COFFEE TO GO!
MODELING THERMOCLINES
IN MULTIVARIABLE CALCULUS

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Abstract: While mathematical modeling is an integral process in applied mathematics, students rarely encounter genuine modeling opportunities in their calculus courses. Here we introduce a laboratory experience as a natural starting point for calculus students to investigate multivariable functions. A layered system of coffee and milk serves as a physical model for temperature gradients in lakes or the atmosphere, where temperature depends on both a temporal and spatial variable. Students create, observe, and collect temperature data of their own, graph the data, and develop mathematical models to fit the data. We require students to write a report about their findings. This article includes details about the class activity conducted in two different college settings and provides our assessment of student interaction with the lab, and how the lab informs student understanding of multivariable functions.

Keywords: Multivariable calculus, mathematical modeling, thermoclines, data collection, inquiry instruction, project-based learning

1 INTRODUCTION

Among problems leading to diminishing mathematical competency in America, is a growing inability of students to relate mathematics to the real world. From kindergarten, mathematics is presented as a series of self-referential rules and tricks, a theoretical edifice, which first graders start to climb and on which older students plateau at levels depending on their chosen profession and stamina. "Applications" and "manipulatives" are tacked on as set-piece word problems or blue boxes parenthetically appended to the main text. The real world, however, does not neatly mirror mathematical structure. Applications that engage students and allow them to build their own understanding do not have mathematics attached in blue boxes.

For mathematics to be a problem-solving tool to model and understand real-world mechanisms, one must view it as a descriptive language [13]. Results must be evaluated not only in terms of intrinsic consistency, but also in terms of their relationship to data. The more students are responsible for that data and connected to the mechanisms that generate it, the more they will raise their capacity to create useful mathematical formulations relevant to real-world situations [15].

The lab we present is one of several class activities we have designed to bring authentic modeling experiences into the mathematics undergraduate curriculum – Laboratory Experiences in Mathematical Biology. We aim to bridge the gap between mathematical theory and scientific practice, building students' modeling and problem-solving skills using labs in mathematics classes. Our goals are to allow students to construct and explore mathematics in the context of observed biological mechanisms and help instructors create a data-driven culture of inquiry in the classroom. Our design principles are summarized by the following framework for creating successful lab experiences [8]. Successful labs exhibit the following characteristics:

- Promote discovery: Labs are open-ended with opportunity to reason inductively, explore concepts and relationships, and discover connections.
- Authentic: Driven by real data, ideally collected by students, labs require models and techniques used by applied mathematicians and scientists.
- Visible success: The plausibility of models is visualized through comparisons of model predictions with collected data.
- Engaging: Labs are based on an accessible, original, scientific question about a natural phenomenon fitting into a broader story line.

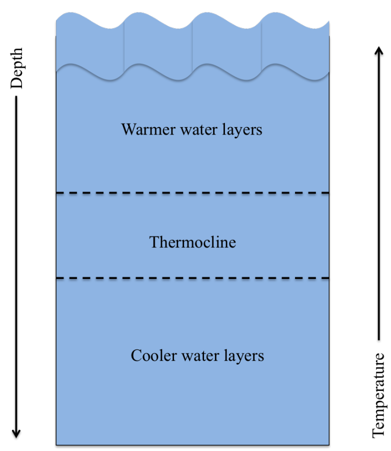
Our labs do not require specialized training or a technical lab setup by the instructor. We try to adhere to the motto: *You won't need anything you can't find at the local hardware store.*

This article describes the Coffee to Go! lab and what we have learned from two implementations at Colorado College (CC) and Utah State University (USU). CC is a highly selective, private, 4-year liberal arts college with 2,000 students on a non-traditional block plan. Students take one course at a time for 3.5 weeks, and there are 8 blocks in the academic year. The Coffee to Go! lab was tested in a Calculus 2 course with 25 students in the fall of 2014. Unlike CC, USU is a large public research university, a land grant institution serving 28,000 students. Courses are offered on a 15-week semester plan. This lab was implemented in a Multivariable Calculus (Calculus 3) class with 34 students in fall 2013 and and 41 students in the spring of 2014. Hence through our implementation we have shown the lab is adaptable to different programs.

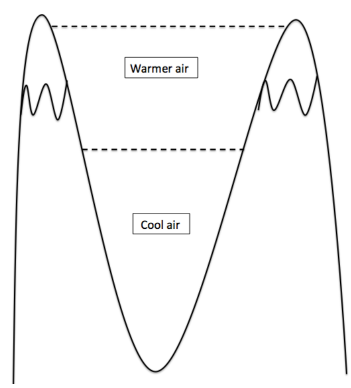
2 CLASS ACTIVITY

The Coffee to Go! lab is inspired by layering phenomena in limnology and atmospheric science (see Figure 1). Lake warming due to climate change can change the mixing dynamics of deep lakes [17]. In the summer months, lakes stratify with warm water layers on top and cold layers on the bottom. The surface water becomes separated from the deep water through a density gradient. The *thermocline* is a layer of rapid transition in temperatures separating warm surface water from deep cold water. In the fall, the surface water cools down, destroying the gradient. As a result the layers mix and algae are washed out. Winter mixing oxygenates the deep water and brings nutrient-rich water to the surface. Climate warming can prevent winter mixing, affecting water quality and local ecosystems.

Another natural layering phenomenon is winter inversions, a condition in which a layer of cold air is trapped under a layer of warmer air, leading to the accumulation of pollutants close to the ground. USU students are familiar with this problem [9]. When driving into the valley through clear blue skies at high elevation one can see the transition layer and hazy polluted air at lower elevations blanketing the region. Storms clear the air from pollution, until cold surface temperatures allow pollutants to accumulate again. People who live in regions affected by winter inversions may experience increased rates of asthma and cardiovascular disease [10].



(a) Summer Lake Stratification: Warmer layers are on top, cooler layers on the bottom, separated by a transition layer, the thermocline.



(b) Winter Inversion: In a valley, cool air is trapped under a layer of warmer air.

Figure 1. Examples of layering phenomena in nature

The Coffee to Go! lab provides a physical model system using milk and coffee for students to investigate the common components of these natural systems, and gain a sense of what scientists do who model natural phenomena. Studying these scientific questions creates a motivation for students to discover and construct the mathematical concepts of (1) multivariable functions, (2) traces and level curves, and (3) partial derivatives. The lab offers an opportunity to discover relationships between mathematical objects in higher dimensions and their single variable counterpart. Students build on their previous understanding of a function of one variable to extend the concept to functions of several variables. They learn to create and interpret various representations of a function and reinforce the notion that functions are relations among variables that exist in the real world. In addition, students learn to use computer algebra systems such as Mathematica or MATLAB to visualize graphs of multivariable functions. Guided by the student task sheet (see Appendix A), the flow of class activities proceeds as follows:

1. We launch the lab by introducing the project, discussing the scientific background, and setting goals for the activity. We ask students to make predictions about how the system will behave.

2. Students gather data.

3. Students plot their data, introduce notation for multivariable functions, and make observations about the data.

4. After an introductory discussion on modeling data with simple functions, students use the Fit command in Mathematica to fit a plane and then higher degree polynomials to the data.

5. Reflecting on the data we introduce partial derivatives. Students tie their observations from the lab to the formal definition and interpretation as a rate of change. We draw on their experiences with rates of change of functions of one variable and lead them to see that here rates of change can be considered with respect to each of the two independent variables.

6. We provide guidance to the written reports.

2.1 Objectives

In order to develop as mathematical modelers, students must engage in various learning activities that address a range of cognition types. The learning objectives and their intended cognition types that guide our instruction and assessment of student work are stated in Table 1.

Table 1. Cognition types and learning objectives to guide instruction and assessment according to the educational research-based methods in [5].

Cognition type	**Learning objectives in this lab**
Construct a concept Use inductive reasoning, distinguish between examples and non-examples.	Explain that functions of 1 or 2 variables are relations between input and output variables. Given a function of 2 variables, distinguish vertical traces from level curves and explain the defining characteristics.
Discover a relationship Use inductive reasoning, discover relationships among concepts.	Explain how objects in single variable calculus generalize to objects in higher dimensions and explain why level curves do not intersect. Observe that making measurements in a physical system changes the system being measured.
Simple knowledge Recall a specified response (not multistep) to a specified stimulus.	State the definition of a partial derivative of a function of two variables.
Comprehension and Communication Extract and interpret meaning, use the language of mathematics.	Explain that a function changes most rapidly where the level curves are close together and recognize that functions are constant along a level curve. Incorporate the following phrases into a working vocabulary: contour plot, trace, level curve, multivariable function, isocline, surface, partial derivative. Integrate quantitative findings and mathematical formulae in a written report of experimental results.
Algorithmic skill Recall and execute a multistep procedure.	Use the appropriate commands to make surface plots from tabular data using Mathematica or MATLAB.
Application Use deductive reasoning, decide if at all mathematical content is relevant.	Given a topographic map, estimate partial derivatives. Decide which mathematical tools and information are relevant in the context of the data and broader scientific narrative.
Creative thinking Use divergent reasoning to view mathematical content in unusual, novel ways.	Create their own notation for expressing a multivariable function. Combine and remix single variable functions into multivariable functions to approximate surfaces.
Appreciation Believe mathematical content has value.	Believe that functions of several variables and rates of change have value beyond the current course. Articulate strengths and weaknesses of mathematical models and the challenges inherent in handling real-world data.
Willingness to try Choose to attempt a mathematical task.	Choose to attempt to compute a partial derivative of a given function of 3 variables (which has not yet been covered in class) based on an extension of ideas from the functions of two variables in the lab.

2.2 Lab Materials and Setup

Students work in groups of 3-4. In addition to hot brewed coffee and refrigerated whole milk, each team needs the following materials pictured in Figure 2:

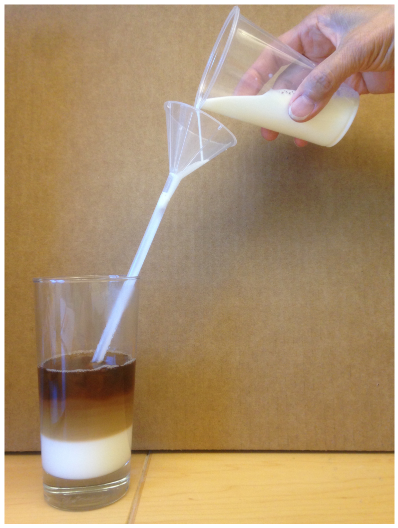
- 1 tall glass
- 1 plastic funnel
- 1 straw
- 1 plastic cup
- 1 ruler, 2 rubber bands
- 1 lab thermometer
- 1 timer (or smart phone)
- paper towels for cleanup

Figure 2. Lab materials.

We instruct the students to attach a ruler to the glass using rubber bands as shown in Figure 3b), so that the zero mark lines up with the bottom of the glass. They fill the glass with 6 cm of coffee, and use the funnel and the straw to pour 6 cm of milk under the layer of coffee. To do this, students put the funnel into the top of the straw and the straw in the coffee so that it touches the bottom of the glass. Then they use the plastic cup to pour the milk into the funnel very slowly to avoid mixing.

Once poured, particular care must be taken to remove the straw slowly. Though we try to avoid mixing, students can observe some interesting turbulent flows of the milk that mixes with the coffee. The pouring process results in the layered system shown in Figure 3b). It is surprising to students that this actually works and that the resulting layers remain quite stable over time.

2.3 Data Collection and Graphing



(a) Pouring the milk under the coffee.

(b) The finished lab setup.

Figure 3. Creating the layered system of coffee and milk for the Coffee to Go! Lab.

With lab thermometers students measure temperatures of the column at 1 cm increments every 5 minutes over a period of 20 minutes. We use thermometers with a sensing bulb of approximately 1 cm in length, yet we still get fairly clean and consistent data. Students can see the thermometer through the side of the glass and wait for the temperature reading to stabilize at each height before recording. Moving the thermometer creates some disturbance, but moving it slowly largely avoids mixing.

An example of data collected by students is in Table 2. After gathering data, students are prompted to visualize the data in the form of a graph. This creates a new challenge since they are used to seeing data tables with only two variables. They aren't sure what to do with all the columns of information. At CC there was time for the instructor to use the computer lab with the students to plot the data with Mathematica. At USU students collected data during class and were provided with MATLAB commands for creating graphs outside of class. In both settings students met with their groups in the computer lab to work collaboratively. Figure 4 shows a graph of the data created by students in Mathematica. While textbook exercises often fail to motivate students to use computer algebra systems since the graphing of a few data points would be easier by hand, here the students are interested in rendering their own data. The lab data creates a need for computer algebra systems to visualize a data set involving three variables.

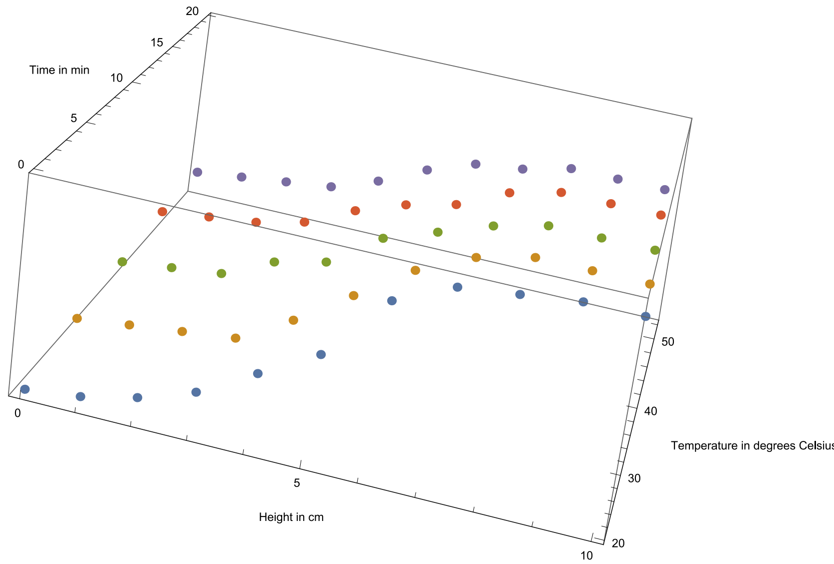
Table 2. An example data set collected by students.

Temp (°C)	Time (min)				
Height (cm)	0	5	10	15	20
0	20	22	23	24	24
1	21	23	24	25	25
2	23	24	25	26	26
3	26	25	29	28	27
4	31	30	31	32	30
5	36	36	37	35	34
6	46	42	40	37	37
7	50	46	43	41	38
8	51	48	45	43	40
9	52	48	45	43	40
10	52	48	45	43	40

2.4 Mathematical Concepts and Modeling Issues

As stated in our objectives, there are two main goals – to introduce important mathematical concepts and relationships, and to stimulate student thinking about challenges and issues in authentic modeling. The mathematical purpose of this lab is to construct the concept of multivariable functions and introduce calculus topics. At the same time, students must learn to carefully distinguishing between the data and the model. We want students to have a clear sense that the points in

Figure 4. An example of a graph drawn by students in Mathematica.



the graph represent data, and not the model. Intuitively temperature changes continuously in time and space, hence we assume an underlying continuous process. The data points show what was sampled. Students may be prompted to choose a time between the times at which data was recorded and guess the temperature at that time. They may justify their guess with an interpolation argument. We may use Mathematica to interpolate and create a continuous representation of the data.

The graphs of functions of two variables are surfaces in 3-space. Figure 4 shows a graph of the temperature data $T_D(x, t)$ where x refers to the height of the fluid in the glass measured in cm, and t denotes time in minutes. To get a better idea about the shape and interpretation of the surface, we ask students to plot a vertical cross section. We formally define a *vertical trace* as the intersection of the graph with a vertical plane parallel to one of the coordinate planes.

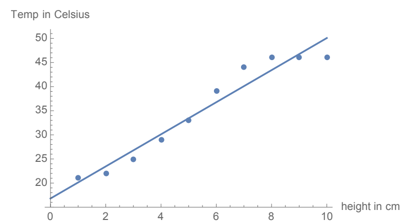
Next, we prompt students to think about what each *horizontal trace* represents physically, namely, the set of all points where the temperature is the same. A horizontal trace, once projected into the xt-plane, is called a *level curve*. If we choose a contour interval then the set of level curves is called a *contour plot*. We prompt students to draw contour plots by hand first to direct their thinking to the process. Some students notice that two different level curves of a function cannot intersect. If they did intersect, an input value would have two distinct output values, which violates the definition of a function.

Drawing on the students' experience with rates of change of a function of one variable we get them to think about rates of change of a function of two variables. We ask them to identify at which points the rate of change of the temperature with respect to time (or height) is the greatest.

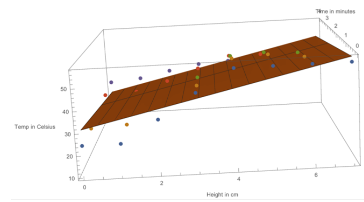
At this point, we have an introductory discussion about modeling. We are looking for a mathematical function that describes the data. We are not looking for a function that passes through every data point. A model is a mathematical description that shares and explains the prevailing shape of the data, explaining some but not all aspects of the data well. Thus every model has strengths as well as weaknesses, of which we should be aware.

Drawing on their experience with data depending on one independent variable, we ask the students fix a time t_0 and consider the corresponding cross section. For example, a line of best fit is a model for the temperature as a function of height at time t_0. The students plot both the data and the model in the same coordinate system and see that the line of best fit describes the increasing trend of the temperature with height (a strength), but not necessarily the S-shape of the data (a weakness, see Figure 5a)).

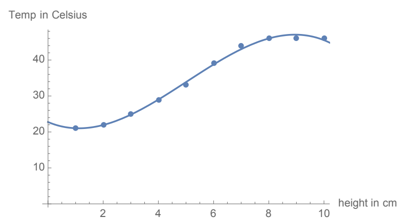
We extend these ideas to our data set that depends on two variables by letting the students figure out that a plane may be a good starting point for modeling the data. The students plot the plane of best fit and the data in one coordinate system and see that it captures some trends in the data, but overall is not a very good fit (see Figure 5b)). Students experiment with the Mathematica Fit command to explore how curved surfaces (polynomial functions) produce a better fit to the data (see Figure 5c),d)).



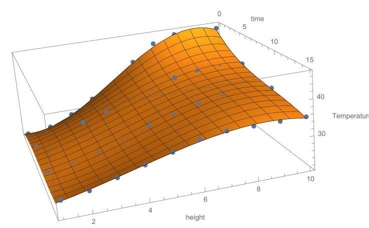
(a) A line of best fit captures the increasing temperature, but not the S-shape of the graph.



(b) A plane of best fit as a 3D analog of the line of best fit.



(c) A cubic function captures the S-shape of the graph.



(d) A polynomial of degree 3 in two variables.

Figure 5. Students draw on their experience with 2D graphs when thinking about multivariable functions.

A mechanistic model using partial differential equations is available on the LEMB website `http://www.digitalcommons.usu.edu/lembs`. This includes class materials that allow instructors to take students through a complete cycle of the modeling process.

2.5 Pedagogical Strategies

We suggest using the pedagogical practices outlined in [16] to implement the lab activity in a way that maintains high levels of cognitive demand. Below are some examples of strategies and questions which instructors may use to guide student learning.

- Scaffold student thinking: Ask questions and minimize direct instruction to give students a chance to do the work and think themselves. *What are the units? Can you sketch your ideas?*

- Sustained press for justification: When students make observations and discoveries, encourage mathematical and deductive explanations. For example, when creating and interpreting contour plots, students discover that while level curves appear to be closer or farther apart from one another, they never intersect. *Can you explain why? How do you make sense of that? Explain the reasoning there.* Some students notice that at a saddle point level curves appear to intersect. *Why does this happen? Does this contradict what we said earlier?*

- Models of high-level performance: Provide example reports from previous years or different labs. Analyze and critique mathematical model descriptions, assumptions, or figures and captions from these reports to clarify expectations for writing. Call students to the board to share creative thinking, interesting observations, and alternate model approaches.

- Sufficient time to explore: Allow time for data gathering and small group discussion. Make time for whole class reflection on the experiment and results.

- Tasks build on prior knowledge: When leading students to describe and interpret traces and level curves, instructors may ask students to think of real-world examples where cross sections can be seen or are important (e.g. a sliced apple, an MRI, 3D-printed objects). *Does this remind you of something?*

- Students have means to monitor their own progress: Provide clear lab instructions and rubrics for evaluation. Instructions can include a time table to help students manage their time, or include instructions such as "be sure you can explain the following before moving on." Another good classroom practice is for students to keep an experimental journal, so they can compare their experiences over time.

A summary of the instructional procedures and some recommendations for implementation are given in Table 3.

Table 3. Summary of the instructional procedures and our recommendations for implementation.

Instructional Procedures	Recommendations
1. Introduction of the Activity Introduce the scientific background related to this lab, and the goal of the activity.	Engage students in a discussion and ask students to make predictions about how the system will behave.
2. Data Gathering Students work in groups to gather and organize their data.	Allow students to get comfortable with the equipment and some flexibility in the way they chose to collect data keeping the goals of the procedure in mind.
3. Graphing the Data Students visualize and imagine their data extending beyond discrete points to a continuous surface. This is often the first time they have interpreted a surface plot with one spatial and one temporal dimension.	Here we have pressed students to first come up with their own notations and represenations for multivariable functions, then we have provided a direct lesson in how such data is graphed using tools like MATLAB or Mathematica.
4. Introductory Modeling Students are challenged to extend what they know about graphs of functions of a single variable to creatively come up with a two-variable function for temperature.	Through discussions encourage creative thinking, and follow through with student ideas. Using the fit command in Mathematica or MATLAB, have students fit a plane and then higher order polynomials to the data.
5. Extract Mathematics Students encounter questions on their task sheet that require thinking about traces of the surface, and rates of temperature change with respect to each variable.	Reflect on data to introduce partial derivatives. Introduce and explain conventional notation with reference to the lab experience.
6. Written Reports Students show evidence of their process and thinking throughout the project.	Give clear instructions on structure and content of the lab report. Provide students with clear guidelines for how their work will be assessed.

3 ASSESSMENT

3.1 Assessment Design Principles

In order to maintain high levels of cognitive demand, it is important that the level of cognition in assessment items align with the level of cognitive demand that was aimed for in class. If cognitive demand in class was high (e.g. application or creative thinking), yet the level of cognition in assessment items remains low (e.g. simple knowledge or algorithmic skill), students tend to perceive the conceptual material from class as less relevant and will focus on tasks with low cognitive demand instead [CITATION HERE (Cangelosi?)]. Below are some examples of assessment items illustrating along with their levels of cognition (compare to Table 1).

Simple knowledge: Given the function $f(x, y)$, state the definition of the partial derivative of f with respect to x.
Comprehension and communication: Explain in your own words what a level curve of a function is.
Application: You are a skier on the top of a mountain. Given the topographic map below (with two possible paths down), which path would you choose? Explain why.
Creative thinking: In class we learned that the level curves of a function of two variables $f(x, y)$ are curves in a plane. Given a function of three variables $f(x, y, z)$, give a geometric description of $f(x, y, z) = k$. What do the level curves of a function of one variable look like?
Willingness to try: Recall what you learned in class about taking partial derivatives of functions of two variables, and extend these ideas to find the partial derivative with respect to x of the function $f(x, y, z) = 3x^2yz$.

3.2 Writing Assignment

In addition to the assessment items in the previous section, we take a project-based assessment approach by creating a writing assignment requiring the students to communicate their results in a lab report. The lab reports are useful in two ways: To help assess student understanding of the concepts and provide an opportunity for students to practice communicating mathematics. The assessment becomes a learning experience itself as students are required to reflect on their findings, interpret, and organize them [1]. Instructions for this assignment are included in the student task sheet in Appendix A. By reviewing the written reports, we gathered qualitative data to inform us what the students had learned and identified concepts that students tend to struggle with. For example, some hand-drawn contour plots show intersecting level curves. Some students write that they "fit the data to the model," which provides us with an opportunity to point out that it was the model that was tweaked to fit the data. We did not make changes to the data to fit the model. A careful assessment of the lab reports allows instructors to give students detailed feedback on their work and point out common misconceptions.

3.3 Assessment of Lab Effectiveness

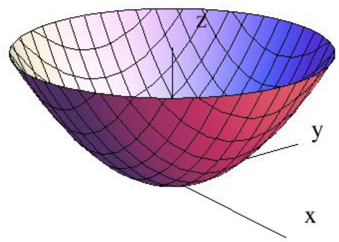
We assessed the effectiveness of the Coffee to Go! instructor support materials lab through pre- and post-tests.

At CC, 27 Calculus 2 students took the pre- and posttest, 24 of whom were in their first year of college, and 1 was in their second, third, and fourth year, respectively. The class comprised 19 male and 8 female students. The pretest was not returned to the students. Every question on the pre- and posttests were graded for correctness out of 4 points.

The following pre- and posttest was given at CC directly before working on the lab in class and after the lab reports were turned in one week later. The pretest was not returned to the students.

1. Give two examples of functions of more than one variable arising in real-world contexts. What are the independent and dependent variables of each function?

2. Consider the following graph. Sketch the trace in the $y, z-$ plane.



3. Explain how the level curves indicate the steepness of a graph of a function of two variables.

4. How useful (if at all) do you think will mathematical models be in your future career? Circle one:

   not at all useful = 0   1   2   3   4 = extremely useful

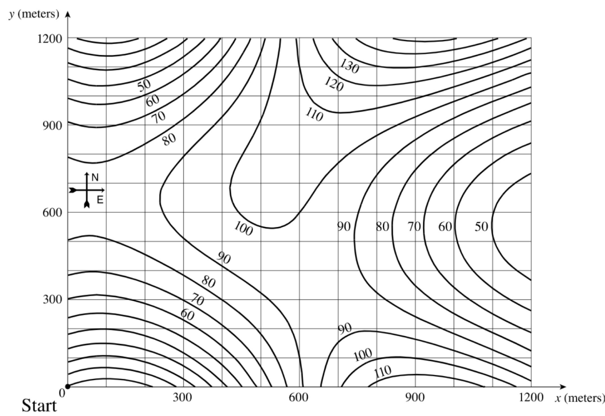
   Explain your response.

At USU, 34 students (29 male, 5 female) completed the semester of Calculus 3: 14 sophomores, 16 juniors and 4 seniors. 43 students took the pretest and 40 took the posttest. Each question on the pre- and posttests were scored out of 4 points. Responses to question 1 received up to 2 points for indicating two examples of multivariable functions, and up to 2 points for correctly identifying dependent and independent variables. For question 2, 2 points were possible for reasonable estimates of the partial derivatives based on the graph, and 2 points were awarded for a correct physical interpretation.

The following pre- and posttest was given at USU directly before working on the lab in class, and after the lab reports were turned in one week later. The pretest was not returned to the students.

1. Provide two examples of multivariable functions arising in real-life contexts. What are the dependent and independent variables in each case?

2. The figure below is a map showing curves of the same elevation of a region in Orangerock National Park ([14], p.738). We define the altitude function $A(x, y)$ as the altitude at a point x meters east and y meters north of the origin ("Start").



   (a) Estimate $A_x(300, 300)$ and $A_y(300, 300)$.

   (b) What do A_x and A_y represent in physical terms?

3. How useful (if at all) do you think reasoning with limits and partial derivatives will be in your future career? Circle one:

   not at all useful = 0   1   2   3   4 = extremely useful

   Explain your response.

4 RESULTS

4.1 Pre- and Posttest Results

The pre- and posttest results from Colorado College are summarized in Table 4.

Table 4. Class mean scores out of 4 points for questions 1-3 in % on the pre- and posttests at Colorado College, rounded to the nearest percentage point. The response to the appreciation question 4 is the average on a scale from 0 to 4.

	Pretest	Posttest	Gain Post − Pretest
Question 1	46%	83%	37%
Question 2	34%	80%	46%
Question 3	0%	85%	85%
Question 4	3.15	3.04	-0.11

Not surprisingly, some students were able to answer Question 1 on examples of models correctly on the pretest. Since some students may have had experience with multivariable calculus in high school, some students answered the procedural Question 2 about the trace of a function correctly on the pretest. No one answered the conceptual Question 3 correctly on the pretest. We see significant gains in the responses to all three questions. Question 3 shows the greatest gain (85% post vs. 0% pre lab) after discussing level curves in class in addition to the lab questions about level curves. Most students made the connection between the steepness of the graph and the spacing between the level curves. Overall, the average score increased from 27% on the pretest to 83% on the posttest. The only score to decrease slightly is the appreciation score in Question 4 (3.15 pre, 3.04 post lab), likely because the students already had a high appreciation of mathematical models before engaging in the lab.

The pre- and posttest results from USU are summarized in Table 5. At USU the most dramatic gains were observed in question 1, where students were able to show their concept of functions of many variables. Modest gains were shown on question 2, however many students still had difficulty making estimates of partial derivatives from graphical information. The notation may have been a point of confusion for students.

Error analysis showed that many incorrect responses resulted from students reporting elevation rather than elevation changes in response to question 2a).

Table 5. Class mean scores out of 4 for questions 1-2 on the pre- and posttests at USU in %, rounded to the nearest percentage point. The response to the appreciation question 3 is the average on a scale from 0 to 4.

	Pretest	Posttest	Gain Post − Pretest
Question 1	34%	73 %	39%
Question 2	19%	44%	25%
Question 3	3.68	3.38	-0.30

4.2 Review of Student Reports

The Coffee to Go! lab clearly sparked student interest. One group wrote "The milk/coffee lab experience was a very good one! The coolest part for me was at the very beginning because you could see the thick white and black layers, and a gray layer where both had combined." In our experience, the lab helps students understand that natural processes can be modeled using mathematics: "This experiment has helped me understand how real life situations may be expressed in mathematical equations. Maybe someday I'll find a better way to model this experiment." The lab reports often show how students make connections between the experiment and the reading about winter mixing in deep lakes. One group wrote: "If the experiment were carried out for longer, the temperature of the entire system would reach an equilibrium. [...] Comparing these observations to the reality of lake warming and mixing, this experiment represents the time of year when the temperature begins to cool."

The writing assignment presents an opportunity for instructors to return to some common misconceptions in class. Some reports show intersecting level curves of functions, to which we responded by posing the question about whether level curves can intersect in class. One group wrote "As

$t \to \infty$ we expect the overall temperature of the mixture to reach 32 degrees Celsius, which was the temperature at the middle of the mixture," showing that they had not thought about the fact that the experiment was carried out at room temperature.

The reports reveal that some groups experimented not only with polynomial models, but also with exponential and logarithmic functions. One group used a shifted arctangent function to model the temperature at a fixed height because it had the right end behavior as time tends to infinity. One report showed a dimensional analysis approach to modeling, by including units in the variables and seeking to match units in equations. Initially, many students are uncomfortable with the idea that the model may not pass through every data point. In the lab reports, we see that students gain a sense that models are approximations and shift their focus to discussing strengths and weaknesses of their models. "The biggest strength of this model is how closely it fits the data. All [data] points ... fall within one degree of our model. Our model has one issue though, which is that it does not accurately describe the end behavior of our graph." Almost all groups tried to improve their first models, and some students ask whether there are other types of models that can be used to produce a better overall fit, which gave us an opportunity to discuss mechanistic models in class.

5 DISCUSSION

Most textbooks present the theory and models first and then tack on applications, giving the impression that the model comes first and then must be applied to given data. However, practitioners begin with the data and then develop a model to fit the data. Illustrating this process, the Coffee to Go! lab provides an open-ended, authentic lab experience for students of multivariate calculus. By collecting data themselves, students are more invested in the activity and more motivated to think deeply about the mathematical and scientific concepts involved. In the framework of the pedagogical strategies that we have outlined, instructors maintain high levels of cognitive demand and develop a culture of inquiry with their students where the focus is on reasoning instead of memorization.

While the project-based approach may take more time than traditional lecture the data collection or graphing can be done outside of class, encouraging students to engage with the material beyond the classroom. The lab may require instructors to re-think their pedagogical strategies, and we hope that our class and instructor materials will help save time in class preparation.

Here we used the lab as an introductory data and mathematical modeling experience and did not take the students through a complete modeling cycle. Instructors may opt to do so by creating models using ordinary differential equations or partial differential equations as outlined on our website `http://digitalcommons.usu.edu/mathsci_edures/1/`. The lab is designed to be adaptable for different classes ranging from calculus to a modeling class or differential equations courses. It is also a good fit for a numerical analysis course, as some interesting numerical issues arise when a PDE model is solved numerically with the appropriate boundary conditions.

The pre- and posttest results show the effectiveness of the Coffee to Go! lab. Beyond the mathematical concepts of traces and level curves, students gain an understanding of where functions of more than one variable are used in real-world contexts. Furthermore, the lab report as a writing assignment offers an opportunity for students to revisit the data collection and models and practice communicating mathematical results. In our experience, the Coffee to Go! data (and data collected by students in general) generates a need to learn mathematical concepts to describe and model the data, and thus students are more motivated to "learn what it takes" to analyze and model the data. To many students, who may have previously viewed mathematics as an abstract logical game and computer labs as useless busywork, mathematical ideas and computer algebra systems become powerful tools to make sense of their data.

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REFERENCES

[1] Bahls, P., 2012. Student Writing in the Quantitative Disciplines: A Guide for College Faculty. Jossey-Bass, San Francisco, CA.

[2] Board on Life Sciences, Division on Earth and Life Studies, National Research Council of the National Academies, 2003. BIO 2010: Transforming Undergraduate Education for Future Research Biologists. National Academies Press, Washington D.C.

[3] Cangelosi, J. S. and Parkinson, S., 1999. Leading students to use their mathematical skills to describe biological phenomena (Episode 1: Questioning strategies for student discovery). Washington: U.S. Department of Education; Logan: Utah State Multimedia and Distance Learning Services.

[4] Cangelosi, J., 2000. Assessment strategies for monitoring students learning. New York: Addison-Wesley/Longman.

[5] Cangelosi, J., 2003. Teaching mathematics in secondary and middle school: An interactive approach (3rd ed.). Upper Saddle River, NJ: Prentice-Hall.

[6] Committee on Developments in the Science of Learning, 2000. How People Learn: Brain, Mind, Experience and School. The National Academies Press, Washington, D.C.

[7] Kohler, B.R., Swank, R., Haefner, J. W., and Powell, J. A., 2010. Leading Students to Investigate Diffusion as a Model of Brine Shrimp Movement. Bulletin of Mathematical Biology, 72, 230–257.

[8] Lewis, M., Modeling zombie outbreaks: A project-based approach to improving mathematics one brain at a time, PRIMUS, 2015.

[9] Malek, E., 2006. Meteorological and environmental aspects of one of the worst national air pollution episodes (January 2004) in Logan, Cache Valley, Utah, USA, Atmospheric Research, 79, 108–122.

[10] Pope and Dockery, 2006. Health Effects of Fine Particulate Air Pollution: Lines that Connect. J. Air Waste Manag., 56(6), 709–742.

[11] Powell, J.A., Kohler, B.R., Haefner, J.W., and Bodily, J., 2012. Carrying BioMath Education in a Leaky Bucket. Bulletin of Mathematical Biology, 74(9), 2232–2264.

[12] Powell, J. A., Cangelosi, J., and Harris, A. M., 1998. Games to teach mathematical modeling. SIAM Review 40, 87–95.

[13] Schoenfeld, A. H., 1992. Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. *Handbook of research on mathematics teaching and learning*, 334–370.

[14] Shaw, D., et al., 2005. Instructors Guide for Stewart's Calculus Concepts and Contexts 3E. Brooks/Cole Thomson Higher Education, Belmont, CA.

[15] Snee, R. D., 1993. What's missing in statistical education?. *The American Statistician*, 47(2), 149–154.

[16] Stein, M.K., et al, 2009. Implementing Standards-Based Mathematics Instruction: A Casebook for Professional Development, 2nd edition. Teachers College Press, New York, NY.

[17] Winder, M., 2012. Lake warming mimics fertilization, Nature Climate Change, Vol. 2, 771–772.

APPENDIX A: Student Task Sheet

Instructions: Discuss in your lab group the mechanism by which winter mixing occurs in deep lakes under healthy conditions as described in the article "Lake warming mimics fertilization" from the journal Nature Climate Change which is posted on the course Canvas page.

Discuss how climate change affects winter mixing and how this contributes to an increased growth of cyanobacteria. Why is this a problem?

Scientists use mathematical models to describe, explain, and predict natural phenomena. To gain a sense of what scientists do who model aspects of the real world, we will create a layered system of cold milk and hot coffee, collect data on its temperature, and describe the data set mathematically (i.e. *model* the system) by finding a function of space and time that resembles the data. Your group should work collaboratively and must submit a two page report on this lab and your results.

1. Experimental setup: Your group will need the following materials: 1 glass, 1 funnel, 1 plastic cup, refrigerated condensed milk, hot coffee, 1 lab thermometer, timer (use your phone). Fill the glass with 6cm of coffee. Use the funnel and the plastic cup to pour 6cm of milk under the layer of coffee. This has to be done slowly to avoid mixing. Very slowly remove the funnel.

2. Collect data: Once both layers are poured, start the timer. Every 5 minutes, measure the temperature at different heights (1cm intervals work well) from the bottom of the glass to the top. Move the thermometer slowly to avoid mixing. Continue making measurements until you have collected at least 20 minutes of data. Be sure to record the data. Before you clean up, measure and list any parameters that may be important.

3. Discuss your observations in your group. Describe the overall phenomenon you observed. How do your observations relate to winter mixing in lakes?

4. In the computer lab, download the file "coffee and milk lab Mathematica" from the course Canvas page. Follow the instructions in the file to plot your data and join the data points. All axes must be clearly labeled and include units of measurement.

(a) Describe the shape of the graph. What are the independent and dependent variables? Introduce some notation to refer to this function.

(b) What does each vertical cross section of the graph parallel to the time axis represent?

(c) What does each vertical cross section of the graph parallel to the height axis represent?

(d) What does each horizontal cross section represent? By hand, draw a sketch of at least 5 level curves. Be sure to label the axes. What are the units on the "height" of the level curves?

(e) At what point is the rate of change of the temperature with respect to time the greatest?

(f) At what point is the rate of change of the temperature with respect to height the greatest?

(g) Choose and fix a certain time. Find a function that (roughly) describes the temperature at every height at this time. Introduce some notation to refer to this function. How many variables does it depend on? Use Mathematica's Plot and ListPlot commands to graph your function and the data on one graph. Be sure to label the axes. We call such a function a model: it describes approximately how the temperature behaves at any height. The model is not required to match the data perfectly. Every model has strengths and weaknesses that we should be aware of. Explain what the strengths and weaknesses of your model are.

(h) What do you expect to happen after a long time, or mathematically speaking, when time goes to infinity?

(i) Choose and fix a certain height. Find a model for the temperature at every time at this height. Introduce some notation to refer to your function. How many variables does it depend on? Use Mathematica's Plot command to graph your model and the data. Be sure to label the axes. Explain what the strengths and weaknesses of your model are.

(j) How many variables does a function have that models the temperature at any height and time? What is the shape of a simple function that does this? Describe what you have to do to improve the model.

(k) Based on your model and the Nature Climate Change article, what consequences will there be for the growth of cyanobacteria in deep lakes when temperatures rise and our winters are no longer as cold as they used to be?

5. Lab report: Your lab report should contain the following sections.

   (a) Cover sheet with the title of the experiment, the date the experiment was performed, your names, and a written statement of the Honor Code, signed by all group members.

   (b) 1-2 paragraphs with some background information about winter mixing in deep lakes.

   (c) Experimental setup: List the materials needed, and describe the procedure in your own words.

   (d) Data: Include a table with your raw data. Be sure to use headers for your rows and columns. Include your graphs. You may attach your Mathematica printouts to your paper.

   (e) Results: Include your answers and detailed explanations to the questions in #3 and #4. Refer to the graphs in the previous section.

   (f) Discussion: Discuss the strengths and weaknesses of your model, and what changes you might make to improve its fit to the data. Generally speaking, in what ways would a model for winter mixing be useful to scientists studying winter mixing in lakes under changing climate conditions?

APPENDIX B: Plotting Data With Mathematica and MATLAB

Listing 1. Example Mathematica code

```
You may make changes to this Mathematica notebook to visualize your data
      from the coffee and milk lab. Note that Mathematica is a
      sophisticated calculator that requires a specific syntax. For example
      , lists of points are always entered in curly brackets { }, and the
      coordinates of each point are entered in its own set of curly
      brackets, see the lists of example data points below. Mathematica
      commands are always capitalized - if we forget to capitalize a
      command, Mathematica will not recognize it and return an error
      message.

Enter your data points in the form of a list of points, such as:

list1 = {{0, 0, 24}, {1, 0, 24}, {2, 0, 34}, {3, 0, 45}};
list2 = {{0, 1, 24}, {1, 1, 25}, {2, 1, 34}, {3, 1, 42}};
list3 = {{0, 2, 25}, {1, 2, 26}, {2, 2, 33}, {3, 2, 40}};
list4 = {{0, 4, 28}, {1, 4, 29.5}, {2, 4, 33}, {3, 4, 36}};

The ListPointPlot3D command creates a 3D graph of your data points in the
      lists. You may add lists and change the axes labels to reflect the
      quantities and units in your data:

plot1 = ListPointPlot3D[{ list1 , list2 , list3 , list4 }, PlotStyle ->
  PointSize[Large], AxesLabel -> {"Distance", "Time", "Concentration"}]

In order to approximate what happens in between the data points, you may
      use the ListPlot3D command and copy and paste all of your data points:

plot2 = ListPlot3D[{{0, 0, 24}, {1, 0, 24}, {2, 0, 34}, {3, 0, 45}, {0, 1,
      24}, {1, 1, 25}, {2, 1, 34}, {3, 1, 42}, {0, 2, 25}, {1, 2, 26}, {2,
      2, 33}, {3, 2, 40}, {0, 4, 28}, {1, 4, 29.5}, {2, 4, 33}, {3, 4,
      36}}, Mesh -> None]

The Show command overlays both graphs. Click on the graph to rotate and
      look at it from different angles:

Show[plot1, plot2]

Below is some code that will help with graphing 2D data. Use the ListPlot
      command to plot data points in 2D:

plot3 = ListPlot[{{1, 2}, {2, 4}, {3, 1}, {4, 7}}, AxesLabel->{"x", "y"}]
```

```

To graph a 2D function, for example f(x)=2x+1, use the Plot command. Here
    , we graph the function on the interval [−5,7] and label the axes:

plot4 = Plot[2*x + 1, {x, −5, 5}, AxesLabel −> {"x", "y"}]

Show[plot3, plot4]
```

Time elapsed

Listing 2. Example MATLAB code

```
% Commands for Staying Cool (Problem from Shaw et al, 2005) MATLAB
    demo

% Set up the domain. This creates matrices for x and y, so that x ranges
    from 0 in steps of 0.5 up to 10, and y ranges from 0 in steps of 1 up
     to 10.
[x,y]=meshgrid(0:0.5:10, 0:1:10);

% The next command creates a matrix (T1) fo the T values for exercise 1.
    The dots before the carrots are important, and they tell MATLAB to
    square the entries of the values in the x matrix. That's different from
    squaring the x matrix with matrix multiplication which is what
    MATLAB would try to do if the dot is left out.
T1=78 − 1/10*(x.^2 + (y−5).^2);

% We do a similar computation for exercise 2.
T2=1/2*x−y+75;

% And now the plotting commands:
figure(1) , surf(x,y,T1)
figure(2), contour(x,y,T1)
figure(3), surf(x,y,T2)
figure(4), contour(x,y,T2)

% The figure command tells MATLAB which figure window to use, the surface
    plot is made with surf, and the contour plot is made with contour.

% For our coffee/milk set up, we have to set up the meshgrid to reflect the
     time and height variables .

[t,x]=meshgrid(0:5:20, 1:1:8)
```

```
Data=[19 23.5 23.0 24 25;
     21.5 25 25.5 27 27.5;
     30.5 32 32.5 32.5 32;
     44 41 41 39.0 37;
     55.5 52 47.5 44.5 41;
     60 54 49 46 43;
     60 54 49 45.5 43;
     54 52.5 47.5 44.5 42];

% model=exp(-t).*x.^3 +22;

subplot(2,2,1)
plot3(t,x,Data, '.','Markersize',3)
xlabel('Time elapsed (min)')
ylabel('Height from bottom of glass (cm)')
zlabel('Temperature')


%
% figure
% surf(t,x,model)
% xlabel('Time elapsed (min)')
% ylabel('Height from bottom of glass (cm)')
% zlabel('Temperature')


figure
contour(t,x,Data)
xlabel('Time elapsed (min)')
ylabel('Height from bottom of glass (cm)')
```