Instructor A peppers the exposition of Torricelli's model with questions, keeping students engaged. Here is an excerpt of the course transcript during the lecture:

Instructor A: So let's say that over a small chunk of time, like . . . a second (it's a fairly big bucket and a fairly small hole so it's not going to drain in just one second), things . . . the height of the fluid above the hole, 'h', goes down. So how much volume was lost?

Student: Pi times the radius squared . . . times the change of the height of the fluid.

Instructor A: Ok, so, what's the change in height? I guess we can call that delta h . . . and I heard pi in there . . .

Student: … times the radius squared, the radius of the bucket squared…
Instructor A: Ok, so if it's …
Student: If it's circular.
Instructor A: If it's circular, what if it was square or what if it was kind of an oblique shape…
Student: Just the area would be…
Instructor A: Ok so, the area, in a sense. So delta 'V'… would be … is it a positive or negative change in volume?
Student: Negative.
Instructor A: So we have negative change in height—oops, I'm mixing stuff up, let's call this A(h)—times the area. I put in area as a function of 'h' because actually buckets are only rarely perfect cylinders. More commonly a bucket slopes in and out, so that area depends on, I mean the area of the surface of the fluid changes as the height of the fluid goes up and down. So in general it could be some function, although if we're lucky it'll be a constant. So that's the area, or that's the volume lost. So what do we expect to happen with the … like if we have a vast tank, a vast deep tank, and I open a spigot at the bottom, what's the water pressure going to be like?
Student: Very high at first.
Instructor A: Pretty high, so the speed of the fluid coming out of there is going to be…
Student: Fast.
Instructor A: Pretty fast, but as that goes down, what's going to happen to that pressure?
Student: Drops.

This line of questioning requires students to follow the reasoning in the derivation and make connections to their existing mathematical knowledge. It is notable that Instructor A adjusts his exposition and notation based on the student responses, including students in the discourse. Instructors are not overwhelming students with a project disconnected from their incoming understanding, nor are they doing an activity for activity's sake, but the purpose and meaning of the laboratory exercise is clearly stated and discussed with the class. The direct exposition of Torricelli's model and its derivation play a role in clarifying task expectations, not only as a means to teach the students the model but also as an example of a clearly explained model where each term has a rationale. Once data collection and investigation commence, lab objectives and the challenge of strong validation are constantly revisited, giving students control of their learning—they have the means to monitor their own progress.

Instructors continually press for justifications and provide sufficient time to explore, create and test ideas by involving students in experimental design and data collection. This next example shows a conversation in which Instructor B probes a lab group to describe the process they will use to estimate the influence of various parameters on the dynamics of the flow.

Student K: So we want to have the same area?
Student D: The same area but with a different shape?
Student K: But with a different shape.

Student D: Sure.
Instructor B: Let's think about this, a little bit. So you can calc—I mean, you can satisfy the requirements for the Torricelli Model ...
Student D: Just by measuring these things?
Instructor B: With—by measuring these things, but what are you going to ... what are you thinking about for the alternative model, because it might influence what sort of holes you do?
Student D: Mmhmm. Okay, so ...
Instructor B: So, you don't need to come up with the actual math equation, but what would be your first step to an alternative?
Student D: So we were talking about doing that statistical, sort of ... where we would have, so we need to know ... so the height, and the time. And then if we were to ...
Student K: I don't know which part you are talking about.
Student D: So if we have a bunch of different shaped holes or whatever, is there some way that we could ... so, height, time ... so we could do like, volume out over the time, and you would still expect shape like this.
Student K: Mmhmm.
Student D: Okay, and the volume is going to be a function of the area ...
Student K: That was our area time velocity?
Student D: ... and the height.
Student K: The volume out? Or the volume that's in there?
Student D: So the volume ... well, any volume is going to be the area times the height. Or, is that ... or some function of the area and the height, so it would be centimeters ...
Student K: Because I thought we talked about the area out, or the volume out is the area of the hole times the time times the velocity.
Student D: Times the time, times the velocity ... volume out.
Student K: Does that seem ...?
Student D: Um ... yeah. That's how we could calculate the volume out.
Student K: I think at the end, [pointing to the illustration they have drawn] that's going to be equal to that.
Instructor B: But how difficult is it going to be to measure the velocity going ...
Student D: Yeah, we can't measure that.
Student K: Yeah you'd have to know the volume out.
Instructor B: How does that help you formulate an alternative model?

Instructor B waits, allows students to discuss things, does not interject, and encourages them to come to a logical agreement. By pressing the students for justification the instructor has lead students to a much deeper understanding as well as allowed their own ideas regarding an alternate model to surface.

In these transcribed vignettes and the discussion of class structure we have provided examples of the pedagogical factors in Table 4: instructor scaffolding, pointing out connections conceptually and with prior knowledge, sustaining a press for justification, modeling high-level performance, leaving time to explore, create and test, and providing students with the means to monitor their own progress. The LB lab (and BAMIM more generally) sets up an environment in which students collect data

Table 5 Distribution of questions asked during LB lab. Memory-level questions are those requiring only factual answers while Reasoning-level questions provoke responses requiring inferences, judgements or conceptual responses. As the flow of the lab moves from more to less hierarchical (Lecture → Group work and Presentation), the number of reasoning-level questions asked both by instructors and students increases dramatically. As the lab moves into Validation, with students testing themselves against independent data following a clear and previously-generated protocol, the questions become more factually oriented and cognition diminishes

	Lecture	Data collection and model creation	Group presentation of models/results	Validation using independent data
Teachers: Memory-level	10	31	21	24
Teachers: Reasoning-level	7	44	19	1
Students: Memory-level	0	93	13	29
Students: Reasoning-level	0	77	29	0
Other questions:	10	22	12	1

and struggle to construct models which explain it. During portion of class that the instructor has allocated to the lab, the burden of judgement and covering a specific syllabus is moved away from the instructors, who are free to serve as facilitators and resources, in keeping with Stein et al. (2009).

4.3 Question Level and Student Responsibility

Evidence for how BAMIM and the LB lab encourage high cognitive function is provided by data we collected on question activity by both students and instructors through the course of the lab (Table 5). Questions (defined as interrogatory statements addressed to other individuals and intended to provoke a response) were counted in the video transcript and categorized as memory- (requiring responses that recall previously learned knowledge) or reasoning-level (provoking responses that require forming inferences and/or judgements) and either teacher- or student-initiated. The classification was based on research on questions used in teaching (Gall 1970). Questions categorized as "other questions" did not pertain to the Leaky Bucket Lab *per se*, but were mainly logistical (like "Where should we meet tomorrow?" or "Would you like to do the timing now?"). This provides evidence on the level of cognition occurring among the participating students.

The Lecture portion, which was not too long, notably did not include any student-initiated questions of any kind. In the data collection and model creation segments, by contrast, student initiated questions were quite plentiful—with 93 memory level and 77 reasoning level questions coming from students. Note that the teacher's lecture had more memory than reasoning level questions, while the data collection and model creation segment included more reasoning than memory level questions as instructors probed students to make sense of the Torricelli model and logically deduce model variations based on their own observations. Conversely, as the students took responsibility for data collection and model creation, their level of questioning (and the cognition behind the questions) was raised. As students made presentations of their model they asked and were asked many reasoning-level questions, largely because the format was of class dialogue as opposed to a final, graded presentation.

Hence, this portion of the lab appears to have been effective at promoting critical thinking among students. At the end of the lab, during the validation challenge, students mostly understood what needed to be done and questions from both teachers and students were nearly exclusively at the memory level.

Overall, the data support our contention that the BAMIM environment of the LB lab encourages high-level cognition (for both instructors and students). The most mentally creative portion of the experience was that most clearly in line with BAMIM: data collection, model creation, and model presentation. To the extent that the lab's educational goals were not about specific facts but aimed at encouraging students to think about the application of quantitative techniques to observational data, the lab was most effective where the students retained the most responsibility and were most engaged in discourse.

5 Conclusion

Rather than attempt to present the LB lab as a narrowly defined procedure, we have tried to convey a pedagogical approach and sufficient breadth of mathematical materials so that instructors can implement the lab in their own instructional context. For maximum impact we argue that responsibility for observations and model development be left as much as possible to teams of students. In our own class, observational evidence clearly supports the idea that the less instructors lecture and judge the more the students are thinking. The optimal role of instructors, from the standpoint of rich student cognition, is to scaffold student accomplishment, highlight connections, press for justification, and promote scientific discourse verbally and in writing. This creates a classroom atmosphere of discussion, investigation and engagement that is critical to BAMIM. When students can approach the lab with an openness to innovation they can both learn new mathematics and gain confidence in their ability to apply it in scientific, data-driven environments.

As outlined above, the LB project can be performed with simple, commonly available materials, does not require expertise in handling biological organisms, and can be performed in most classrooms, although a laboratory setting is preferred. Moreover, it has clear potential for rich analytic investigation, from elementary phenomenological modeling approaches to fully mechanistic fluid models. All of the approaches discussed above (with the exception of free streamlines) were initially 'discovered' by some group of students in one of our previous classes. Which alternate model a group will develop is unpredictable; however, they will definitely learn something about model development, connecting parameters with observations, model competition with data, and how mathematics, statistics and observational science interact.

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References

Barak, M., & Dori, Y. (2004). Enhancing undergraduate students chemistry understanding through project-based learning in an IT environment. *Sci. Educ.*, *89*(1), 117139.

Batchelor, G. K. (1985). *An introduction to fluid mechanics*. Cambridge: Cambridge University Press.

BIO 2010 (2003). *Transforming undergraduate education for future research biologists*. Washington: National Academies Press. Board on Life Sciences, Division on Earth and Life Studies, National Research Council of the National Academies.

Bressoud, D. M. (2001). What's been happening to undergraduate mathematics? *J. Chem. Educ.*, *78*(5), 578–581.

Chazan, D., & Ball, D. (1999). Beyond being told not to tell. *For Learn. Math.*, *19*(2), 2.

Chin, C., & Chia, L.-G. (2005). Problem-based learning: using Ill-structured problems in biology project work. *Sci. Educ.*, *90*(1), 44–67.

DebBurman, S. K. (2002). Learning how scientists work: experiential research projects to promote cell biology learning and scientific process skills. *Cell Biol. Educ.*, *1*, 154–172.

Gall, M. D. (1970). The use of questions in teaching. *Rev. Educ. Res.*, *40*(5), 707–721.

Haefner, J. W., & Bowen, M. D. (2002). Physical-based model of fish movement in fish extraction facilities. *Ecol. Model.*, *152*, 227–245.

Helle, L., Tynjälä, P., & Olkinuora, E. (2006). Project-based learning in post-secondary education—theory, practice and rubber sling shots. *High. Educ.*, *51*, 287–314.

Hilborn, R., & Mangel, M. (1997). *The ecological detective*. Princeton: Princeton University Press.

Hmelo-Silver, C. E. (2004). Problem-based learning: what and how do students learn? *Educ. Psychol. Rev.*, *16*, 235–266.

Keener, J. P., & Sneyd, J. (1998). *Mathematical physiology*. New York: Springer.

Keyes, C. W. (1999). Revitalizing instruction in scientific genres: connecting knowledge production with writing to learn in science. *Sci. Educ.*, *83*(2), 18–30.

Kohler, B. R., Swank, R. J., Haefner, J. W., & Powell, J. A. (2009). Leading students to investigate diffusion as a model of brine shrimp movement. *Bull. Math. Biol.*, *72*(1), 230257. doi:10.1007/s11538-009-9444-4.

Kuhn, T. (1962). *The structure of scientific revolutions*. Chicago: University of Chicago Press.

Lemasson, B. H., Haefner, J. W., & Bowen, M. D. (2008). The effect of avoidance behavior on predicting fish passage rates through water diversion structures. *Ecol. Model.*, *219*, 178–188.

Logan, J. D. (2006). *Applied mathematics* (3rd ed.). Hoboken: Wiley.

NCTM (1991). Professional Standards for Teaching Mathematics. Reston, VA: National Council of Teachers of Mathematics.

Platt, J. R. (1964). Strong inference. *Science*, *146*, 3642.

Powell, J., Cangelosi, J., & Harris, A. M. (1998). Games to teach mathematical modelling. *SIAM Rev.*, *40*, 87–95.

Powell, J. A., & Engelhardt, K. A. (2000). Optimal trajectories for the short-distance foraging flights of swans. *J. Theor. Biol.*, *204*, 415–430.

Rigley, M. (2009). *An intermediate-complexity biological modeling framework for nutrient cycling in lakes based on physical structure*. Masters Thesis, Utah State University.

Schwartz, C. V., Reiser, B. J., Davis, E. A., Kenyon, L., Acher, A., Fortus, D., Shwartz, Y., Hug, B., & Krajcik, J. (2009). Developing a learning progression for scientific modeling: making scientific modeling accessible and meaningful for learners. *J. Res. Sci. Teach.*, *46*(6), 632654.

Smith, H. L., & Waltman, P. (1995). *The theory of the chemostat*. London: Cambridge University Press.

Smith, K. A., Sheppard, S. D., Johnson, D. W., & Johnson, R. T. (2005). Pedagogies of engagement: classroom-based practices. *J. Eng. Educ.*, *94*(1), 87–102. Special Issue on the State of the Art and, Practice of Engineering Education Research.

Springer, L., Stanne, M. E., & Donovan, S. S. (1999). Effects of small-group learning on undergraduates in science, mathematics, engineering, and technology: a meta-analysis. *Rev. Educ. Res.*, *69*(1), 21–51.

Stein, M. K., Smith, M. S., Henningsen, M. A., & Silver, E. A. (2009). *Implementing standards-based mathematics instruction: a casebook for professional development* (2nd ed., p. 16). New York: Teachers College Press.

Tynjälä, P. (1998). Writing as a tool for constructive learning: students learning experiences during an experiment. *High. Educ.*, *36*, 209–230.

Usiskin, Z. (1991). Building mathematics curricula with applications and modeling. In M. Niss, W. Blum, & I. D. Huntley (Eds.), *Mathematical modeling and applications* (pp. 30–45). Chichester: Horwood. 1991.

Wurtsbaugh, W. A., Baker, M. A., Gross, H. P., & Brown, P. D. (2005). Lakes as nutrient "sources" for watersheds: a landscape analysis of the temporal flux of nitrogen through sub-alpine lakes and streams. *Verh. Int. Ver. Limnol.*, *29*(2), 645–649.