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Lee Samuel Finn, Shane L. Larson and Joseph D. Romano, "Detecting a Stochastic Gravitational-Wave Background: The Overlap Reduction Function", Physical Review D, 79, 062003 (2009)

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PHYSICAL REVIEW D **79,** 062003 (2009)

Detecting a stochastic gravitational-wave background: The overlap reduction function

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(Received 21 November 2008; published 30 March 2009)

Detection of a gravitational-wave stochastic background via ground or space-based gravitational-wave detectors requires the cross correlation of the response of two or more independent detectors. The cross correlation involves a frequency-dependent factor—the so-called *overlap reduction function* or *Hellings-Downs curve*—that depends on the relative geometry of each detector pair, i.e., the detector separations and the relative orientation of their antenna patterns (beams). An incorrect formulation of this geometrical factor has appeared in the literature, leading to incorrect conclusions regarding the sensitivity of proposed detectors to a stochastic gravitational-wave background. To rectify these errors and as a reference for future work we provide here a complete, first-principles derivation of the overlap reduction function and assess the nature of the errors associated with the use of the incorrect expression that has appeared in the literature. We describe the behavior of the overlap reduction function in different limiting regimes, and show how the difference between the correct and incorrect expressions can be understood physically.

DOI: 10.1103/PhysRevD.79.062003 PACS numbers: 04.80.Nn, 04.30.−w, 95.55.Ym

I. INTRODUCTION

The measured response of a single gravitational-wave detector to a stationary stochastic gravitational-wave signal is indistinguishable from unidentified instrumental noise. The gravitational-wave contribution to the measured response of two or more independent detectors will, however, be correlated between detector pairs in ways that other technical noises will not. The relationship between the power in a stochastic gravitational-wave background and the cross-correlated response of a detector pair depends on the response of the individual detectors and their relative geometry, i.e., their separation and the relative orientation of their respective detector antenna patterns, or beams. In the context of ground or space-based laser interferometric detectors [1–7] or resonant acoustic gravitational-wave detectors [8] this geometrical factor is referred to as the *overlap reduction function* [9–13]; in the context of pulsar timing arrays [14] or spacecraft doppler tracking [15] it is called the *Hellings-Downs curve* [16].

Incorrect expressions for the overlap reduction function have appeared in the recent literature [17,18] and, with them, incorrect conclusions regarding the sensitivity of proposed gravitational-wave detectors to stochastic gravitational waves. These errors have lead to significantly flawed appraisals of the high-frequency sensitivity of the Big Bang Observer (BBO) to a stochastic gravitational-wave background, including spurious nulls in the frequency-dependent detector response and a reduced estimate of the signal-to-noise ratio as a function of the gravitational-wave power. To rectify these errors and as a reference for future work we provide here a complete, first-principles derivation of the overlap reduction function and assess the nature and physical interpretation of the errors associated with the use of the incorrect expression that has appeared in the literature.

II. THE OVERLAP REDUCTION FUNCTION

The *overlap reduction function* of a pair of gravitational-wave detectors is the collection of geometric factors, associated with the relative position and orientation of the detector pair that appear in the cross-correlation of the detector pair's response. Here, we derive an expression for the overlap reduction function by *deconstructing* the cross correlation, identifying those contributions that depend only on the radiation and those that depend only on the detectors, which are then identified as the overlap reduction function. This approach has the virtue of clearly illustrating the physical origins of the overlap reduction function and making less likely mistakes of the kind that may have led to the existing errors in the literature.

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A. Interdetector cross correlation

Consider two gravitational-wave detectors. A stochastic gravitational-wave "background" will manifest itself in a nonvanishing cross correlation between the measurements $m(t)$ made at the two detectors, calculated as a time average over the product of the measurements

$$C_T(\Delta t, t) = \frac{1}{T}\int_{-T/2}^{T/2} dt' m_1(t + \Delta t + t')m_2(t + t'). \quad (2.1)$$

The signature of a stochastic gravitational-wave signal $h_{ij}(t, \vec{x})$ is just the expectation value of $C_T(\Delta t, t)$ in the presence of $h_{ij}(t, \vec{x})$. Write the measurement $m_I(t)$ made at detector I as the sum of a noise contribution $n_I(t)$ and a signal contribution $r_I(t)$, corresponding to the detector response to $h_{ij}(t, \vec{x})$ [19]. Assume that the noise in each detector is independent and that there are no nongravitational-wave effects that might lead to a correlation in the measurements made at each detector. Under these assumptions the expectation value of the product $n_1(t + \Delta t)n_2(t)$ vanishes, implying that the expectation value of $C_T(\Delta t, t)$ is just the expectation value of the product $r_1(t + \Delta t)r_2(t)$

$$\bar{C}(\Delta t) \equiv \overline{C_T(\Delta t, t)} = \overline{r_1(t + \Delta t)r_2(t)}, \quad (2.2)$$

where the overbar denotes the expectation value, and we have assumed that the expectation value of $C_T(\Delta t, t)$ is independent of t. This is equivalent to assuming that the background is *stationary*, as we will describe in more detail in Sec. II C.

Turn now to the detector response. Gravitational waves are weak. Even the most sensitive detectors respond linearly to the local field $h_{ij}(t, \vec{x})$. Correspondingly, we write the detector response as a convolution, in time *and* space, of an impulse response function $R_I^{ij}(t, \vec{x})$ with the field $h_{ij}(t, \vec{x})$

$$\begin{aligned} r_I(t) &\equiv r_I(t, \vec{x}_I) \\ &= \int_{-\infty}^{\infty} d\tau \int_{R^3} d^3x h_{ij}(t - \tau, \vec{x}_I - \vec{x})R_I^{ij}(\tau, \vec{x}), \end{aligned} \quad (2.3)$$

where $\vec{x}_I$ is the spatial location of detector I about which its response $R_I^{ij}(t, \vec{x})$ is defined. Causality requires that $R_I^{ij}(t, \vec{x})$ vanishes outside the future light cone of $(0, \vec{0})$. Exploiting the convolution theorem we can also write

$$r_I(t) = (2\pi)^3 \int_{-\infty}^{\infty} df \int_{R^3} d^3k \tilde{h}_{ij}(f, \vec{k})\tilde{R}_I^{ij}(f, \vec{k})e^{i(2\pi f t - \vec{k}\cdot\vec{x}_I)}, \quad (2.4)$$

where

$$\tilde{h}_{ij}(f, \vec{k}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} dt \int_{R^3} d^3x h_{ij}(t, \vec{x})e^{-i(2\pi f t - \vec{k}\cdot\vec{x})}, \quad (2.5a)$$

$$\tilde{R}_I^{ij}(f, \vec{k}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} dt \int_{R^3} d^3x R_I^{ij}(t, \vec{x})e^{-i(2\pi f t - \vec{k}\cdot\vec{x})} \quad (2.5b)$$

are the field Fourier modes and detector transfer function. Note that for real h_{ij} and R^{ij}

$$\tilde{h}_{ij}^*(f, \vec{k}) = \tilde{h}_{ij}(-f, -\vec{k}), \quad (2.6a)$$

$$\tilde{R}_I^{ij*}(f, \vec{k}) = \tilde{R}_I^{ij}(-f, -\vec{k}). \quad (2.6b)$$

With the above representations of the detector response, we can express $\bar{C}(\Delta t)$ in terms of the detector response and the field

$$\begin{aligned} \bar{C}(\Delta t) = &\int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' \int_{R^3} d^3x \int_{R^3} d^3x' \\ &\times \overline{h_{ij}(t + \Delta t - \tau, \vec{x}_1 - \vec{x})h_{kl}(t - \tau', \vec{x}_2 - \vec{x}')} \\ &\times R_1^{ij}(\tau, \vec{x}), R_2^{kl}(\tau', \vec{x}') \end{aligned} \quad (2.7a)$$

or, equivalently,

$$\begin{aligned} \bar{C}(\Delta t) = (2\pi)^6 &\int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df' \int_{R^3} d^3k \int_{R^3} d^3k' \\ &\times \{e^{2\pi i(f-f')t}e^{2\pi i f\Delta t}\overline{\tilde{h}_{ij}(f, \vec{k})\tilde{h}_{kl}^*(f', \vec{k}')} \\ &\times [\tilde{R}_1^{ij}(f, \vec{k})e^{-i\vec{k}\cdot\vec{x}_1}][\tilde{R}_2^{kl}(f', \vec{k}')e^{-i\vec{k}'\cdot\vec{x}_2}]^*\}. \end{aligned} \quad (2.7b)$$

Note particularly how the detector location and the transfer function appear together in the combination $\tilde{R}_I^{ij}(f, \vec{k})e^{-i\vec{k}\cdot\vec{x}_I}$. The form of this combination will be critical when we come to understand the physical character of the errors made in earlier calculations of the overlap reduction function.

B. Plane-wave representation of stochastic signal

Focus attention on gravitational-wave fields $h_{ij}(t, \vec{x})$. These are conveniently represented as a superposition of plane waves

$$h_{ij}(t, \vec{x}) = \int_{-\infty}^{\infty} df \int_{S^2} d^2\Omega_{\hat{k}} e^{2\pi i f(t - \hat{k}\cdot\vec{x})}\mathcal{H}_A(f, \hat{k})\mathbf{e}_{ij}^A(\hat{k}), \quad (2.8)$$

where $\hat{k}$ is the unit vector direction of wave propagation, and $\mathbf{e}_{ij}^A(\hat{k})$ are the two orthogonal polarization tensors

$$2\delta^{AA'} = \mathbf{e}_{ij}^A(\hat{k})\mathbf{e}_{ij}^{A'}(\hat{k}), \quad (2.9a)$$

$$\mathbf{e}_{ij}^A(\hat{k}) = \mathbf{e}_{ji}^A(\hat{k}). \quad (2.9b)$$

Note that $\mathcal{H}_A^*(f, \hat{k}) = \mathcal{H}_A(-f, \hat{k})$ as a consequence of the reality of h_{ij}. The plane-wave field amplitudes $\mathcal{H}_A(f, \hat{k})$

are related to the field's Fourier modes $\tilde{h}_{ij}(f,\vec{k})$ by

$$\mathcal{H}_A(f,\hat{k})\mathbf{e}^A_{ij}(\hat{k}) = \begin{cases} (2\pi f)^2 \mathsf{H}^{(+)}_{ij}(f,\hat{k}), & f \geq 0 \\ (2\pi f)^2 \mathsf{H}^{(-)}_{ij}(f,-\hat{k}), & f < 0 \end{cases}, \tag{2.10}$$

where

$$\tilde{h}_{ij}(f,\vec{k}) = \mathsf{H}^{(+)}_{ij}(f,\hat{k})\delta(|\vec{k}| - 2\pi f) + \mathsf{H}^{(-)}_{ij}(f,\hat{k})\delta(|\vec{k}| + 2\pi f). \tag{2.11}$$

Here, we have introduced separate amplitudes $\mathsf{H}^{(\pm)}_{ij}$ for the positive and negative frequency solutions to the dispersion relations $|\vec{k}|^2 = (2\pi f)^2$ for a plane wave.

Using expansion (2.8), we can write the detector response $r_I(t)$ as

$$r_I(t) = \int_{-\infty}^{\infty} df \int_{S^2} d^2\Omega_{\hat{k}} \mathcal{H}_A(f,\hat{k})\mathcal{R}^A_I(f,\hat{k}) e^{2\pi i f(t-\hat{k}\cdot\vec{x}_I)}, \tag{2.12}$$

where

$$\mathcal{R}^A_I(f,\hat{k}) = (2\pi)^3 \mathbf{e}^A_{ij}(\hat{k})\tilde{R}^{ij}_I(f, 2\pi f\hat{k}). \tag{2.13}$$

The expectation value $\bar{C}(\Delta t)$ can also be written as

$$\bar{C}(\Delta t) = \int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df' \int_{S^2} d^2\Omega_{\hat{k}} \int_{S^2} d^2\Omega_{\hat{k}'} \times \{e^{2\pi i(f-f')t} e^{2\pi i f\Delta t} \overline{\mathcal{H}_A(f,\hat{k})\mathcal{H}^*_{A'}(f',\hat{k}')} \times [\mathcal{R}^A_1(f,\hat{k}) e^{-2\pi i f\hat{k}\cdot\vec{x}_1}][\mathcal{R}^{A'}_2(f',\hat{k}') e^{-2\pi i f'\hat{k}'\cdot\vec{x}_2}]^*\}. \tag{2.14}$$

Again make note of how the detector locations $\vec{x}_1$, $\vec{x}_2$ are associated with the respective transfer functions $\mathcal{R}^A_1$ and $\mathcal{R}^A_2$.

C. Stationarity, isotropy, and polarization correlations

The statistical properties of the stochastic signal are encoded in the expectation values of products of the gravitational field

$$\overline{h_{ij}(t,\vec{x})}, \qquad \overline{h_{ij}(t,\vec{x})h_{kl}(t',\vec{x}')}, \qquad \overline{h_{ij}(t,\vec{x})h_{kl}(t',\vec{x}')h_{mn}(t'',\vec{x}'')}, \cdots \tag{2.15}$$

Without loss of generality, we will assume that any nonzero mean has been absorbed in the background spacetime, so that $\overline{h_{ij}(t,\vec{x})} = 0$. Furthermore, for Gaussian-distributed fields, knowledge of the quadratic correlations will suffice as all higher-order moments can be constructed from these.

In our problem we expect that the gravitational-wave background is effectively stationary, i.e., that $\overline{h_{ij}(t,\vec{x})h_{kl}(t',\vec{x}')}$ depends on t and t' only through their difference $t - t'$. In terms of the plane-wave components $\mathcal{H}_A(f,\hat{k})$, this condition becomes

$$\overline{\mathcal{H}_A(f,\hat{k})\mathcal{H}^*_{A'}(f',\hat{k}')} = \mathsf{H}_{AA'}(f,\hat{k},\hat{k}')\delta(f-f'). \tag{2.16}$$

Thus, the different frequency components of a stationary stochastic background are statistically independent, but they can contribute differently to the cross-correlated power through the f dependence in $\mathsf{H}_{AA'}$. Note that the $\delta(f-f')$ factor in (2.16) eliminates the t dependence in $\bar{C}(\Delta t)$, cf., Eq. (2.14).

If the background is *isotropic*—i.e., the gravitational-wave specific intensity is independent of the direction of propagation $\hat{k}$—then the most general form of the quadratic expectation value of the plane-wave components $\mathcal{H}_A(f,\hat{k})$ is

$$\overline{\mathcal{H}_A(f,\hat{k})\mathcal{H}^*_{A'}(f',\hat{k}')} = \mathsf{H}_{AA'}(f,f',\hat{k}\cdot\hat{k}'), \tag{2.17a}$$

where $\mathsf{H}_{AA'}$ depends on $\hat{k}$ and $\hat{k}'$ only through the angle between them. If we further assume that the components corresponding to different propagation directions are statistically independent, then

$$\overline{\mathcal{H}_A(f,\hat{k})\mathcal{H}^*_{A'}(f',\hat{k}')} = \mathsf{H}_{AA'}(f,f')\delta^2(\hat{k},\hat{k}'), \tag{2.17b}$$

where $\delta^2(\hat{k},\hat{k}') \equiv \delta(\cos\theta - \cos\theta')\delta(\phi - \phi')$ is the covariant Dirac delta function on the two-sphere. This latter, more restrictive, condition is the definition of isotropy for gravitational-wave stochastic backgrounds typically assumed in the literature, e.g., [10–13].

Finally, if the background is *unpolarized*, by which we will mean that the different polarization components are statistically independent and contribute equally to the cross-correlated power, then

$$\overline{\mathcal{H}_A(f,\hat{k})\mathcal{H}^*_{A'}(f',\hat{k}')} = \mathsf{H}(f,f',\hat{k},\hat{k}')\delta_{AA'}. \tag{2.18}$$

Putting all these conditions together, we have that an unpolarized, stationary, isotropic stochastic gravitational-wave background satisfies

$$\overline{\mathcal{H}_A(f,\hat{k})\mathcal{H}^*_{A'}(f,\hat{k}')} = \mathsf{H}(f)\delta(f-f')\delta^2(\hat{k},\hat{k}')\delta_{AA'}. \tag{2.19}$$

Here, $\mathsf{H}(f)$ is a real-valued function proportional to the gravitational-wave energy density. This is, in turn, directly related to $\Omega_{\rm gw}(f)$, the ratio of the gravitational-wave energy density to the cosmological closure density [13]

$$\mathsf{H}(f) = \frac{3H_0^2}{32\pi^3}\frac{\Omega_{\rm gw}(f)}{|f|^3}, \tag{2.20}$$

where H_0 is the Hubble expansion rate at the present epoch.

D. The overlap reduction function

Combining the results of the previous subsections we find that we can express the expectation value $\bar{C}(\Delta t)$ of the interdetector cross correlation in the presence of an unpolarized, stationary, isotropic gravitational-wave back-

ground as

$$\bar{C}(\Delta t) = \int_{-\infty}^{\infty} df e^{2\pi i f \Delta t} \mathsf{H}(f) \Gamma_{12}(f), \tag{2.21a}$$

where

$$\Gamma_{12}(f) = \int_{S^2} d^2\Omega_{\hat{k}} \mathcal{R}_1^A(f,\hat{k}) \mathcal{R}_2^{A*}(f,\hat{k}) e^{-2\pi i f \hat{k}\cdot(\vec{x}_1-\vec{x}_2)}, \tag{2.21b}$$

$$\mathcal{R}_I^A(f,\hat{k}) = \mathbf{e}_{ij}^A(\hat{k}) \int_{-\infty}^{\infty} dt \int_{R^3} d^3x R_I^{ij}(t,\vec{x}) e^{-2\pi i f(t-\hat{k}\cdot\vec{x})}. \tag{2.21c}$$

The quantity $\Gamma_{12}(f)$ is the overlap reduction function. It is often convenient to define a *normalized* overlap reduction function $\gamma_{12}(f) \propto \Gamma_{12}(f)$ with $\gamma_{12}(0) = 1$ for two coincident and coaligned identical detectors. For identical interferometers with opening angle β this leads to the normalized form

$$\gamma_{12}(f) = \frac{5}{8\pi \sin^2\beta} \Gamma_{12}(f). \tag{2.22}$$

Some symmetry properties of $\gamma_{12}(f)$ follow from immediately from its definition, in particular,

$$\gamma_{12}^*(f) = \gamma_{21}(f), \tag{2.23a}$$

$$\gamma_{12}^*(f) = \gamma_{12}(-f). \tag{2.23b}$$

III. DISCUSSION

Summarizing the results of the previous section, the overlap reduction function normalized for interferometric detectors is

$$\gamma_{12}(f) = \frac{5}{8\pi \sin^2\beta} \times \int_{S^2} d^2\Omega_{\hat{k}} \mathcal{R}_1^A(f,\hat{k}) \mathcal{R}_2^{A*}(f,\hat{k}) e^{-2\pi i f \hat{k}\cdot(\vec{x}_1-\vec{x}_2)}, \tag{3.1a}$$

where

$$\mathcal{R}_I^A(f,\hat{k}) = \mathbf{e}_{ij}^A(\hat{k}) \int_{-\infty}^{\infty} dt \int_{R^3} d^3x R_I^{ij}(t,\vec{x}) e^{-2\pi i f(t-\hat{k}\cdot\vec{x})}. \tag{3.1b}$$

Adjusting where necessary for differences in the plane-wave expansion and detector numbering conventions, Refs. [17,18] give the overlap reduction function as

$$\gamma'_{12}(f) = \frac{5}{8\pi \sin^2\beta} \times \int_{S^2} d^2\Omega_{\hat{k}} \mathcal{R}_1^A(f,\hat{k}) \mathcal{R}_2^{A*}(f,\hat{k}) e^{2\pi i f \hat{k}\cdot(\vec{x}_1-\vec{x}_2)}. \tag{3.2}$$

Comparing this expression with our Eq. (3.1a) we see that they differ by the substitution of $e^{2\pi i f \hat{k}\cdot(\vec{x}_1-\vec{x}_2)}$ for $e^{-2\pi i f \hat{k}\cdot(\vec{x}_1-\vec{x}_2)}$. This difference and when it is significant can be understood physically; doing so provides the occasion for a deeper discussion of the overlap reduction function.

A. Detector locations

At the end of Sec. II B, we observed that the response of detector I, located at $\vec{x}_I$, to a field of plane gravitational waves is given by [cf. Equation (2.12)]

$$r_I(t) = \int_{-\infty}^{\infty} df \int_{S^2} d^2\Omega_{\hat{k}} \mathcal{H}_A(f,\hat{k}) \mathcal{R}_I^A(f,\hat{k}) e^{2\pi i f(t-\hat{k}\cdot\vec{x}_I)}. \tag{3.3}$$

The detector location appears here in the form $e^{-2\pi i f \hat{k}\cdot\vec{x}_I}$. Referring to Eqs. (3.1a) and (3.2) for γ_{12} and γ'_{12} it is clear the substitution of γ'_{12} for γ_{12} is equivalent to simply exchanging the locations of detectors 1 and 2 keeping the rest of the configuration of the detectors fixed, i.e., γ'_{12} is the overlap reduction function for the detector configuration consisting of detector 1 at location $\vec{x}_2$ and detector 2 at location $\vec{x}_1$. With this understanding we now ask when that exchange is significant and when it is not.

B. Radiation wavelength and detector separation

An intuitive understanding of $\gamma_{12}(f)$ recognizes that its behavior in different frequency regimes is governed by several independent dimensionless parameters that can be created from the radiation wavelength, the separation between the detectors, and several intrinsic properties of the detectors as they are represented in the detector impulse response functions.

Referring to Eq. (3.1a) we note that when $f|\vec{x}_1 - \vec{x}_2| \ll 1$ the exponential term may be replaced by unity. As this is the only place where the detector separation appears, in this limit the detector separation plays no role in determining the value or behavior of $\gamma_{12}(f)$. Defining

$$\delta = f|\vec{x}_1 - \vec{x}_2| \tag{3.4}$$

we refer to $\delta \ll 1$ as the *small separation limit*. In the small separation limit, then, the difference between γ_{12} and γ'_{12} is negligible under all circumstances.

Now consider the case $\delta \gtrsim 1$. As we have observed, the difference between γ_{12} and γ'_{12} is the difference between locating detector 1 at $\vec{x}_1$ or $\vec{x}_2$, and detector 2 at $\vec{x}_2$ or $\vec{x}_1$. When the two detectors are identical in all other aspects, so that $\mathcal{R}_1^A = \mathcal{R}_2^A$, this exchange leaves the physical configuration unchanged and, again, there will be no difference between γ_{12} and γ'_{12}.

To understand the case $\delta \gtrsim 1$ when the two detectors are not identical we must consider the detector impulse response functions as they appear in Eqs. (3.1).

C. Radiation wavelength and detector impulse response

The impulse response of a detector has finite support, i.e., $R_I^{ij}(t,\vec{x}) \simeq 0$ for sufficiently large $t > 0$ or $|\vec{x}|$. A detector does not sample the field beyond its physical extent, so the support in $\vec{x}$ will be on order the detector's size ℓ. Referring to Eq. (3.1b) it is apparent that $\mathcal{R}_I^A(f,\hat{k})$ depends on $\hat{k}$ only through $\mathbf{e}_A^{ij}(\hat{k})$ when $f\ell \ll 1$

$$\mathcal{R}_I^A(f,\hat{k}) \simeq (2\pi)^3 \mathbf{e}_{ij}^A(\hat{k})\tilde{R}_I^{ij}(f,\vec{0}). \tag{3.5}$$

Introducing the parameters

$$\epsilon_I = f\ell_I \tag{3.6}$$

for the two detectors $I = 1, 2$ we refer to $\epsilon_I \ll 1$ as the *small antenna* limit for detector I.

Return now to the difference between γ_{12} and γ'_{12} when $\delta \gtrsim 1$ and the detectors are not identical. When both ϵ_1 and ϵ_2 are $\ll 1$ and noting that $\mathbf{e}_{ij}^A(-\hat{k}) = \pm\mathbf{e}_{ij}^A(\hat{k})$ we find

$$\begin{aligned}\gamma_{12}(f) &\simeq \frac{5}{8\pi\sin^2\beta}(2\pi)^6\tilde{R}_1^{ij}(f,\vec{0})\tilde{R}_2^{kl*}(f,\vec{0})\int_{S^2} d^2\Omega_{\hat{k}}\mathbf{e}_{ij}^A(\hat{k})\mathbf{e}_{kl}^A(\hat{k})e^{-2\pi i f\hat{k}\cdot(\vec{x}_1-\vec{x}_2)}\\ &= \frac{5}{8\pi\sin^2\beta}(2\pi)^6\tilde{R}_1^{ij}(f,\vec{0})\tilde{R}_2^{kl*}(f,\vec{0})\int_{S^2} d^2\Omega_{-\hat{k}}\mathbf{e}_{ij}^A(-\hat{k})\mathbf{e}_{kl}^A(-\hat{k})e^{2\pi i f\hat{k}\cdot(\vec{x}_1-\vec{x}_2)}\\ &= \frac{5}{8\pi\sin^2\beta}(2\pi)^6\tilde{R}_1^{ij}(f,\vec{0})\tilde{R}_2^{kl*}(f,\vec{0})\int_{S^2} d^2\Omega_{\hat{k}}\mathbf{e}_{ij}^A(\hat{k})\mathbf{e}_{kl}^A(\hat{k})e^{2\pi i f\hat{k}\cdot(\vec{x}_1-\vec{x}_2)} \simeq \gamma'_{12}(f);\end{aligned} \tag{3.7}$$

i.e., in the small antenna limit there is no distinction between γ_{12} and γ'_{12}.

D. Large separations and large detectors

Finally, consider the case $\delta \gtrsim 1$ and, without loss of generality, $\epsilon_1 \gtrsim 1$. In this case, the variation of the field across the spatial extent of detector 1 is important to the detector response and, in turn to γ_{12}. Exchanging the detector locations changes the relationship between the spatial *extent and orientation* of detector 1 relative to the location of detector 2. Correspondingly, in this limit the distinction between γ_{12} and γ'_{12} is significant.

An example of the case $\delta \gtrsim 1$, $\epsilon_1, \epsilon_2 \gtrsim 1$ is the anticipated sensitivity to a stochastic gravitational-wave background of the BBO, a space-based follow-on to LISA that has been the subject of recent study [7,17,20–22]. The principal results of this study reported in the literature [17,18] make use of the incorrect form of the overlap reduction function, thus misestimating this proposed detector's sensitivity to a stochastic gravitational-wave signal in the higher frequency regime. Figure 1 illustrates the physical effect of using γ'_{12} in place of γ_{12} when analyzing the cross correlation of the two BBO detectors as described in [17]. On the left are the two interferometric detectors as they are actually arranged in space; on the right are the effective location and orientation of the detectors when γ'_{12} is used in place of γ_{12}, i.e., when detector 1 is translated to $\vec{x}_2$ and detector 2 is translated to $\vec{x}_1$. Under this transformation the spatial extent of the two detector pairs is much greater than is actually the case; correspondingly, we expect γ'_{12} to be a much more sensitive function of frequency than γ_{12}.

To calculate γ'_{12} for comparison with γ_{12}, we need an explicit expression for the transfer function of the detectors. This is derived in [17] (Eqs. 5, 7, 11). In our notation,

$$\mathcal{R}^A(f,\hat{k}) = \mathbf{e}_{ij}^A(\hat{k})\tfrac{1}{2}(u^i u^j \mathcal{T}(\hat{u}\cdot\hat{k},f) - v^i v^j \mathcal{T}(\hat{v}\cdot\hat{k},f)), \tag{3.8a}$$

where

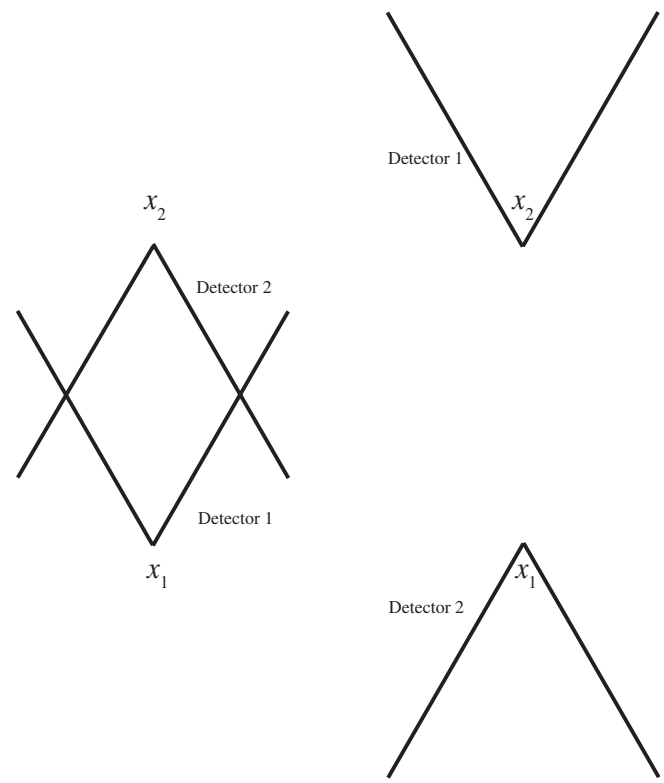

FIG. 1. The arrangement of the two BBO detectors described [17] and the arrangement actually analyzed when γ'_{12} is substituted for γ_{12}. Owing to its larger spatial extent, the system actually studied is much more sensitive to frequency than the system whose study was intended.

$$\mathcal{T}(\hat{u}\cdot\hat{k}, f) = \frac{1}{2}\Big[\mathrm{sinc}\Big(\frac{f}{2f_*}(1-\hat{u}\cdot\hat{k})\Big) \times \exp\Big(-i\frac{f}{2f_*}(3+\hat{u}\cdot\hat{k})\Big) + \mathrm{sinc}\Big(\frac{f}{2f_*}(1+\hat{u}\cdot\hat{k})\Big) \times \exp\Big(-i\frac{f}{2f_*}(1+\hat{u}\cdot\hat{k})\Big)\Big]. \tag{3.8b}$$

Here, $\hat{u}$ and $\hat{v}$ are unit vectors pointing in the direction of the detector arms, $f_* = c/(2\pi L)$ is the *transfer* frequency of the detectors (arm length $L = 5\times 10^9$ m), and $\mathrm{sinc}(x) = \sin(x)/x$. Figure 2 compares γ'_{12} (cf. [17] Fig. 5) and γ_{12} [Eq. (3.1)] for this configuration. As expected, γ'_{12} decreases much more rapidly with frequency and has its nulls more closely spaced than those of γ_{12}. Figure 3 shows the fractional error $|1-\gamma'_{12}(f)/\gamma_{12}(f)|$ as a function of frequency. The large amplitude spikes in the error occur where nulls of γ'_{12} do not coincide with nulls of γ_{12}.

It is apparent from Fig. 2 that the use of an incorrect overlap reduction function has, in this case, led to an underestimate of BBO's sensitivity to a stochastic gravitational-wave background. Estimating the background and detector noise power spectral density as white, the magnitude of the error is just the ratio of integrated squared magnitudes $|\gamma_{12}(f)|^2$ and $|\gamma'_{12}(f)|^2$,

$$\frac{\int df|\gamma'_{12}(f)|^2}{\int df|\gamma_{12}(f)|^2} = 1 - 0.28; \tag{3.9}$$

i.e., the BBO estimates of [17,18] underestimate the sensitivity of BBO by nearly 30% across the entire band, and substantially larger if interest is focused on the higher frequencies. Achieving BBO's goals of detecting the stochastic gravitational-wave relics of the inflationary epoch depend on the accurate identification and subtraction of contributions owing to compact binary systems. Underestimating BBO's response to a gravitational-wave background leads to an overestimate of the accuracy required in this identification and subtraction [20]. Recognizing and correcting the underestimate in BBO sensitivity thus relaxes the analysis problem associated with the identification of these foreground sources.

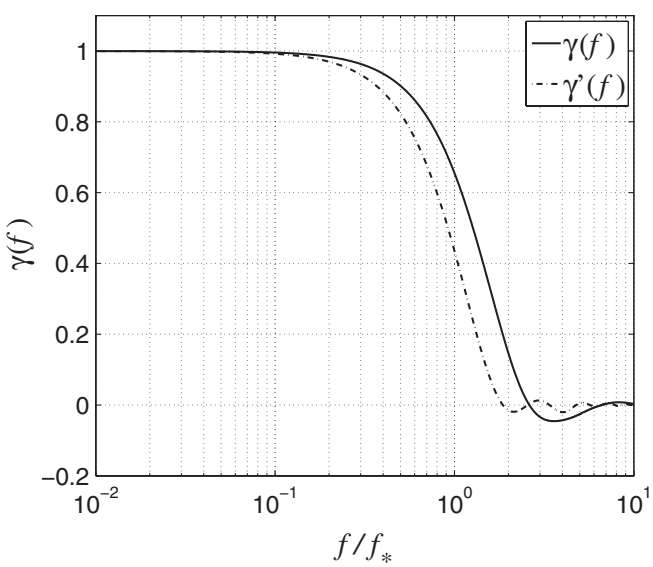

FIG. 2. Plots of the overlap reduction functions $\gamma_{12}(f)$ (solid line) and $\gamma'_{12}(f)$ (dotted-dashed line) for the BBO configurations in Fig. 1.

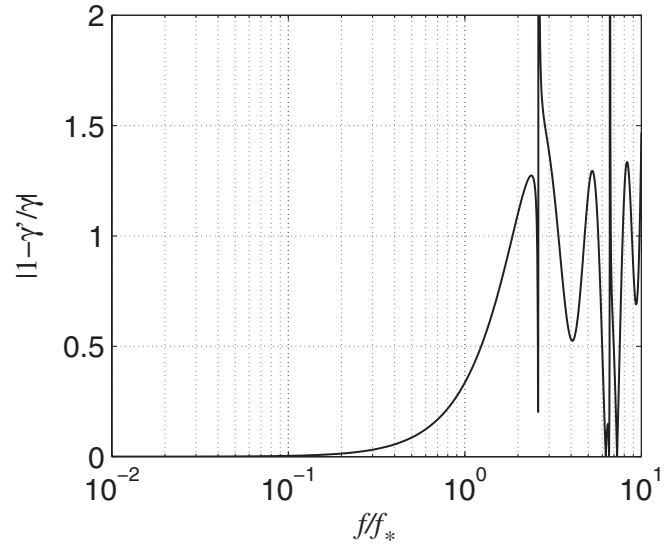

FIG. 3. Fractional difference between $\gamma_{12}(f)$ and $\gamma'_{12}(f)$, relative to the correct $\gamma_{12}(f)$ given by Eq. (3.1).

IV. CONCLUSIONS

Detection of a gravitational-wave stochastic background relies on the cross-correlated response of one or more pairs of gravitational-wave detectors. The separation and relative orientation of the two detectors plays a crucial role in determining the frequency-dependent sensitivity of each detector pair to the stochastic background. Recent studies [17,18] of the sensitivity of the BBO and related future generation gravitational-wave detectors have used an incorrect expression for this geometrical factor. The errors committed may be physically interpreted as an exchange in space of the two detectors, leaving their absolute orientations fixed. In the case of the BBO, this error leads to an approximately 30% underestimate in its sensitivity to relic gravitational waves associated with, e.g., the inflationary epoch. Since achieving BBO's goals of detecting this background requires the accurate identification and subtraction of gravitational-wave foreground contributions from compact binary systems [20], this underestimate has lead to commensurate overestimate of the difficulty of this analysis problem. Recognizing and correcting this error

thus improves, in two ways, the prospects for the BBO missions' main goal as a Big Bang Observer.

ACKNOWLEDGMENTS

We most gratefully acknowledge the hospitality of the Aspen Center for Physics and the 2008 Workshop on Gravitational-Wave Astronomy, where most of the work reported here was done. J. D. R. also acknowledges B. Allen, W. Anderson, A. Lazzarini, and J. T. Whelan for discussions. Some of the results in this paper were performed using the HEALPIX [23] package [24]. The research was supported in part by NSF Grant No. PHY-0555842 to the University of Texas at Brownsville, NSF Grant No. PHY-0653462, and NASA Grant No. NNG05GF71G to The Pennsylvania State University, and the Center for Gravitational Wave Physics, which was supported by the NSF under Cooperative Agreement No. PHY-014375.

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