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Abstract

We extend $2n$-dim biconformal gauge theory by including Lorentz-scalar matter fields of arbitrary conformal weight. We show that for a massless scalar field of conformal weight zero in a torsion-free biconformal geometry, the solution is determined by the Einstein equation on an n-dim submanifold, with the stress-energy tensor of the scalar field as source. The matter field satisfies the n-dim Klein-Gordon equation.

1 Introduction

Recently, we developed a new gauge theory of the conformal group, which solved many of the problems typically associated with scale invariance [1]. In particular, this new class of *biconformal geometries* has been shown to resolve the problem of writing scale-invariant vacuum gravitational actions in arbitrary dimension *without* the use of compensating fields [2]. In the cited work, we wrote the most general linear vacuum action and completely solved the resulting field equations subject only to a minimal torsion assumption. We found that all such solutions were foliated by equivalent n-dimensional Ricci-flat Riemannian spacetimes.

Reference [2] left an open question: how are matter fields coupled to biconformal gravity? *A priori*, it is not at all obvious that any action for biconformal matter permits the same embedded n-dim Riemannian structure that occurs for the vacuum case since biconformal fields are $2n$-dimensional. Indeed, in the case of standard n-dim conformal gauging ([3]-[5]), we generally require compensating fields to recover the Einstein equation with matter (see, for example, [6]-[9]). To answer this question for biconformal space, in the present work we extend the results of [2] by introducing a set of Klein-Gordon-type fields ϕ^m of conformal weight m into the theory. Using the Killing metric intrinsic to biconformal space, we write the natural kinetic term in the biconformally covariant derivatives of ϕ^m and find the resulting gravitationally coupled field equations. Then, for the case of one scalar field ϕ of conformal weight zero, we completely solve the field equations, under the assumption of vanishing torsion. We find that, as before, the solutions are foliated by equivalent n-dim Riemannian spacetime submanifolds whose curvatures now satisfy the usual Einstein equations with scalar matter. The field ϕ, which *a priori* depended on all $2n$ biconformal coordinates, is completely determined by the n coordinates of the submanifolds and satisfies the submanifold massless Klein-Gordon equation $\eta^{cd}D_cD_d\phi = 0$.

Thus, the new gauging establishes a clear connection between conformal gauge theory and general relativity with scalar matter, without the use of compensating fields.

The structure of the paper is as follows. In the next section, we extend the applicability of the biconformal dual introduced in [2] and establish how to write the usual kinetic action for scalar fields of arbitrary conformal weight using differential forms. Then, in Sec.(3) we find the field equations resulting from this action coupled to the linear gravity action introduced in [2]. Interestingly, these equations together with previous results show that pairs of scalar matter fields of conjugate conformal weight provide a source for the Weyl vector. Next, restricting to the case of a zero-weight scalar field, Secs.(4) and (5) give the solutions for the curvature and connection, respectively, of the background geometry in the case of vanishing torsion. Finally, in Sec.(6), we examine the field equations and other constraints on the matter field.

2 The biconformal dual and inner product

For full detail on the new conformal gauging we refer to [1]. We will use the same notation as in [2].

The Minkowski metric is written as $\eta_{ab} = diag(1\dots 1, -1)$, where $a, b, \dots = 1, \dots, n$. We denote the connection components (gauge fields) associated with the Lorentz, translation, co-translation, and dilation generators of the conformal group $O(n,2)$, $n > 2$, as the spin-connection $\boldsymbol{\omega}^a_b$, the solder-form $\boldsymbol{\omega}^a$, the co-solder-form $\boldsymbol{\omega}_a$, and the Weyl vector $\boldsymbol{\omega}^0_0$, respectively. The corresponding $O(n,2)$ curvatures $\boldsymbol{\Omega}^{\tilde{a}}_{\tilde{b}}$ $(\tilde{a}, \tilde{b}, \dots = 0, 1, \dots, n)$ are referred to as the curvature $\boldsymbol{\Omega}^a_b$, torsion $\boldsymbol{\Omega}^a \equiv \boldsymbol{\Omega}^a_0$, co-torsion $\boldsymbol{\Omega}_a \equiv \boldsymbol{\Omega}^0_a$, and dilation $\boldsymbol{\Omega}^0_0$, respectively, and are defined by the biconformal structure equations,

$$\begin{aligned}
\boldsymbol{\Omega}^a_b &= \mathbf{d}\boldsymbol{\omega}^a_b - \boldsymbol{\omega}^c_b\boldsymbol{\omega}^a_c - \Delta^{ad}_{cb}\boldsymbol{\omega}_d\boldsymbol{\omega}^c && (1)\\
\boldsymbol{\Omega}^a &= \mathbf{d}\boldsymbol{\omega}^a - \boldsymbol{\omega}^b\boldsymbol{\omega}^a_b - \boldsymbol{\omega}^0_0\boldsymbol{\omega}^a && (2)\\
\boldsymbol{\Omega}_a &= \mathbf{d}\boldsymbol{\omega}_a - \boldsymbol{\omega}^b_a\boldsymbol{\omega}_b - \boldsymbol{\omega}_a\boldsymbol{\omega}^0_0 && (3)\\
\boldsymbol{\Omega}^0_0 &= \mathbf{d}\boldsymbol{\omega}^0_0 - \boldsymbol{\omega}^a\boldsymbol{\omega}_a, && (4)
\end{aligned}$$

where $\Delta^{ab}_{cd} \equiv \delta^a_c\delta^b_d - \eta^{ab}\eta_{cd}$. In all cases differential forms are bold and the wedge product is assumed between adjacent forms. The position of any lowercase Latin index corresponds to the associated conformal weight: each upper index contributes $+1$ to the weight, while each lower index contributes -1.

Biconformal space is the $2n$-dimensional base space of the $O(n,2)$ principal bundle with homothetic fiber, first constructed in [1]. Each biconformal curvature may be expanded in the $(\boldsymbol{\omega}^a, \boldsymbol{\omega}_b)$ basis as

$$\boldsymbol{\Omega}^{\tilde{a}}_{\tilde{b}} = \tfrac{1}{2}\Omega^{\tilde{a}}_{\tilde{b}cd}\boldsymbol{\omega}^{cd} + \Omega^{\tilde{a}c}_{\tilde{b}d}\boldsymbol{\omega}_c\boldsymbol{\omega}^d + \tfrac{1}{2}\Omega^{\tilde{a}cd}_{\tilde{b}}\boldsymbol{\omega}_{cd}, \qquad (5)$$

where we use the convention of writing

$$\boldsymbol{\omega}^{ab\ldots c} \equiv \boldsymbol{\omega}^a \boldsymbol{\omega}^b \ldots \boldsymbol{\omega}^c = \boldsymbol{\omega}^a \wedge \boldsymbol{\omega}^b \wedge \ldots \wedge \boldsymbol{\omega}^c.$$

The three terms of eq.(5) will be called the spacetime-, cross-, and momentum-term, respectively, of the corresponding curvature.

Any r-form $\mathbf{U}$ $(r \leq 2n)$ defined on the cotangent bundle to biconformal space can be uniquely decomposed into a sum of (p,q)-forms,

$$\mathbf{U} = \sum_{p=0}^{r} \mathbf{U}_{p,r-p},$$

each of which is of the form

$$\mathbf{U}_{p,q} = \tfrac{1}{p!q!} U_{a_1 \cdots a_p}{}^{b_1 \cdots b_q} \boldsymbol{\omega}^{a_1 \cdots a_p} \boldsymbol{\omega}_{b_1 \cdots b_q} \qquad (p, q \leq n,\ p+q=r) \tag{6}$$

and has conformal weight $p-q$. For example, a 1-form can be written as

$$\mathbf{U} = U_a \boldsymbol{\omega}^a + U^a \boldsymbol{\omega}_a \equiv U_A \boldsymbol{\omega}^A, \tag{7}$$

where capital Latin indices denote both upper and lower lower-case Latin indices.

Biconformal space possesses a natural metric,

$$K^{AB} = \begin{pmatrix} 0 & \delta^a_b \\ \delta^a_b & 0 \end{pmatrix}$$

which is obtained when the non-degenerate Killing form of the conformal group $O(n,2)$ is restricted to the biconformal base space. The Killing metric defines a natural inner product between two 1-forms $\mathbf{U}$ and $\mathbf{V}$:

$$\langle \mathbf{U}, \mathbf{V} \rangle \equiv \tfrac{1}{2} K^{AB} U_A V_B = \tfrac{1}{2} \left(U^a V_a + V^a U_a \right) \tag{8}$$

Notice that, because the metric is essentially δ^a_b, whenever we sum an upper with a lower index we have implicitly used the Killing metric.

In [2], we demonstrated that when the indices of the $2n$-dimensional Levi-Civita symbol are sorted by weight, it may be written as the product of two n-dimensional Levi-Civita symbols of opposite weights:

$$\varepsilon_{a_1 \cdots a_n}{}^{b_1 \cdots b_n} = \varepsilon_{a_1 \cdots a_n}\ \varepsilon^{b_1 \cdots b_n},$$

where the mixed index positioning indicates the scaling weight of the indices, and <u>not</u> any use of the metric. The Levi-Civita tensor is normalized such that traces are given by

$$\varepsilon_{a_1 \cdots a_p c_{p+1} \cdots c_n} \varepsilon^{b_1 \cdots b_p c_{p+1} \cdots c_n} = p!(n-p)! \delta^{b_1 \cdots b_p}_{a_1 \cdots a_p},$$

where the antisymmetric δ-symbol is defined as

$$\delta^{b_1 \cdots b_p}_{a_1 \cdots a_p} \equiv \delta^{[b_1}_{a_1} \cdots \delta^{b_p]}_{a_p}.$$

Using the Levi-Civita tensor, the scale-invariant volume form of biconformal space is given by:

$$\mathbf{\Phi} = \varepsilon_{a_1\cdots a_n}{}^{b_1\cdots b_n}\boldsymbol{\omega}^{a_1\ldots a_n}\boldsymbol{\omega}_{b_1\cdots b_n}.$$

Notice that despite the mixed index positions, $\varepsilon_{a_1\cdots a_n}{}^{b_1\cdots b_n}$ is totally antisymmetric on all $2n$ indices.

We now define the biconformal dual of a general r-form. Let $\mathbf{U}_{p,q}$ be an arbitrary (p,q)-form as denoted in (6). Then the dual of $\mathbf{U}_{p,q}$ is an $(n-q, n-p)$-form, also of weight $p-q$, defined as

$$^*\mathbf{U}_{p,q} \equiv \frac{\tau(p,q)}{p!q!(n-p)!(n-q)!}U_{a_1\cdots a_p}{}^{b_1\cdots b_q}\varepsilon_{b_1\cdots b_n}{}^{a_1\cdots a_n}\boldsymbol{\omega}^{b_{q+1}\cdots b_n}\boldsymbol{\omega}_{a_{p+1}\cdots a_n},$$

where

$$\tau(p,q) = \begin{cases} 1 & if \quad p \geq q \\ (-1)^{n(p+q)} & if \quad p < q \end{cases}$$

This choice of $\tau(p,q)$ guarantees that for any r-form $\mathbf{U}$,

$$^{**}\mathbf{U} = (-1)^{r(n-r)}\mathbf{U}$$

and that for two arbitrary (p,q)- and (q,p)-forms, $\mathbf{U}_{p,q}$ and $\mathbf{V}_{q,p}$, respectively,

$$\mathbf{U}_{p,q}{}^*\mathbf{V}_{q,p} = \mathbf{V}_{q,p}{}^*\mathbf{U}_{p,q},$$

where we again assume wedge products between forms. It is then easy to show that for any two r-forms $\mathbf{U}$ and $\mathbf{V}$, the product $\mathbf{U}^*\mathbf{V}$ is proportional to the volume form $\mathbf{\Phi}$ and

$$\mathbf{U}^*\mathbf{V} = \sum_{p=0}^{r}\mathbf{U}_{p,r-p}{}^*\mathbf{V}_{r-p,p} = \sum_{p=0}^{r}\mathbf{V}_{p,r-p}{}^*\mathbf{U}_{r-p,p} = \mathbf{V}^*\mathbf{U},$$

because $\mathbf{U}_{p,r-p}{}^*\mathbf{V}_{r-q,q}$ vanishes unless $p = q$.

Now let $\mathbf{U}$ be a general 1-form as in (7). Then

$$^{**}\mathbf{U} = (-1)^{n-1}\mathbf{U}$$

and

$$\tfrac{1}{2}\mathbf{U}^*\mathbf{U} = \tfrac{(-1)^n}{n!^2}U_aU^a\mathbf{\Phi}. \tag{9}$$

Thus, by eq.(8), the term $\mathbf{U}^*\mathbf{U}$ is proportional to the inner product $\langle\mathbf{U},\mathbf{U}\rangle$:

$$\mathbf{U}^*\mathbf{U} = \tfrac{2(-1)^n}{n!^2}\langle\mathbf{U},\mathbf{U}\rangle\,\mathbf{\Phi} = \tfrac{(-1)^n}{n!^2}K^{AB}U_AU_B\mathbf{\Phi}. \tag{10}$$

We are now ready to build the biconformal theory of scalar matter. Let ϕ^m be a set of massless Lorentz-scalar fields of conformal weight $m \in \mathbf{Z}$ [10] and $\mathbf{D}\phi^m$ be their biconformally covariant derivatives defined by

$$\mathbf{D}\phi^m \equiv \mathbf{d}\phi^m + m\boldsymbol{\omega}_0^0\phi^m. \tag{11}$$

This covariant derivative is a 1-form which is also of weight m and can be expanded as

$$\mathbf{D}\phi^m = \boldsymbol{\omega}^A D_A \phi^m.$$

Since the dual operator preserves the conformal weight, ${}^*\mathbf{D}\phi^{-m}$ must be of weight $-m$, so that every term in the infinite sum

$$\sum_{m\in\mathbf{Z}} \mathbf{D}\phi^{m*}\mathbf{D}\phi^{-m}$$

is of conformal weight zero. Using (10) we see that

$$\mathbf{D}\phi^{m*}\mathbf{D}\phi^{-m} = \tfrac{(-1)^n}{n!^2} K^{AB} D_A\phi^m D_B\phi^{-m}\mathbf{\Phi}$$

for every m. We have thus arrived at the Weyl-scalar-valued action

$$S_M = \tfrac{1}{2}\bar{\lambda}\sum_m \int \mathbf{D}\phi^{m*}\mathbf{D}\phi^{-m} = \tfrac{1}{2}\bar{\lambda}\tfrac{(-1)^n}{n!^2}\sum_m \int K^{AB} D_A\phi^m D_B\phi^{-m}\mathbf{\Phi}. \quad (12)$$

We shall make use of the 'dual' form of S_M when we vary the action with respect to the field and with respect to the connection, whereas the form of S_M that explicitly displays the dependence on the Killing metric proves more useful in varying the base forms.

3 The linear scalar action

In a $2n$-dimensional biconformal space the most general Lorentz and scale-invariant action which is linear in the biconformal curvatures and structural invariants is

$$S_G = \int (\alpha\mathbf{\Omega}^{a_1}_{b_1} + \beta\delta^{a_1}_{b_1}\mathbf{\Omega}^0_0 + \gamma\boldsymbol{\omega}^{a_1}\boldsymbol{\omega}_{b_1})\boldsymbol{\omega}^{a_2\ldots a_n}\boldsymbol{\omega}_{b_2\ldots b_n}\varepsilon^{b_1\ldots b_n}\varepsilon_{a_1\ldots a_n},$$

first introduced in [2]. We will always assume non-vanishing α, β, and γ. For a set of massless Lorentz scalar fields ϕ^m of weight m with kinetic term S_M given by (12), we have

$$S = S_M + S_G. \quad (13)$$

Variation of this action with respect to the scalar fields yields the equation

$$0 = \mathbf{D}^*\mathbf{D}\phi^m \quad (14)$$

for every m, where

$$\mathbf{D}^*\mathbf{D}\phi^m \equiv \mathbf{d}^*\mathbf{D}\phi^m + m\boldsymbol{\omega}^{0*}_0\mathbf{D}\phi^m.$$

Variation with respect to the connection one-forms gives rise to the following field equations:

$$\begin{aligned}
\beta(\Omega^a{}_{ba} - 2\Omega_{ca}{}^{d}\delta^{ca}_{db}) &= -\lambda\Theta_b &(15)\\
\beta(\Omega_a{}^{ba} - 2\Omega^{cd}{}_{a}\delta^{ab}_{dc}) &= \lambda\Theta^b &(16)\\
\alpha(-\Delta^{af}_{eg}\Omega^b{}_{ab} + 2\Delta^{cf}_{eb}\delta^{ab}_{dg}\Omega_{ac}{}^{d}) &= 0 &(17)\\
\alpha(-\Delta^{gf}_{eb}\Omega_a{}^{ab} + 2\Delta^{af}_{ed}\delta^{gc}_{ab}\Omega^{bd}{}_{c}) &= 0 &(18)\\
\alpha\Omega^a_{bac} + \beta\Omega^0_{0bc} &= -\lambda\Upsilon_{bc} &(19)\\
2(\alpha\Omega^{ec}_{cd} + \beta\Omega^{0e}_{0d})\delta^{ad}_{eb} + \Lambda^a_b &= \lambda\Upsilon^a_b &(20)\\
\alpha\Omega_a^{bac} + \beta\Omega_0^{0bc} &= \lambda\Upsilon^{bc} &(21)\\
2(\alpha\Omega^{ce}_{dc} + \beta\Omega^{0e}_{0d})\delta^{ad}_{eb} + \Lambda^a_b &= \lambda\Upsilon^a_b &(22)
\end{aligned}$$

where the matter sources are given by

$$\begin{aligned}
\Upsilon_{ab} &\equiv \sum_m D_a\phi^m D_b\phi^{-m} = \Upsilon_{ba}\\
\Upsilon^a_b &\equiv \sum_m \left(D^a\phi^m D_b\phi^{-m} + D^c\phi^m D_c\phi^{-m}\delta^a_b\right)\\
\Upsilon^{ab} &\equiv \sum_m D^a\phi^m D^b\phi^{-m} = \Upsilon^{ba}\\
\Theta_b &= \sum_m m\phi^m D_b\phi^{-m}\\
\Theta^b &= \sum_m m\phi^m D^b\phi^{-m}
\end{aligned}$$

and we have defined

$$\begin{aligned}
\Lambda^a_b &\equiv (\alpha(n-1) - \beta + \gamma n^2)\delta^a_b\\
\lambda &\equiv \frac{1}{(n-1)!^2}\bar{\lambda}.
\end{aligned}$$

Note that since the spin connection does not occur in the covariant derivative (11), $\delta_{\boldsymbol{\omega}^a_b} S_M \equiv 0$, and there is no matter contribution to eq.(17) or (18). Combining equations (20) and (22) we see that the latter can be replaced by

$$\Omega^{ac}_{cd} = \Omega^{ca}_{dc}. \tag{23}$$

We remark that the biconformal structure equations together with eqs.(15)-(18) may be used to express the torsion and co-torsion in terms of the connection, the Weyl vector, and (here) the matter fields ϕ^m. In [2] it was observed that constraining the torsion to vanish also forces the Weyl vector to vanish, but this conclusion no longer holds with matter present. This suggests that setting the torsion to zero is not an undue constraint as assumed in [2], but rather, that the Weyl vector vanishes unless there are appropriate matter fields present. Taking this view, we are free to assume $\boldsymbol{\Omega}^a = 0$. Then, there exists a gauge in which the Weyl vector is given in terms of covariant derivatives of the fields ϕ^m by

$$\boldsymbol{\omega}_0^0 = \frac{1}{(n-1)(n-2)}\frac{\lambda}{\beta}\sum_m m\phi^m \mathbf{D}\phi^{-m},$$

Notice that $\boldsymbol{\omega}_0^0 = 0$ unless conjugate weights, $+m$ and $-m$ are both present.

We will explore such dilational sources further elsewhere. Here, since our goal is to derive the usual form of the Einstein equations with scalar matter, it is sufficient to restrict our attention to the case $m = 0$. Thus, for the remainder of this paper we restrict to the case of a scalar field ϕ of conformal weight zero. Then, the covariant derivative is simply the exterior derivative

$$\mathbf{D}\phi \equiv \mathbf{d}\phi = \boldsymbol{\omega}^A d_A \phi,$$

so the field equations reduce to

$$\begin{aligned} 0 &= {}^*\mathbf{d}^*\mathbf{d}\phi && (24)\\ 0 &= \Omega^a{}_{ba} - 2\Omega_{ca}^{\ \ d}\delta^{ca}_{db} && (25)\\ 0 &= \Omega_a^{\ ba} - 2\Omega^{cd}_{\ \ a}\delta^{ab}_{dc} && (26) \end{aligned}$$

together with eqs.(17)-(23), where $\Theta_b = 0$, $\Theta^b = 0$ and

$$\begin{aligned} \Upsilon_{ab} &\equiv d_a\phi\ d_b\phi = \Upsilon_{ba}\\ \Upsilon^a_b &\equiv d^a\phi\ d_b\phi + d^c\phi\ d_c\phi\ \delta^a_b\\ \Upsilon^{ab} &\equiv d^a\phi\ d^b\phi = \Upsilon^{ba}. \end{aligned}$$

4 Solution for the curvatures

We now find the most general solution to these equations subject only to the constraint of vanishing torsion. As discussed above, this condition no longer implies a vanishing Weyl vector. The Weyl vector nonetheless vanishes because of our choice to consider only zero conformal weight matter.

Despite vanishing torsion, the general approach to solving the field equations follows that of [2]. Starting with a general ansatz for the spin connection and Weyl vector, we first solve the torsion and co-torsion equations, eqs.(25, 26) and eqs.(17, 18). The Bianchi identity following from the vanishing torsion constraint and field equations (19)-(23) then determines the form of the curvature and dilation. In Sec.(5), we show that the vanishing torsion constraint also leads to a foliation by n-dimensional flat Riemannian manifolds. By invoking the gauge freedom on each of these manifolds, we show the existence of a second foliation by n-dimensional Riemannian spacetimes satisfying the Einstein equations with scalar matter.

To begin, we write the spin connection $\boldsymbol{\omega}^a_b$ as

$$\boldsymbol{\omega}^a_b = \boldsymbol{\alpha}^a_b + \boldsymbol{\beta}^a_b + \boldsymbol{\gamma}^a_b \qquad (27)$$

with $\boldsymbol{\alpha}^a_b$ and $\boldsymbol{\beta}^a_b$ defined by

$$\begin{aligned}
\mathbf{d}\boldsymbol{\omega}^a &= \boldsymbol{\omega}^b \boldsymbol{\alpha}^a_b + \tfrac{1}{2}\Omega^{abc}\boldsymbol{\omega}_{bc} \\
\mathbf{d}\boldsymbol{\omega}_a &= \boldsymbol{\beta}^b_a \boldsymbol{\omega}_b + \tfrac{1}{2}\Omega_{abc}\boldsymbol{\omega}^{bc}.
\end{aligned}$$

Using this ansatz in structure equations (2) and (3), Ω^{abc} and Ω_{abc} remain related to derivatives of the solder- and co-solder forms, whereas the other torsion and co-torsion terms are algebraic in the components of $\boldsymbol{\alpha}^a_b, \boldsymbol{\beta}^a_b$, $\boldsymbol{\gamma}^a_b$, and $\boldsymbol{\omega}^0_0$. Thus, the separation of the connection allows us to solve the torsion/co-torsion field equations (25), (26), (17), and (18) algebraically.

We simply state the result of this reduction here. More detail is available in [2]. Defining

$$\boldsymbol{\sigma}^a_b \equiv \boldsymbol{\alpha}^a_b - \boldsymbol{\beta}^a_b \equiv \sigma^a_{bc}\boldsymbol{\omega}^c + \sigma^{ac}_b \boldsymbol{\omega}_c$$

and setting

$$\boldsymbol{\Omega}^a = 0,$$

field equations (25), (26), (17), and (18) imply

$$\begin{aligned}
\boldsymbol{\omega}^0_0 &= 0 \\
\sigma^a_{bc} &= 0 \\
\sigma^{ba}_a &= 0
\end{aligned}$$

with no assumption concerning the co-torsion, curvature, or dilation. From these we find

$$\boldsymbol{\omega}^a_b = \boldsymbol{\alpha}^a_b.$$

The co-torsion cross- and momentum terms reduce to

$$\begin{aligned}
\Omega_{ac}^{\ \ b} &= 0 \\
\Omega_a^{\ bc} &= \sigma^{bc}_a - \sigma^{cb}_a,
\end{aligned}$$

so that the full co-torsion is

$$\boldsymbol{\Omega}_a = \tfrac{1}{2}\Omega_{acd}\boldsymbol{\omega}^{cd} + \sigma^{bc}_a \boldsymbol{\omega}_{bc}$$

with

$$\sigma^{ba}_a = 0.$$

The spacetime co-torsion Ω_{abc} remains undetermined.

Next, we turn our attention to the curvature and dilation equations, eqs.(19)-(23). The vanishing torsion constraint makes it possible to obtain an algebraic condition on the curvatures from the Bianchi identity associated with eq.(2). Taking the exterior derivative of eq.(2) gives

$$\boldsymbol{\omega}^b \boldsymbol{\Omega}^a_b = \boldsymbol{\omega}^a \boldsymbol{\Omega}^0_0, \tag{28}$$

which implies for the curvature components

$$\begin{aligned}
\Omega^{0cd}_{0} &= 0 & (29)\\
\Omega^{acd}_{b} &= 0 & (30)\\
\Omega^{0c}_{0d} &= -\tfrac{1}{n-1}\Omega^{ac}_{da} & (31)\\
\Omega^{ab}_{cd} &= \tfrac{1}{n-1}\Delta^{fa}_{cd}\Omega^{eb}_{fe} = -\Delta^{fa}_{cd}\Omega^{0b}_{0f} & (32)\\
\Omega^{0}_{0cd} &= \tfrac{1}{n-2}(\Omega^{a}_{dac} - \Omega^{a}_{cad}) & (33)
\end{aligned}$$

Next, we impose field equations (19)-(23) onto these conditions and see that eq. (21) is satisfied by virtue of eqs.(29) and (30) if and only if

$$\Upsilon^{ab} = 0,$$

so that

$$\begin{aligned}
d^a\phi &= 0\\
\Upsilon^{a}_{b} &= 0.
\end{aligned}$$

Imposing eqs.(20) and (22) onto (31) now yields for $\beta \neq (n-1)\alpha$

$$\Omega^{0a}_{0b} = \tfrac{1}{\alpha(n-1)-\beta}(\lambda(-d^a\phi d_b\phi + \tfrac{2}{n-1}d^c\phi d_c\phi\delta^a_b) - \tfrac{1}{n-1}(\alpha(n-1)-\beta+\gamma n^2)\delta^a_b$$

Since $d^a\phi = 0$, this reduces to

$$\Omega^{0a}_{0b} = -\frac{(\alpha(n-1)-\beta+\gamma n^2)}{(n-1)(\alpha(n-1)-\beta)}\delta^a_b \equiv -\chi\delta^a_b. \qquad (34)$$

Imposing eq.(19) onto (33) yields

$$\left.\begin{aligned}
\Omega^{0}_{0bc} &= 0\\
\Omega^{a}_{bac} &= \Omega^{a}_{cab}\\
\alpha\Omega^{a}_{bac} &= -\lambda\Upsilon_{bc}
\end{aligned}\right. \qquad if \qquad \beta \neq \tfrac{1}{2}(n-2)\alpha$$

If the first of these conditions is substituted back into (28), we obtain the cyclic identity on the spacetime curvature:

$$\Omega^{a}_{[bcd]} = 0.$$

Summarizing the forms of the dilation and curvature so far we have for the generic case

$$\begin{aligned}
\boldsymbol{\Omega}^{0}_{0} &= -\chi\boldsymbol{\omega}_a\boldsymbol{\omega}^a & (35)\\
\boldsymbol{\Omega}^{a}_{b} &= \tfrac{1}{2}\Omega^{a}_{bcd}\boldsymbol{\omega}^{cd} + \chi\Delta^{ca}_{bd}\boldsymbol{\omega}_c\boldsymbol{\omega}^d, & (36)
\end{aligned}$$

with

$$\begin{aligned}
\Omega^{a}_{[bcd]} &= 0 & (37)\\
\alpha\Omega^{a}_{bac} &= -\lambda\Upsilon_{bc}. & (38)
\end{aligned}$$

We now use the vanishing of the Weyl vector to obtain further constraints on $\mathbf{\Omega}^0_0$ and $\mathbf{\Omega}^a_b$. Substituting the restricted form of the dilation (35) into structure equation (4),

$$\mathbf{\Omega}^0_0 = \Omega^{0a}_{0b}\boldsymbol{\omega}_a\boldsymbol{\omega}^b = \boldsymbol{\omega}_a\boldsymbol{\omega}^a,$$

we see that

$$\Omega^{0a}_{0b} = \delta^a_b. \tag{39}$$

The last equation has to be equal to the dilation crossterm as given by eq.(34), which implies $\chi = -1$, i.e. a relationship between the constants α, β, and γ:

$$\frac{\gamma n}{\alpha(n-1)-\beta} = -1.$$

Hence, the volume (γ) term must necessarily be present in the action. Thus, in the generic case, where

$$\beta \neq (n-1)\alpha;\ \beta \neq \tfrac{1}{2}(n-2)\alpha,$$

we have a two-parameter class of allowed actions, differing only in the constant $\frac{\lambda}{\alpha}$. Through eq.(32), eq.(39) also implies that

$$\Omega^{ac}_{bd} = -\Delta^{ac}_{db}.$$

We have now satisfied all of field equations (15)-(23). The curvatures take the form

$$\begin{aligned}
\mathbf{\Omega}^a &= 0 && (40)\\
\mathbf{\Omega}_a &= \sigma^{bc}_a\boldsymbol{\omega}_{bc} + \tfrac{1}{2}\Omega_{abc}\boldsymbol{\omega}^{bc} && (41)\\
\mathbf{\Omega}^0_0 &= \boldsymbol{\omega}_a\boldsymbol{\omega}^a && (42)\\
\mathbf{\Omega}^a_b &= \tfrac{1}{2}\Omega^a_{bcd}\boldsymbol{\omega}^{cd} - \Delta^{ca}_{bd}\boldsymbol{\omega}_c\boldsymbol{\omega}^d && (43)
\end{aligned}$$

subject to the constraints

$$\begin{aligned}
\Omega^a_{[bcd]} &= 0\\
\alpha\Omega^a_{bac} &= -\lambda\Upsilon_{bc}\\
\sigma^{ba}_a &= 0\\
d^a\phi &= 0
\end{aligned}$$

Notice that the dilation is necessarily non-degenerate, but may not be closed.

In the next section, we find further constraints on the curvatures arising from the structure equations.

5 Solution for the connection

While eqs.(40)-(43) for the curvatures satisfy all of the field equations, they do not fully incorporate the form of the biconformal structure equations as embodied in the Bianchi identities. Therefore, in this section, we turn to the consequences of the form of the curvatures on the connection.

So far, we have established that in the as yet unspecified original gauge the Weyl vector vanishes and the spin connection is fully determined by the solder form $\boldsymbol{\omega}^a$:

$$\begin{aligned} \boldsymbol{\omega}_0^0 &= 0 \\ \boldsymbol{\omega}_b^a &= \boldsymbol{\alpha}_b^a = \alpha_{bc}^a \boldsymbol{\omega}^c + \alpha_b^{ac} \boldsymbol{\omega}_c. \end{aligned}$$

Substituting the reduced curvatures into eqs.(1)-(3) (eq.(4) is identically satisfied by (42)), the structure equations now take the form

$$\mathbf{d}\boldsymbol{\alpha}_b^a = \boldsymbol{\alpha}_b^c \boldsymbol{\alpha}_c^a + \tfrac{1}{2}\Omega_{bcd}^a \boldsymbol{\omega}^{cd} \tag{44}$$
$$\mathbf{d}\boldsymbol{\omega}^a = \boldsymbol{\omega}^b \boldsymbol{\alpha}_b^a \tag{45}$$
$$\mathbf{d}\boldsymbol{\omega}_a = \boldsymbol{\alpha}_a^b \boldsymbol{\omega}_b + \sigma_a^{bc} \boldsymbol{\omega}_{bc} + \tfrac{1}{2}\Omega_{abc} \boldsymbol{\omega}^{bc}. \tag{46}$$

We observe that eq.(45) is in involution. By the Frobenius theorem, we can consistently set $\boldsymbol{\omega}^a$ to zero and obtain a foliation by submanifolds, where the spin connection and the co-solder form reduce to

$$\begin{aligned} \mathbf{f}_a &\equiv \boldsymbol{\omega}_a|_{\boldsymbol{\omega}^a=0} \\ \hat{\boldsymbol{\alpha}}_b^a &\equiv \boldsymbol{\alpha}_b^a|_{\boldsymbol{\omega}^a=0} = \alpha_b^{ac}\mathbf{f}_c \\ \hat{\sigma}_a^{bc} &\equiv \sigma_a^{bc}|_{\boldsymbol{\omega}^a=0}. \end{aligned}$$

Then each submanifold is described by the reduced structure equations

$$\begin{aligned} \mathbf{d}\hat{\boldsymbol{\alpha}}_b^a &= \hat{\boldsymbol{\alpha}}_b^c \hat{\boldsymbol{\alpha}}_c^a \\ \mathbf{d}\mathbf{f}_a &= \hat{\boldsymbol{\alpha}}_a^b \mathbf{f}_b + \hat{\sigma}_a^{bc} \mathbf{f}_{bc}. \end{aligned}$$

Since the spin-connection is involute, there exists a Lorentz gauge transformation such that $\hat{\boldsymbol{\alpha}}_b^a = 0$ on each submanifold, i.e. $\alpha_b^{ac} = 0$. With this gauge choice the system reduces to simply

$$\mathbf{d}\mathbf{f}_a = \hat{\sigma}_a^{bc} \mathbf{f}_{bc}. \tag{47}$$

This can be solved in the usual way giving $\hat{\sigma}_a^{bc}$ in terms of $\mathbf{f}_a$ and $\mathbf{d}\mathbf{f}_a$. Since this solution has the same form on each leaf of the foliation, the expression for σ_a^{bc} remains valid when it is extended back to the full space, i.e. σ_a^{bc} depends on the $2n$ biconformal coordinates only through its dependence on $\mathbf{f}_a$.

The existence of an $\hat{\boldsymbol{\alpha}}_b^a = 0$ gauge depends only on vanishing torsion, which leads to an involution of the solder form $\boldsymbol{\omega}^a$ and the resulting Bianchi identity, eq.(28). Therefore, the results of Sec.(4) remain valid in the $\hat{\boldsymbol{\alpha}}_b^a = 0$ gauge.

Returning to the full biconformal space, we now have a gauge

$$\boldsymbol{\omega}^a \equiv \mathbf{e}^a$$

such that the spin connection is

$$\boldsymbol{\alpha}^a_b = \alpha^a_{bc}\mathbf{e}^c,$$

while the co-solder form may be written in terms of $\mathbf{f}_a$ and an additional term linear in the solder form,

$$\boldsymbol{\omega}_a = \mathbf{f}_a + h_{ab}\mathbf{e}^b, \tag{48}$$

Notice that while $\mathbf{f}_a$ depends on all $2n$ coordinates of this extension, it remains independent of the 1-forms $\mathbf{e}^a$. This means that $\mathbf{df}_a$ remains at least linear in $\mathbf{f}_a$, and is consequently involute. We can therefore turn the problem around, setting $\mathbf{f}_a = 0$ to obtain a second foliation of the biconformal space. We can define $\mathbf{h}_a$ in terms of this involution, setting

$$\mathbf{h}_a \equiv \boldsymbol{\omega}_a|_{\mathbf{f}_a=0} = h_{ab}\mathbf{e}^b,$$

with h_{ab} arbitrary. The structure equations for the $\mathbf{f}_a = 0$ geometry are

$$\begin{aligned}
\mathbf{d}\boldsymbol{\alpha}^a_b &= \boldsymbol{\alpha}^c_b\boldsymbol{\alpha}^a_c + \tfrac{1}{2}\Omega^a_{bcd}\mathbf{e}^{cd} && (49)\\
\mathbf{de}^a &= \mathbf{e}^b\boldsymbol{\alpha}^a_b && (50)\\
\mathbf{dh}_a &= \alpha^b_{ac}\mathbf{e}^c\mathbf{h}_b + \sigma^{bc}_a\mathbf{h}_{bc} + \tfrac{1}{2}\Omega_{abc}\mathbf{e}^{bc} && (51)
\end{aligned}$$

Eq.(51) determines $\mathbf{h}_a$ once the spacetime co-torsion, Ω_{abc}, is given, with little consequence for the rest of the geometry. We focus our attention on the first two equations. Since they are unchanged from their full biconformal form, the curvature

$$\mathbf{R}^a_b \equiv \tfrac{1}{2}\Omega^a_{bcd}\mathbf{e}^{cd}$$

and connection $\boldsymbol{\alpha}^a_b$ (and of course $\mathbf{e}^a$, by the first involution) are fully determined on the n-dimensional $\mathbf{f}_a = 0$ submanifold. Thus, $\boldsymbol{\alpha}^a_b$ is the usual spin connection compatible with $\mathbf{e}^a$, while $\mathbf{R}^a_b$ is its curvature. If we let

$$R_{ab} \equiv \Omega^c_{acb},$$

then the Bianchi identity following from (49),

$$\mathbf{DR}^a_b = 0,$$

implies that the tensor

$$G_{ab} \equiv R_{ab} - \tfrac{1}{2}\eta_{ab}R$$

is divergence-free.

Now that Ω^c_{acb} is seen to be the Ricci tensor of an underlying n-dim submanifold, it follows from the remaining condition (38),

$$\begin{aligned}
R_{ab} &= \kappa\Upsilon_{ab}\\
\kappa &\equiv -\tfrac{\lambda}{\alpha}
\end{aligned}$$

that these submanifolds satisfy the Einstein equations,

$$G_{ab} = \kappa T_{ab}, \tag{52}$$

with the divergence-free stress-energy tensor given by derivatives of the matter field:

$$T_{ab} \equiv d_a\phi d_b\phi - \frac{1}{2}\eta_{ab}\eta^{ce}d_c\phi d_e\phi. \tag{53}$$

Even though the co-torsion has a nonvanishing spacetime projection, the curvature is the one computed from the solder form $\boldsymbol{\omega}^a$ alone. This is our principal result, establishing a direct connection between general relativity with scalar matter and the more general structure of biconformal gauge theory with scalar matter.

6 Constraints on the matter field

We have seen that the Bianchi identity associated with eq.(2) with vanishing torsion together with field equation (21) imply

$$d^a\phi = 0$$

or

$$\mathbf{d}\phi = \boldsymbol{\omega}^a d_a\phi. \tag{54}$$

We now show that this implies that the scalar field ϕ depends only on the n coordinates spanning each leaf of the $\mathbf{f}_a = 0$ foliation and identically satisfies its own field equation.

Based on the involution for $\boldsymbol{\omega}^a$ there exist n coordinates x^μ of weight $+1$ such that

$$\boldsymbol{\omega}^a = e_\mu^{\ a}\mathbf{d}x^\mu$$

with the component matrices $e_\mu^{\ a}$ necessarily invertible. From eq.(45), we immediately find that $e_\mu^{\ a} = e_\mu^{\ a}(x)$. Similarly, it can be shown from the $\mathbf{f}_a$ involution that there exist n complementary coordinates y_ν of weight -1 such that $\mathbf{f}_a$ takes the form

$$\mathbf{f}_a = f_a^{\ \mu}\mathbf{d}y_\mu.$$

Since both $\mathbf{f}_a$ and $\mathbf{d}y_\mu$ completely span the co-tangent bundles of the $\boldsymbol{\omega}^a = 0$ submanifolds, the component matrices $f_a^{\ \mu}$ are also necessarily invertible. Thus, (x^μ, y_μ) form a complete set of local coordinates on biconformal space. If we expand $\mathbf{d}\phi$ in this coordinate basis, we obtain

$$\mathbf{d}\phi = \partial_\mu\phi\mathbf{d}x^\mu + \partial^\mu\phi\mathbf{d}y_\mu,$$

where $(\partial_\mu, \partial^\mu)$ denote derivatives with respect to (x^μ, y_ν). Equating this general form with the derived form (54),

$$\mathbf{d}\phi = e_\mu^{\ a}(x)d_a\phi\mathbf{d}x^\mu,$$

we see that $\phi = \phi(x)$. Hence, the scalar field is entirely defined on the n-dimensional Riemannian spacetime.

Using eq.(48) to expand the co-solder-form as

$$\boldsymbol{\omega}_a = f_a^{\ \mu}\mathbf{d}y_\mu + h_{ab}e_\mu^{\ b}\mathbf{d}x^\mu,$$

it is now easy to show that

$$d^a\phi = 0 \Rightarrow d^a d_a\phi = 0.$$

This result, together with the vanishing torsion constraint and eqs.(45)-(46), imply that the field equation for ϕ, eq.(24), is identically satisfied.

However, the field is constrained by the fact that the stress-energy tensor (53) is, by eq.(52), proportional to the divergence-free Einstein tensor and hence must be itself divergence-free. Since

$$\begin{aligned}
\eta^{be}D_eT_{ab} &= \eta^{be}D_e\left(D_a\phi D_b\phi - \tfrac{1}{2}\eta_{ab}\eta^{cf}D_c\phi D_f\phi\right) \\
&= \eta^{cd}(D_cD_d\phi)D_a\phi
\end{aligned}$$

the stress-energy tensor is divergence-free if and only if

$$\eta^{cd}D_cD_d\phi = 0.$$

This establishes that the scalar field ϕ satisfies the free Klein-Gordon (wave) equation on the n-dimensional spacetime.

7 Conclusions

We have developed aspects of the theory of scalar matter in biconformal space. Using the existence of an inner product of 1-forms and a dual operator, we constructed an action for a scalar matter field ϕ^m coupled to gravity and found the field equations. We solved them for the case of a scalar field of conformal weight zero in a torsion-free biconformal geometry. As in the vacuum case, the generic solutions are foliated by equivalent n-dimensional Riemannian spacetime manifolds. The curvature of each submanifold satisfies the usual Einstein equations with scalar matter. The scalar field is entirely defined on the submanifold and satisfies the n-dimensional massless Klein-Gordon equation.

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