A One-Dimensional Finite-Difference Solver for Fully-Developed Pipe and Channel Flows

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An algorithm is developed to solve the fundamental flow cases of fully-developed turbulent flow in a pipe and in a channel. The algorithm uses second-order finite-difference approximations for nonuniform grid spacing and is developed in such a way as to easily facilitate the implementation of several two-equation, Reynolds-Averaged-Navier-Stokes turbulence models. Results are included for the Wilcox 1998 *k-ω* model.

Introduction

Fully-developed flow in a pipe or channel is a very fundamental flow scenario for testing turbulence models. The degree to which a turbulence model is seen to be valid is partially based on the ability of the model to predict the wall shear stress of turbulent flows in pipes and channels as a function of Reynolds number. This paper presents the development of a second-order algorithm that can be easily adapted to several two-equation turbulence models.

Governing Equations

The governing steady-state, incompressible continuity and Boussinesq-RANS equations can be written in vector form as

$$\nabla \cdot \overline{\mathbf{V}} = 0 \tag{1}$$

$$(\overline{\mathbf{V}} \cdot \nabla)\overline{\mathbf{V}} = -\nabla \hat{\overline{p}} / \rho + \nabla \cdot [2(\nu + \nu_t) \overline{\overline{\mathbf{S}}}(\overline{\mathbf{V}})] \tag{2}$$

For fully-developed flow in a pipe, the gradients of transport properties along the axis of the pipe are zero. Additionally, all gradients with respect to θ are zero and there is no component of velocity in the azimuthal direction. Therefore, Eqs. (1) and (2) can be written in cylindrical coordinates as

$$\frac{d(r\overline{V}_r)}{dr} = 0 \tag{3}$$

$$\overline{V}_r \frac{d\overline{V}_r}{dr} = -\frac{1}{\rho}\frac{\partial \hat{\overline{p}}}{\partial r} + \frac{1}{r}\frac{d}{dr}\left[2(\nu + \nu_t)r\frac{d\overline{V}_r}{dr}\right] - \frac{2(\nu + \nu_t)}{r^2}\overline{V}_r \tag{4}$$

$$\overline{V}_r \frac{d\overline{V}_z}{dr} = -\frac{1}{\rho}\frac{\partial \hat{\overline{p}}}{\partial z} + \frac{1}{r}\frac{d}{dr}\left[(\nu + \nu_t)r\frac{d\overline{V}_z}{dr}\right] \tag{5}$$

where the coordinate r is the normal coordinate measured outward from the center of the pipe, and z is the coordinate along the axis of the pipe. The no-slip boundary conditions at the wall of the pipe, $r=R$, are

$$\begin{aligned} \overline{V}_r(R) &= 0 \\ \overline{V}_z(R) &= 0 \end{aligned} \tag{6}$$

Applying the $\overline{V}_r$ boundary condition to the integral of Eq. (3) gives $\overline{V}_r = 0$ and the governing equations can be written as

$$\overline{V}_r = 0 \tag{7}$$

$$\frac{\partial \hat{\overline{p}}}{\partial r} = 0 \tag{8}$$

$$\frac{1}{r}\frac{d}{dr}\left[(\nu + \nu_t)r\frac{d\overline{V}_z}{dr}\right] = \frac{1}{\rho}\frac{d\hat{\overline{p}}}{dz} \tag{9}$$

The results of Eq. (8) show that $\hat{\overline{p}} = \hat{\overline{p}}(z)$ which has been used in Eq. (9). The velocity boundary condition at the center of the pipe, $r=0$, is

$$\frac{d\overline{V}_z}{dr}(0) = 0 \tag{10}$$

Integrating Eq. (9) from the centerline to some arbitrary distance, r, gives

$$\int_{r=0}^{r}\frac{d}{dr}\left[(\nu + \nu_t)r\frac{d\overline{V}_z}{dr}\right] = \int_{r=0}^{r}\frac{r}{\rho}\frac{d\hat{\overline{p}}}{dz}$$

$$(\nu + \nu_t)r\frac{d\overline{V}_z}{dr} = \frac{1}{\rho}\frac{d\hat{\overline{p}}}{dz}\left(\frac{r^2}{2} + C\right) \tag{11}$$

where C is an arbitrary constant. Applying the boundary condition given in Eq. (10) gives $C = 0$ and Eq. (11) can be written

$$(\nu + \nu_t)\frac{d\overline{V}_z}{dr} = \frac{1}{2\rho}\frac{d\hat{\overline{p}}}{dz}r \tag{12}$$

The left-hand side of Eq. (12) evaluated at the wall is related to the shear stress at the wall which can also be written in terms of the friction velocity

$$(\nu + \nu_t)\frac{d\overline{V}_z}{dr}(R) = -\frac{\tau_w}{\rho} = -u_\tau^2 \tag{13}$$

Using this in Eq. (12) evaluated at $r=R$ gives the relationship

$$\frac{1}{2\rho}\frac{d\hat{\overline{p}}}{dz} = -\frac{u_\tau^2}{R} \tag{14}$$

Using this in Eq. (12) gives the formulation and the remaining boundary condition

$$\frac{d\overline{V}_z}{dr} = -\frac{u_\tau^2}{(\nu + \nu_t)}\frac{r}{R}, \qquad \overline{V}_z(R) = 0 \tag{15}$$

This can be nondimensionalized using the following parameters

$$\hat{r} \equiv \frac{r}{R}, \quad R_\tau \equiv \frac{u_\tau R}{\nu}, \quad u^+ \equiv \frac{\overline{V}_z}{u_\tau}, \quad v^+ \equiv \frac{\nu_t}{\nu},$$

$$y^+ \equiv \frac{u_\tau(R-r)}{\nu} = \frac{u_\tau R(1-r/R)}{\nu} = R_\tau(1-\hat{r}), \qquad (16)$$

$$p^+ \equiv \frac{d\hat{p}}{dz}\frac{\nu}{\rho u_\tau^3} = -\frac{2\nu}{u_\tau^3}\frac{u_\tau^2}{R} = -\frac{2\nu}{u_\tau R} = -\frac{2}{R_\tau}$$

Using these nondimensional parameters, Eq. (15) can be written as

$$\frac{du^+}{d\hat{r}} = -\frac{R_\tau \hat{r}}{(1+v^+)}, \qquad u^+(1) = 0 \qquad (17)$$

Two-equation RANS-based turbulence models can be coupled with Eq. (17). A general form for the complete system of equations including boundary conditions can often be written in the form

$$\frac{du^+}{d\hat{r}} = -\frac{R_\tau \hat{r}}{(1+v^+)}, \qquad v^+ = f_\nu$$

$$\frac{1}{\hat{r}}\frac{d}{d\hat{r}}\left[-(1+v^+/\sigma_k)\hat{r}\frac{dk^+}{d\hat{r}}\right] = S_k$$

$$\frac{1}{\hat{r}}\frac{d}{d\hat{r}}\left[-(1+v^+/\sigma_h)\hat{r}\frac{dh^+}{d\hat{r}}\right] = S_h \qquad (18)$$

$$u^+(1) = 0, \quad k^+(1) = 0,$$

$$\frac{dk^+}{d\hat{r}}(1) = 0, \quad \frac{dk^+}{d\hat{r}}(0) = 0, \quad \frac{dh^+}{d\hat{r}}(0) = 0$$

where h^+ represents the second turbulence variable, and f_ν, S_k, and S_h are functions of model constants, the turbulence variables, and their derivatives with respect to $\hat{r}$.

Finite-Difference Discretization

A general formulation for fully-developed pipe and channel flow can be written to allow implementation of several two-equation turbulence models. This formulation takes the form

$$y^+ = R_\tau(1-\hat{r}), \quad v^+ = f_\nu, \quad u^{+'} = -\frac{R_\tau \hat{r}}{(1+v^+)}$$

$$-\frac{v^{+'}}{\sigma_k}k^{+'} - \frac{1}{\hat{r}}\left(1+\frac{v^+}{\sigma_k}\right)k^{+'} - \left(1+\frac{v^+}{\sigma_k}\right)k^{+''} = S_k$$

$$-\frac{v^{+'}}{\sigma_\omega}h^{+'} - \frac{1}{\hat{r}}\left(1+\frac{v^+}{\sigma_h}\right)h^{+'} - \left(1+\frac{v^+}{\sigma_h}\right)h^{+''} = S_h \qquad (19)$$

$$u^+(1) = 0, \quad k^+(1) = 0,$$

$$k^{+'}(1) = 0, \quad k^{+'}(0) = 0, \quad h^{+'}(0) = 0$$

where the primes represent derivatives with respect to $\hat{r}$.

Equation (19) can be solved numerically using the finite-difference method on a non-uniform grid. Here we define a one-dimensional domain discretized by m nodes. Node 1 is located at $\hat{r} = 0$, and node m is located at $\hat{r} = 1$. Applying second-order forward difference approximations, the first and second derivatives of any variable, ϕ, can be approximated at node 1 as

$$\delta\hat{r}_a \equiv \hat{r}_2 - \hat{r}_1, \quad \delta\hat{r}_b \equiv \hat{r}_3 - \hat{r}_1, \quad \delta\hat{r}_c \equiv \hat{r}_4 - \hat{r}_1$$

$$\phi_1' = \frac{\delta\hat{r}_b^2 - \delta\hat{r}_a^2}{\delta\hat{r}_a^2\delta\hat{r}_b - \delta\hat{r}_b^2\delta\hat{r}_a}\phi_1 + \frac{-\delta\hat{r}_b^2}{\delta\hat{r}_a^2\delta\hat{r}_b - \delta\hat{r}_b^2\delta\hat{r}_a}\phi_2$$

$$+ \frac{\delta\hat{r}_a^2}{\delta\hat{r}_a^2\delta\hat{r}_b - \delta\hat{r}_b^2\delta\hat{r}_a}\phi_3$$

$$\phi_1'' = \frac{2(\delta\hat{r}_a + \delta\hat{r}_b + \delta\hat{r}_c)}{\delta\hat{r}_a\delta\hat{r}_b\delta\hat{r}_c}\phi_1 + \frac{2(\delta\hat{r}_b + \delta\hat{r}_c)}{\delta\hat{r}_a(\delta\hat{r}_a - \delta\hat{r}_b)(\delta\hat{r}_c - \delta\hat{r}_a)}\phi_2 \qquad (20)$$

$$+ \frac{2(\delta\hat{r}_c + \delta\hat{r}_a)}{\delta\hat{r}_b(\delta\hat{r}_b - \delta\hat{r}_c)(\delta\hat{r}_a - \delta\hat{r}_b)}\phi_3$$

$$+ \frac{2(\delta\hat{r}_a + \delta\hat{r}_b)}{\delta\hat{r}_c(\delta\hat{r}_b - \delta\hat{r}_c)(\delta\hat{r}_c - \delta\hat{r}_a)}\phi_4$$

Applying second-order forward difference approximations, the first and second derivatives of any variable, ϕ, can be approximated at node 2 as

$$\delta\hat{r}_a \equiv \hat{r}_1 - \hat{r}_2, \quad \delta\hat{r}_b \equiv \hat{r}_3 - \hat{r}_2, \quad \delta\hat{r}_c \equiv \hat{r}_4 - \hat{r}_2$$

$$\phi_2' = \frac{-\delta\hat{r}_b^2}{\delta\hat{r}_a^2\delta\hat{r}_b - \delta\hat{r}_b^2\delta\hat{r}_a}\phi_1 + \frac{\delta\hat{r}_b^2 - \delta\hat{r}_a^2}{\delta\hat{r}_a^2\delta\hat{r}_b - \delta\hat{r}_b^2\delta\hat{r}_a}\phi_2$$

$$+ \frac{\delta\hat{r}_a^2}{\delta\hat{r}_a^2\delta\hat{r}_b - \delta\hat{r}_b^2\delta\hat{r}_a}\phi_3$$

$$\phi_2'' = \frac{2(\delta\hat{r}_b + \delta\hat{r}_c)}{\delta\hat{r}_a(\delta\hat{r}_a - \delta\hat{r}_b)(\delta\hat{r}_c - \delta\hat{r}_a)}\phi_1 + \frac{2(\delta\hat{r}_a + \delta\hat{r}_b + \delta\hat{r}_c)}{\delta\hat{r}_a\delta\hat{r}_b\delta\hat{r}_c}\phi_2 \qquad (21)$$

$$+ \frac{2(\delta\hat{r}_c + \delta\hat{r}_a)}{\delta\hat{r}_b(\delta\hat{r}_b - \delta\hat{r}_c)(\delta\hat{r}_a - \delta\hat{r}_b)}\phi_3$$

$$+ \frac{2(\delta\hat{r}_a + \delta\hat{r}_b)}{\delta\hat{r}_c(\delta\hat{r}_b - \delta\hat{r}_c)(\delta\hat{r}_c - \delta\hat{r}_a)}\phi_4$$

Applying second-order finite difference approximations, the first and second derivatives of any variable, ϕ, can be approximated at an interior node, j, as

$$\delta\hat{r}_a \equiv \hat{r}_{j-1} - \hat{r}_j, \quad \delta\hat{r}_b \equiv \hat{r}_{j+1} - \hat{r}_j, \quad \delta\hat{r}_c \equiv \hat{r}_{j-2} - \hat{r}_j$$

$$\phi_j' = \frac{-\delta\hat{r}_b^2}{\delta\hat{r}_a^2 \delta\hat{r}_b - \delta\hat{r}_b^2 \delta\hat{r}_a}\phi_{j-1} + \frac{\delta\hat{r}_b^2 - \delta\hat{r}_a^2}{\delta\hat{r}_a^2 \delta\hat{r}_b - \delta\hat{r}_b^2 \delta\hat{r}_a}\phi_j$$

$$+ \frac{\delta\hat{r}_a^2}{\delta\hat{r}_a^2 \delta\hat{r}_b - \delta\hat{r}_b^2 \delta\hat{r}_a}\phi_{j+1}$$

$$\phi_j'' = \frac{2(\delta\hat{r}_a + \delta\hat{r}_b)}{\delta\hat{r}_c(\delta\hat{r}_b - \delta\hat{r}_c)(\delta\hat{r}_c - \delta\hat{r}_a)}\phi_{j-2} \tag{22}$$

$$+ \frac{2(\delta\hat{r}_b + \delta\hat{r}_c)}{\delta\hat{r}_a(\delta\hat{r}_a - \delta\hat{r}_b)(\delta\hat{r}_c - \delta\hat{r}_a)}\phi_{j-1}$$

$$+ \frac{2(\delta\hat{r}_a + \delta\hat{r}_b + \delta\hat{r}_c)}{\delta\hat{r}_a \delta\hat{r}_b \delta\hat{r}_c}\phi_j + \frac{2(\delta\hat{r}_c + \delta\hat{r}_a)}{\delta\hat{r}_b(\delta\hat{r}_b - \delta\hat{r}_c)(\delta\hat{r}_a - \delta\hat{r}_b)}\phi_{j+1}$$

Applying second-order backward difference approximations, the first and second derivatives of any variable, ϕ, can be approximated at node m as

$$\delta\hat{r}_a \equiv \hat{r}_{m-1} - \hat{r}_m, \quad \delta\hat{r}_b \equiv \hat{r}_{m-2} - \hat{r}_m, \quad \delta\hat{r}_c \equiv \hat{r}_{m-3} - \hat{r}_m$$

$$\phi_m' = \frac{\delta\hat{r}_a^2}{\delta\hat{r}_a^2 \delta\hat{r}_b - \delta\hat{r}_b^2 \delta\hat{r}_a}\phi_{m-2} + \frac{-\delta\hat{r}_b^2}{\delta\hat{r}_a^2 \delta\hat{r}_b - \delta\hat{r}_b^2 \delta\hat{r}_a}\phi_{m-1}$$

$$+ \frac{\delta\hat{r}_b^2 - \delta\hat{r}_a^2}{\delta\hat{r}_a^2 \delta\hat{r}_b - \delta\hat{r}_b^2 \delta\hat{r}_a}\phi_m$$

$$\phi_m'' = \frac{2(\delta\hat{r}_a + \delta\hat{r}_b)}{\delta\hat{r}_c(\delta\hat{r}_b - \delta\hat{r}_c)(\delta\hat{r}_c - \delta\hat{r}_a)}\phi_{m-3} \tag{23}$$

$$+ \frac{2(\delta\hat{r}_c + \delta\hat{r}_a)}{\delta\hat{r}_b(\delta\hat{r}_b - \delta\hat{r}_c)(\delta\hat{r}_a - \delta\hat{r}_b)}\phi_{m-2}$$

$$+ \frac{2(\delta\hat{r}_b + \delta\hat{r}_c)}{\delta\hat{r}_a(\delta\hat{r}_a - \delta\hat{r}_b)(\delta\hat{r}_c - \delta\hat{r}_a)}\phi_{m-1}$$

$$+ \frac{2(\delta\hat{r}_a + \delta\hat{r}_b + \delta\hat{r}_c)}{\delta\hat{r}_a \delta\hat{r}_b \delta\hat{r}_c}\phi_m$$

Any number of algorithms can be used for grid generation. An algorithm that works well for the present study clusters the nodes near the wall and can be written as

$$\eta_j = \frac{j-1}{m-1}$$

$$\hat{r}_{m+1-j} = 1 - \frac{(\beta+1) - (\beta-1)[(\beta+1)/(\beta-1)]^{1-\eta_j}}{1 + [(\beta+1)/(\beta-1)]^{1-\eta_j}} \tag{24}$$

$$y_{m+1-j}^+ = R_\tau(1 - \hat{r}_{m+1-j})$$

where β is a clustering parameter.

Once the $\hat{r}$ array has been filled out, the following coefficient arrays can also be calculated

$$\delta\hat{r}_a \equiv \hat{r}_2 - \hat{r}_1, \quad \delta\hat{r}_b \equiv \hat{r}_3 - \hat{r}_1, \quad \delta\hat{r}_c \equiv \hat{r}_4 - \hat{r}_1$$

$$C_{L2,1} \equiv 0, \; C_{L1,1} \equiv 0, \; C_{D,1} \equiv \frac{(\delta\hat{r}_a + \delta\hat{r}_b)}{\delta\hat{r}_a \delta\hat{r}_b},$$

$$C_{U1,1} \equiv \frac{-\delta\hat{r}_b}{\delta\hat{r}_a(\delta\hat{r}_a - \delta\hat{r}_b)}, \; C_{U2,1} \equiv \frac{\delta\hat{r}_a}{\delta\hat{r}_b(\delta\hat{r}_a - \delta\hat{r}_b)}$$

$$A_{L3,1} \equiv 0, \; A_{L2,1} \equiv 0, \; A_{L1,1} \equiv 0, \; A_{D,1} \equiv \frac{2(\delta\hat{r}_a + \delta\hat{r}_b + \delta\hat{r}_c)}{\delta\hat{r}_a \delta\hat{r}_b \delta\hat{r}_c} \tag{25}$$

$$A_{U1,1} \equiv \frac{2(\delta\hat{r}_b + \delta\hat{r}_c)}{\delta\hat{r}_a(\delta\hat{r}_a - \delta\hat{r}_b)(\delta\hat{r}_c - \delta\hat{r}_a)}$$

$$A_{U2,1} \equiv \frac{2(\delta\hat{r}_c + \delta\hat{r}_a)}{\delta\hat{r}_b(\delta\hat{r}_b - \delta\hat{r}_c)(\delta\hat{r}_a - \delta\hat{r}_b)}$$

$$A_{U3,1} \equiv \frac{2(\delta\hat{r}_a + \delta\hat{r}_b)}{\delta\hat{r}_c(\delta\hat{r}_b - \delta\hat{r}_c)(\delta\hat{r}_c - \delta\hat{r}_a)}$$

$$\delta\hat{r}_a \equiv \hat{r}_1 - \hat{r}_2, \quad \delta\hat{r}_b \equiv \hat{r}_3 - \hat{r}_2, \quad \delta\hat{r}_c \equiv \hat{r}_4 - \hat{r}_2$$

$$C_{L2,2} \equiv 0, \; C_{L1,2} \equiv \frac{-\delta\hat{r}_b}{\delta\hat{r}_a(\delta\hat{r}_a - \delta\hat{r}_b)}, \; C_{D,2} \equiv \frac{-(\delta\hat{r}_a + \delta\hat{r}_b)}{\delta\hat{r}_a \delta\hat{r}_b},$$

$$C_{U1,2} \equiv \frac{\delta\hat{r}_a}{\delta\hat{r}_b(\delta\hat{r}_a - \delta\hat{r}_b)}, \; C_{U2,2} \equiv 0$$

$$A_{L3,2} \equiv 0, \; A_{L2,2} \equiv 0, \; A_{L1,2} \equiv \frac{2(\delta\hat{r}_b + \delta\hat{r}_c)}{\delta\hat{r}_a(\delta\hat{r}_a - \delta\hat{r}_b)(\delta\hat{r}_c - \delta\hat{r}_a)} \tag{26}$$

$$A_{D,2} \equiv \frac{2(\delta\hat{r}_a + \delta\hat{r}_b + \delta\hat{r}_c)}{\delta\hat{r}_a \delta\hat{r}_b \delta\hat{r}_c},$$

$$A_{U1,2} \equiv \frac{2(\delta\hat{r}_c + \delta\hat{r}_a)}{\delta\hat{r}_b(\delta\hat{r}_b - \delta\hat{r}_c)(\delta\hat{r}_a - \delta\hat{r}_b)}$$

$$A_{U2,2} \equiv \frac{2(\delta\hat{r}_a + \delta\hat{r}_b)}{\delta\hat{r}_c(\delta\hat{r}_b - \delta\hat{r}_c)(\delta\hat{r}_c - \delta\hat{r}_a)}, \; A_{U3,2} \equiv 0$$

$$\delta\hat{r}_a \equiv \hat{r}_{j-1} - \hat{r}_j, \quad \delta\hat{r}_b \equiv \hat{r}_{j+1} - \hat{r}_j, \quad \delta\hat{r}_c \equiv \hat{r}_{j-2} - \hat{r}_j$$

$$C_{L2,j} \equiv 0, \; C_{L1,j} \equiv \frac{-\delta\hat{r}_b}{\delta\hat{r}_a(\delta\hat{r}_a - \delta\hat{r}_b)}, \; C_{D,j} \equiv \frac{-(\delta\hat{r}_a + \delta\hat{r}_b)}{\delta\hat{r}_a \delta\hat{r}_b},$$

$$C_{U1,j} \equiv \frac{\delta\hat{r}_a}{\delta\hat{r}_b(\delta\hat{r}_a - \delta\hat{r}_b)}, \; C_{U2,j} \equiv 0, \; A_{L3,j} \equiv 0$$

$$A_{L2,j} \equiv \frac{2(\delta\hat{r}_a + \delta\hat{r}_b)}{\delta\hat{r}_c(\delta\hat{r}_b - \delta\hat{r}_c)(\delta\hat{r}_c - \delta\hat{r}_a)}$$

$$A_{L1,j} \equiv \frac{2(\delta\hat{r}_b + \delta\hat{r}_c)}{\delta\hat{r}_a(\delta\hat{r}_a - \delta\hat{r}_b)(\delta\hat{r}_c - \delta\hat{r}_a)}, \tag{27}$$

$$A_{D,j} \equiv \frac{2(\delta\hat{r}_a + \delta\hat{r}_b + \delta\hat{r}_c)}{\delta\hat{r}_a \delta\hat{r}_b \delta\hat{r}_c}$$

$$A_{U1,j} \equiv \frac{2(\delta\hat{r}_c + \delta\hat{r}_a)}{\delta\hat{r}_b(\delta\hat{r}_b - \delta\hat{r}_c)(\delta\hat{r}_a - \delta\hat{r}_b)}$$

$$A_{U2,j} \equiv 0, \; A_{U3,j} \equiv 0$$

$$\delta\hat{r}_a \equiv \hat{r}_{m-1} - \hat{r}_m, \quad \delta\hat{r}_b \equiv \hat{r}_{m-2} - \hat{r}_m, \quad \delta\hat{r}_c \equiv \hat{r}_{m-3} - \hat{r}_m$$

$$C_{L2,m} \equiv \frac{\delta\hat{r}_a}{\delta\hat{r}_b(\delta\hat{r}_a - \delta\hat{r}_b)}, \quad C_{L1,m} \equiv \frac{-\delta\hat{r}_b}{\delta\hat{r}_a(\delta\hat{r}_a - \delta\hat{r}_b)}$$

$$C_{D,m} \equiv \frac{-(\delta\hat{r}_a + \delta\hat{r}_b)}{\delta\hat{r}_a\,\delta\hat{r}_b}, \quad C_{U1,m} \equiv 0, \; C_{U2,m} \equiv 0$$

$$A_{L3,m} \equiv \frac{2(\delta\hat{r}_a + \delta\hat{r}_b)}{\delta\hat{r}_c(\delta\hat{r}_b - \delta\hat{r}_c)(\delta\hat{r}_c - \delta\hat{r}_a)}$$

$$A_{L2,m} \equiv \frac{2(\delta\hat{r}_c + \delta\hat{r}_a)}{\delta\hat{r}_b(\delta\hat{r}_b - \delta\hat{r}_c)(\delta\hat{r}_a - \delta\hat{r}_b)}, \tag{28}$$

$$A_{L1,m} \equiv \frac{2(\delta\hat{r}_b + \delta\hat{r}_c)}{\delta\hat{r}_a(\delta\hat{r}_a - \delta\hat{r}_b)(\delta\hat{r}_c - \delta\hat{r}_a)}$$

$$A_{D,m} \equiv \frac{2(\delta\hat{r}_a + \delta\hat{r}_b + \delta\hat{r}_c)}{\delta\hat{r}_a\,\delta\hat{r}_b\,\delta\hat{r}_c},$$

$$A_{U1,m} \equiv 0, \; A_{U2,m} \equiv 0, \; A_{U3,m} \equiv 0$$

Using Eqs. (25)–(28) in Eqs. (20)–(23) gives second-order approximations for the first and second derivatives of any variable, ϕ

$$\phi_1' = C_{D,1}\phi_1 + C_{U1,1}\phi_2 + C_{U2,1}\phi_3$$

$$\phi_1'' = A_{D,1}\phi_1 + A_{U1,1}\phi_2 + A_{U2,1}\phi_3 + A_{U3,1}\phi_4$$

$$\phi_2' = C_{L1,2}\phi_1 + C_{D,2}\phi_2 + C_{U1,2}\phi_3$$

$$\phi_2'' = A_{L1,2}\phi_1 + A_{D,2}\phi_2 + A_{U1,2}\phi_3 + A_{U2,2}\phi_4$$

for $3 \le j \le m-1$ 　　　　　　　　　　　　(29)

$$\phi_j' = C_{L1,j}\phi_{j-1} + C_{D,j}\phi_j + C_{U1,j}\phi_{j+1}$$

$$\phi_j'' = A_{L2,j}\phi_{j-2} + A_{L1,j}\phi_{j-1} + A_{D,j}\phi_j + A_{U1,j}\phi_{j+1}$$

$$\phi_m' = C_{L2,m}\phi_{m-2} + C_{L1,m}\phi_{m-1} + C_{D,m}\phi_m$$

$$\phi_m'' = A_{L3,m}\phi_{m-3} + A_{L2,m}\phi_{m-2} + A_{L1,m}\phi_{m-1} + A_{D,m}\phi_m$$

Beginning with initial estimates for k^+ and h^+ along with the known arrays of $\hat{r}$ and y^+, initial estimates for v^+ and u^+ can be found from

$$v_j^+ = f_v(\mathbf{C}, \mathbf{y}^+, \mathbf{k}^+, \mathbf{h}^+), \quad u_j^{+'} = -\frac{R_\tau \hat{r}_j}{(1+v_j^+)} \tag{30}$$

If the model requires the second derivative of u^+ to be calculated, it can be estimated using Eq. (29)

$$u_1^{+''} = C_{D,1}u_1^{+'} + C_{U1,1}u_2^{+'} + C_{U2,1}u_3^{+'}$$

for $2 \le j \le m-1$ 　　　　　　　　　　　(31)

$$u_j^{+''} = C_{L1,j}u_{j-1}^{+'} + C_{D,j}u_j^{+'} + C_{U1,j}u_{j+1}^{+'}$$

$$u_m^{+''} = C_{L2,m}u_{m-2}^{+'} + C_{L1,m}u_{m-1}^{+'} + C_{D,m}u_m^{+'}$$

Likewise, in light of the boundary conditions, the first derivatives for k^+, h^+, and v^+ can be estimated

$$k_1^{+'} = 0, \quad h_1^{+'} = 0, \quad v_1^{+'} = 0$$

for $2 \le j \le m-1$

$$k_j^{+'} = C_{L1,j}k_{j-1}^+ + C_{D,j}k_j^+ + C_{U1,j}k_{j+1}^+$$

$$h_j^{+'} = C_{L1,j}h_{j-1}^+ + C_{D,j}h_j^+ + C_{U1,j}h_{j+1}^+$$

$$v_j^{+'} = C_{L1,j}v_{j-1}^+ + C_{D,j}v_j^+ + C_{U1,j}v_{j+1}^+ \tag{32}$$

$$k_m^{+'} = 0$$

$$h_m^{+'} = C_{L2,m}h_{m-2}^+ + C_{L1,m}h_{m-1}^+ + C_{D,m}h_m^+$$

$$v_m^{+'} = C_{L2,m}v_{m-2}^+ + C_{L1,m}v_{m-1}^+ + C_{D,m}v_m^+$$

The k-transport equation can now be written in terms of the second-order finite-difference approximations expressed above including the boundary conditions on k

$$C_{D,1}k_1^+ + C_{U1,1}k_2^+ + C_{U2,1}k_3^+ = 0$$

$$-\frac{v_2^{+'}}{\sigma_k}(C_{L1,2}k_1^+ + C_{D,2}k_2^+ + C_{U1,2}k_3^+)$$

$$-\frac{1}{\hat{r}_2}\left(1 + \frac{v_2^+}{\sigma_k}\right)(C_{L1,2}k_1^+ + C_{D,2}k_2^+ + C_{U1,2}k_3^+)$$

$$-\left(1 + \frac{v_2^+}{\sigma_k}\right)(A_{L1,2}k_1^+ + A_{D,2}k_2^+ + A_{U1,2}k_3^+ + A_{U2,2}k_4^+) = S_k$$

for $3 \le j \le m-2$

$$-\frac{v_j^{+'}}{\sigma_k}(C_{L1,j}k_{j-1}^+ + C_{D,j}k_j^+ + C_{U1,j}k_{j+1}^+)$$

$$-\left(1 + \frac{v_j^+}{\sigma_k}\right)(A_{L2,j}k_{j-2}^+ + A_{L1,j}k_{j-1}^+ + A_{D,j}k_j^+ + A_{U1,j}k_{j+1}^+)$$

$$-\frac{1}{\hat{r}_j}\left(1 + \frac{v_j^+}{\sigma_k}\right)(C_{L1,j}k_{j-1}^+ + C_{D,j}k_j^+ + C_{U1,j}k_{j+1}^+) = S_k$$

$$C_{L2,m}k_{m-2}^+ + C_{L1,m}k_{m-1}^+ + C_{D,m}k_m^+ = 0$$

$$k_m^+ = 0 \tag{33}$$

The h-transport equation can also be written in terms of the second-order finite-difference approximations expressed above including the boundary conditions on h

$$C_{D,1}h_1^+ + C_{U1,1}h_2^+ + C_{U2,1}h_3^+ = 0$$

$$-\frac{v_2^{+'}}{\sigma_h}\left(C_{L1,2}h_1^+ + C_{D,2}h_2^+ + C_{U1,2}h_3^+\right)$$

$$-\frac{1}{\hat{r}_2}\left(1+\frac{v_2^+}{\sigma_h}\right)\left(C_{L1,2}h_1^+ + C_{D,2}h_2^+ + C_{U1,2}h_3^+\right)$$

$$-\left(1+\frac{v_2^+}{\sigma_h}\right)\left(A_{L1,2}h_1^+ + A_{D,2}h_2^+ + A_{U1,2}h_3^+ + A_{U2,2}h_4^+\right) = S_h$$

for $3 \le j \le m - m_h$

$$-\frac{v_j^{+'}}{\sigma_h}\left(C_{L1,j}h_{j-1}^+ + C_{D,j}h_j^+ + C_{U1,j}h_{j+1}^+\right)$$

$$-\left(1+\frac{v_j^+}{\sigma_h}\right)\left(A_{L2,j}h_{j-2}^+ + A_{L1,j}h_{j-1}^+ + A_{D,j}h_j^+ + A_{U1,j}h_{j+1}^+\right)$$

$$-\frac{1}{\hat{r}_j}\left(1+\frac{v_j^+}{\sigma_h}\right)\left(C_{L1,j}h_{j-1}^+ + C_{D,j}h_j^+ + C_{U1,j}h_{j+1}^+\right) = S_h$$

for $m - m_h + 1 \le j \le m$

$$h_j^+ = f_{h0} \tag{34}$$

where f_{h0} is the near-wall asymptotic solution for h^+ at node j, and m_h is an integer that determines how many nodes are calculated from the near-wall solution. The integer $m_h \ge 7$ should be used if the asymptotic solution of h^+ is singular. Otherwise, $m_h = 1$ can generally be used.

The formulations given in Eqs. (33) and (34) are for fully-developed flow in a pipe. This formulation can be used to calculate fully-developed flow in a channel by making minor modifications. Here we define **P** as an array that can be conditionally evaluated and used for the terms in the pipe formulation that are different than those in the channel formulation. For example, in the k-equation P_j can be evaluated as follows

for $1 \le j \le m$

$$\text{if pipe flow;} \quad P_j = \frac{1}{\hat{r}_j}\left(1+\frac{v_j^+}{\sigma_k}\right) \tag{35}$$

$$\text{if channel flow;} \quad P_j = 0$$

Solution Procedure

The systems of equations given in Eqs. (33) and (34) can be solved through an iterative process by lagging certain terms. The systems reduce to tridiagonal systems if any terms other than the tridiagonal terms are lagged (moved to the right-hand side of the system) along with the source terms. The solution process becomes more stable if part of the off-diagonals of the resulting tridiagonal system are also lagged. In the following algorithms, Ω is a relaxation factor, and Γ is a blending factor. Given initial estimates for $\mathbf{k}^+$ and $\mathbf{h}^+$, the following arrays can be calculated

for $1 \le j \le m$; $v_j^+ = f_v$; $u_j^{+'} = -R_\tau \hat{r}_j / (1+v_j^+)$

$$u_1^{+''} = C_{D,1}u_1^{+'} + C_{U1,1}u_2^{+'} + C_{U2,1}u_3^{+'}$$

$$k_1^{+'} = 0; \quad h_1^{+'} = 0; \quad v_1^{+'} = 0$$

for $2 \le j \le m-1$

$$u_j^{+''} = C_{L1,j}u_{j-1}^{+'} + C_{D,j}u_j^{+'} + C_{U1,j}u_{j+1}^{+'}$$

$$k_j^{+'} = C_{L1,j}k_{j-1}^+ + C_{D,j}k_j^+ + C_{U1,j}k_{j+1}^+ \tag{36}$$

$$h_j^{+'} = C_{L1,j}h_{j-1}^+ + C_{D,j}h_j^+ + C_{U1,j}h_{j+1}^+$$

$$v_j^{+'} = C_{L1,j}v_{j-1}^+ + C_{D,j}v_j^+ + C_{U1,j}v_{j+1}^+$$

$$u_m^{+''} = C_{L2,m}u_{m-2}^{+'} + C_{L1,m}u_{m-1}^{+'} + C_{D,m}u_m^{+'}; \quad k_m^{+'} = 0$$

$$h_m^{+'} = C_{L2,m}h_{m-2}^+ + C_{L1,m}h_{m-1}^+ + C_{D,m}h_m^+$$

$$v_m^{+'} = C_{L2,m}v_{m-2}^+ + C_{L1,m}v_{m-1}^+ + C_{D,m}v_m^+$$

Using the notation

$$T_{Lj}k_{j-1}^+ + T_{Dj}k_j^+ + T_{Uj}k_{j+1}^+ = B_j$$

as row j in the tridiagonal system, the following algorithm can be used to obtain an improved estimate for $\mathbf{k}^+$

for $1 \le j \le m$

  if pipe flow $P_j = (1 + v_j^+/\sigma_k)/\hat{r}_j$

  else $P_j = 0$

$T_{L1} = 0;\quad T_{D1} = C_{D,1};\quad T_{U1} = C_{U1,1};\quad B_1 = -C_{U2,1}k_3^+$

$$T_{L2} = -\left(\frac{v_2^{+'}}{\sigma_k} + P_2\right)C_{L1,2} - \left(1 + \frac{v_2^+}{\sigma_k}\right)A_{L1,2}$$

$$T_{D2} = -\left(\frac{v_2^{+'}}{\sigma_k} + P_2\right)C_{D,2} - \left(1 + \frac{v_2^+}{\sigma_k}\right)A_{D,2};$$

$$T_{U2} = -\left(\frac{v_2^{+'}}{\sigma_k} + P_2\right)C_{U1,2} - \left(1 + \frac{v_2^+}{\sigma_k}\right)A_{U1,2}$$

$$B_2 = S_k + \left(1 + \frac{v_2^+}{\sigma_k}\right)A_{U2,2}k_4^+$$

for $3 \le j \le m-2$

$$T_{Lj} = -\left(\frac{v_j^{+'}}{\sigma_k} + P_j\right)C_{L1,j} - \left(1 + \frac{v_j^+}{\sigma_k}\right)A_{L1,j}$$

$$T_{Dj} = -\left(\frac{v_j^{+'}}{\sigma_k} + P_j\right)C_{D,j} - \left(1 + \frac{v_j^+}{\sigma_k}\right)A_{D,j};$$

$$T_{Uj} = -\left(\frac{v_j^{+'}}{\sigma_k} + P_j\right)C_{U1,j} - \left(1 + \frac{v_j^+}{\sigma_k}\right)A_{U1,j};$$

$$B_j = S_k + \left(1 + \frac{v_j^+}{\sigma_k}\right)A_{L2,j}k_{j-2}^+ \tag{37}$$

$T_{Lm-1} = C_{L2,m};\quad T_{Dm-1} = C_{L1,m};\quad T_{Um-1} = C_{D,m}$

$B_{m-1} = 0;\quad T_{Lm} = 0;\quad T_{Dm} = 1;\quad T_{Um} = 0;\quad B_m = 0$

for $2 \le j \le m-1$

  $T_{Lj} = \Gamma_k T_{Lj};\quad T_{Uj} = \Gamma_k T_{Uj}$

  $B_j = B_j - (1 - \Gamma_k)(T_{Lj}k_{j-1}^+ + T_{Uj}k_{j+1}^+)$

$\mathbf{k}_{old} = \mathbf{k};\quad \mathbf{k} = [\mathbf{T}]^{-1}\mathbf{B};\quad \mathbf{k} = \mathbf{k}_{old} + \Omega_k(\mathbf{k} - \mathbf{k}_{old})$

The new estimate for $\mathbf{k}^+$ is used in Eq. (36) to update the arrays in that equation. A similar tridiagonal algorithm can then be used to evaluate an improved estimate for $\mathbf{h}^+$

for $1 \le j \le m$

  if pipe flow $P_j = (1 + v_j^+/\sigma_h)/\hat{r}_j$

  else $P_j = 0$

$T_{L1} = 0;\quad T_{D1} = C_{D,1};\quad T_{U1} = C_{U1,1};\quad B_1 = -C_{U2,1}h_3^+$

$$T_{L2} = -\left(\frac{v_2^{+'}}{\sigma_h} + P_2\right)C_{L1,2} - \left(1 + \frac{v_2^+}{\sigma_h}\right)A_{L1,2}$$

$$T_{D2} = -\left(\frac{v_2^{+'}}{\sigma_h} + P_2\right)C_{D,2} - \left(1 + \frac{v_2^+}{\sigma_h}\right)A_{D,2};$$

$$T_{U2} = -\left(\frac{v_2^{+'}}{\sigma_h} + P_2\right)C_{U1,2} - \left(1 + \frac{v_2^+}{\sigma_h}\right)A_{U1,2}$$

$$B_2 = S_h + \left(1 + \frac{v_2^+}{\sigma_h}\right)A_{U2,2}h_4^+$$

for $3 \le j \le m - m_h$

$$T_{Lj} = -\left(\frac{v_j^{+'}}{\sigma_h} + P_j\right)C_{L1,j} - \left(1 + \frac{v_j^+}{\sigma_h}\right)A_{L1,j}$$

$$T_{Dj} = -\left(\frac{v_j^{+'}}{\sigma_h} + P_j\right)C_{D,j} - \left(1 + \frac{v_j^+}{\sigma_h}\right)A_{D,j};$$

$$T_{Uj} = -\left(\frac{v_j^{+'}}{\sigma_h} + P_j\right)C_{U1,j} - \left(1 + \frac{v_j^+}{\sigma_h}\right)A_{U1,j};$$

$$B_j = S_h + \left(1 + \frac{v_j^+}{\sigma_h}\right)A_{L2,j}h_{j-2}^+ \tag{38}$$

for $m - m_h + 1 \le j \le m;$

  $T_{Lj} = 0;\quad T_{Dj} = 1;\quad T_{Uj} = 0;\quad B_j = f_{h0}$

for $2 \le j \le m - m_h$

  $T_{Lj} = \Gamma_h T_{Lj};\quad T_{Uj} = \Gamma_h T_{Uj}$

  $B_j = B_j - (1 - \Gamma_h)(T_{Lj}h_{j-1}^+ + T_{Uj}h_{j+1}^+)$

$\mathbf{h}_{old} = \mathbf{h};\quad \mathbf{h} = [\mathbf{T}]^{-1}\mathbf{B};\quad \mathbf{h} = \mathbf{h}_{old} + \Omega_h(\mathbf{h} - \mathbf{h}_{old})$

The algorithms given in Eqs. (36)–(38) are repeated until a converged solution is obtained. Upon completion, $\mathbf{u}^+$ can be found by direct numerical integration using the trapezoidal rule

$u_m^+ = 0$

for $m-1 \ge j \ge 1;\quad u_j^+ = u_{j+1}^+ + \frac{1}{2}(\hat{r}_j - \hat{r}_{j+1})(u_j^{+'} + u_{j+1}^{+'})$ (39)

Results

Results are included here for the Wilcox 1998 k-ω model [1]. This model can be written as

$$\frac{du^+}{d\hat{r}} = -\frac{R_\tau \hat{r}}{(1+v^+)}, \quad v^+ = f_\mu k^+ / \omega^+$$

$$\frac{1}{\hat{r}}\frac{d}{d\hat{r}}\left[-(1+v^+/\sigma_k)\hat{r}\frac{dk^+}{d\hat{r}}\right] = v^+\left(\frac{du^+}{d\hat{r}}\right)^2 - C_\mu f_k R_\tau^2 k^+ \omega^+$$

$$\frac{1}{\hat{r}}\frac{d}{d\hat{r}}\left[-(1+v^+/\sigma_\omega)\hat{r}\frac{d\omega^+}{d\hat{r}}\right] = C_{\omega1} f_1 f_\mu \left(\frac{du^+}{d\hat{r}}\right)^2 - C_{\omega2} f_2 R_\tau^2 \omega^{+2}$$

$$u^+(1) = 0, \quad k^+(1) = 0$$

$$\frac{dk^+}{d\hat{r}}(1) = 0, \quad \frac{dk^+}{d\hat{r}}(0) = 0, \quad \frac{d\omega^+}{d\hat{r}}(0) = 0$$

where

$$R_t \equiv \frac{k}{v\omega}, \quad \chi_k \equiv \frac{\nabla k \cdot \nabla \omega}{\omega^3}, \quad f_\mu = \frac{0.024 + R_t/6}{1 + R_t/6}$$

$$f_k = \frac{4/15 + (R_t/8)^4}{1 + (R_t/8)^4}\begin{Bmatrix} 1, & \chi_k \le 0 \\ \dfrac{1+680\chi_k^2}{1+400\chi_k^2}, & \chi_k > 0 \end{Bmatrix}$$

$$f_1 = \frac{1/9 + R_t/2.95}{f_\mu(1 + R_t/2.95)}, \quad f_2 = 1 \tag{40}$$

$$C_\mu = 0.09, \quad C_{\omega1} = 0.52, \quad C_{\omega2} = 0.072$$

$$\sigma_k = 2.0, \quad \sigma_\omega = 2.0$$

The shear-flow correction term in the damping equations can be written in cylindrical coordinates as

$$\chi_k \equiv \frac{\nabla k \cdot \nabla \omega}{\omega^3} = \frac{1}{\omega^3}\left(\frac{\partial k}{\partial r}\frac{\partial \omega}{\partial r} + \frac{1}{r^2}\frac{\partial k}{\partial \theta}\frac{\partial \omega}{\partial \theta} + \frac{\partial k}{\partial z}\frac{\partial \omega}{\partial z}\right) \tag{41}$$

For fully-developed pipe flow, derivatives with respect to θ and z are zero. Applying these simplifications to Eq. (41) and using the nondimensional definitions

$$k^+ \equiv \frac{k}{u_\tau^2}, \quad \omega^+ \equiv \frac{\omega v}{u_\tau^2} \tag{42}$$

along with those included in Eq. (16) gives

$$\chi_k \equiv \frac{\nabla k \cdot \nabla \omega}{\omega^3} = \frac{1}{\omega^3}\frac{\partial k}{\partial r}\frac{\partial \omega}{\partial r} = \frac{1}{\omega^{+3} R_\tau^2}\frac{\partial k^+}{\partial \hat{r}}\frac{\partial \omega^+}{\partial \hat{r}} \tag{43}$$

The Wilcox 1998 k-ω model is a special case of Eq. (18) where

$$h^+ = \omega^+, \quad f_v = f_\mu k^+ / \omega^+, \quad R_t \equiv \frac{k^+}{\omega^+}$$

$$S_k = v^+\left(\frac{du^+}{d\hat{r}}\right)^2 - C_\mu f_k R_\tau^2 k^+ \omega^+$$

$$S_h = C_{\omega1} f_1 f_\mu \left(\frac{du^+}{d\hat{r}}\right)^2 - C_{\omega2} f_2 R_\tau^2 \omega^{+2}$$

$$\chi_k = \frac{1}{\omega^{+3} R_\tau^2}\frac{\partial k^+}{\partial \hat{r}}\frac{\partial \omega^+}{\partial \hat{r}}, \quad f_\mu = \frac{0.024 + R_t/6}{1 + R_t/6} \tag{44}$$

$$f_k = \frac{4/15 + (R_t/8)^4}{1 + (R_t/8)^4}\begin{Bmatrix} 1, & \chi_k \le 0 \\ \dfrac{1+680\chi_k^2}{1+400\chi_k^2}, & \chi_k > 0 \end{Bmatrix}$$

$$f_1 = \frac{1/9 + R_t/2.95}{f_\mu(1 + R_t/2.95)}, \quad f_2 = 1$$

$$C_\mu = 0.09, \quad C_{\omega1} = 0.52, \quad C_{\omega2} = 0.072$$

$$\sigma_k = 2.0, \quad \sigma_\omega = 2.0$$

The algorithms given in Eqs. (36)–(38) can be used to solve this model where the functions are defined as

$$R_t \equiv \frac{k_j^+}{\omega_j^+}, \quad \chi_k = \frac{k_j^{+'}\omega_j^{+'}}{\omega_j^{+3} R_\tau^2}, \quad f_v = \frac{k_j^+}{\omega_j^+}\frac{0.024 + R_t/6}{1 + R_t/6}$$

$$f_k = \frac{4/15 + (R_t/8)^4}{1 + (R_t/8)^4}\begin{Bmatrix} 1, & \chi_k \le 0 \\ \dfrac{1+680\chi_k^2}{1+400\chi_k^2}, & \chi_k > 0 \end{Bmatrix}$$

$$S_k = v_j^+ u_j^{+'2} - 0.09 R_\tau^2 k_j^+ \omega_j^+ f_k \tag{45}$$

$$S_h = 0.52 \frac{1/9 + R_t/2.95}{1 + R_t/2.95} u_j^{+'2} - 0.072 R_\tau^2 \omega_j^{+2}$$

$$f_{h0} = \frac{6}{0.072 y_j^{+2}}$$

and $m_h = 7$ can be used because ω^+ is singular at the wall.

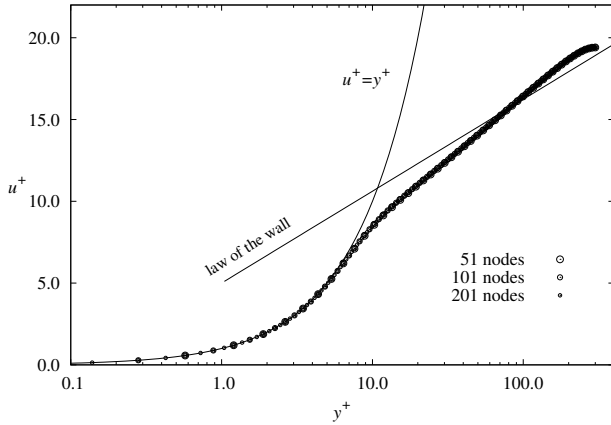Figure 1 shows the grid convergence results of a case for $R_\tau = 300$.

Figure 1. Grid convergence results for the Wilcox 1998 k-ω model.

Figure 2 shows the results of the algorithm presented here compared to results from code distributed by the author of the model, Dr. David C. Wilcox.
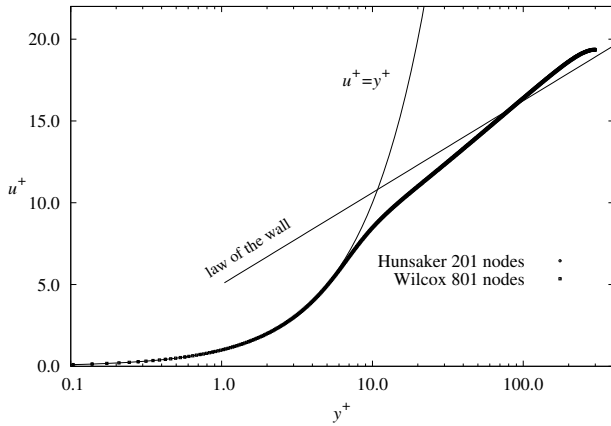

Figure 2. Results of the algorithm compared to results from code by Dr. David C. Wilcox.

These results show that the models grid-converge to the same solution.

Conclusion

A concise second-order algorithm has been presented that can be used to implement several two-equation turbulence models. The results of the Wilcox 1998 k-ω model have been included here.

Refernces

[1] Wilcox D. C., (1998), "The k-ω Model," *Turbulence Modeling for CFD*, Second Edition, DCW Industries, Inc., La Cañada, California, pp. 119–122.