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SYMMETRIC SOLUTIONS TO THE GAUSS-BONNET EXTENDED EINSTEIN EQUATIONS

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ABSTRACT

Low energy limits of string theories suggest that gravity lagrangians should include quadratic and higher order curvature terms, in the form of dimensionally continued Gauss-Bonnet densities. In an arbitrary number of dimensions, we consider the static, spherically symmetric solutions to the lowest order Gauss-Bonnet extended Einstein equations. We also find isotropic, homogeneous cosmological solutions with an ideal fluid source.

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In the low energy limit, string theories give rise to effective field theories of gravity. As shown by Scherk and Schwarz [1], the lagrangians for these theories contain terms of quadratic and higher order in the curvature in addition to the usual scalar curvature term. In particular, Zwiebach [2] showed that the lagrangian

$$\mathcal{L}_{1+2} = c_1\mathcal{L}_1 + c_2\mathcal{L}_2 = (-g)^{1/2}\{c_1R + c_2[R_{ab}{}^{cd}R_{cd}{}^{ab} - 4R^{ab}R_{ab} + R^2]\} \quad (1)$$

where $c_1 = (-16\pi G)^{-1}$, $c_2 = -c_1(3\alpha'(8\pi)^{3/2})$, and α' is the slope parameter, agrees with the three graviton scattering amplitude predicted by the Virasoro-Shapiro model. For arbitrary c_2, $\mathcal{L}_{1+2}$ is free of ghosts [2], [3]. Certain solutions of the $\mathcal{L}_{1+2}$ system and their stability have been studied independently by Boulware and Deser [4].

Zumino [3] showed that this ghost-free character is shared by all of the so-called "dimensionally-extended Euler characteristics", that is, all densities of the form

$$\mathcal{L}_k = \mathfrak{R}^{ab}\mathfrak{R}^{cd}\ldots\mathfrak{R}^{ef}\mathfrak{e}^{g}\ldots\mathfrak{e}^{h}\epsilon_{abcdfg\ldots h} \quad (2)$$

where there are k factors of the curvature two-form, $\mathfrak{R}^{ab}$; $\mathfrak{e}^g$ is the vielbein one-form, and $\epsilon_{abc\ldots h}$ is the D-dimensional Levi-Civita tensor. These densities can be written without the Levi-Civita symbol, and thereby extended to higher dimensions. Zwiebach conjectured that the full string lagrangian is of the form

$$\mathcal{L} = \sum_{k=0}^{k_m} c_k\mathcal{L}_k \quad (3)$$

where k_m is the integer part of (D-2)/2. The series must terminate at k_m since $\mathcal{L}_k$ vanishes for $2k \geq D$. For generality we have added the cosmological term c_0.

In section I below, the field equations following from $\mathcal{L}_{1+2}$ are derived. In section II, a particular solution to $\mathcal{L}_{1+2}$ is found for the case of a static, spherically symmetric vacuum in arbitrary dimension and in section III a cosmological solution is derived. Section IV contains a brief discussion of these results.

I. The Field Equations

As described by Zumino [3], [5], and mentioned above, the lagrangian density $\mathcal{L}_{1+2}$ can be written as the sum of the forms

$$\mathcal{L}_1 = \mathfrak{R}^{ab}\mathbf{e}^c\mathbf{e}^d \cdots \mathbf{e}^f\epsilon_{abcd\ldots f}$$

$$\mathcal{L}_2 = \mathfrak{R}^{ab}\mathfrak{R}^{cd}\mathbf{e}^f\mathbf{e}^g \cdots \mathbf{e}^h\epsilon_{abcdfg\ldots h}$$

Zumino varied this expression with respect to the connection and vielbein. The connection variation implies the vanishing of the torsion, while the veilbein variation gives

$$c_1(D-2)\mathfrak{R}^{ab}(\delta\mathbf{e}^c)\mathbf{e}^d \cdots \mathbf{e}^f\epsilon_{abcd\ldots f} + c_2(D-4)\mathfrak{R}^{ab}\mathfrak{R}^{cd}(\delta\mathbf{e}^f)\mathbf{e}^g \cdots \mathbf{e}^h\epsilon_{abcdfg\ldots h} = 0$$

Theseexpressions are D-forms in D dimensions, and therefore each in turn has components proportional to a Levi-Civita tensor. Using the identity

$$\epsilon_{\mu\upsilon\ldots\omega}\,\epsilon^{ab\ldots c} = e\, e_{[\mu}{}^a\, e_\upsilon{}^b \cdots e_{\omega\,]}{}^c$$

gives

$$\begin{aligned}(D-2)c_1(R_b{}^a - (1/_2)R\delta_b{}^a)&\\ + 80(D-4)c_2\big[R_{cb}{}^{de}R_{de}{}^{ca} - 2R_d{}^cR_{cb}{}^{da} - 2R_b{}^cR_c{}^a + RR_b{}^a&\\ - {}^1/_4\,\delta_b{}^a\big(R_{cd}{}^{ef}R_{ef}{}^{cd} - 4R_{cd}R^{cd} + R^2\big)\big] = 0&\end{aligned} \tag{4}$$

The trace of the field equations bears a simple relationship to the lagrangian

$$c_1(D-2)^2R + 40c_2(D-4)^2\big[R_{ab}{}^{cd}R_{cd}{}^{ab} - 4R^{ab}R_{ab} + R^2\big] = 0$$

Therefore, when the field equations are satisfied, the lagrangian becomes a multiple of the scalar curvature. This provides a simple check on solutions.

II. Static Spherically Symmetric Solution

For a static, spherically symmetric space, the metric may be put into the form

$$g_{ab} = \begin{pmatrix} -f^2 & & \\ & h^{-2} & \\ & & r^2\{\ \delta_{ij} + x_i x_j/(\ 1 - x^2\ )\ \} \end{pmatrix} \qquad i,j = 1, 2, \ldots ,d \qquad (5)$$

where the x_i are dimensionless angular coordinates of the maximally symmetric $d = (D-2)$ - dimensional subspace of radius r, and h and f are functions of r alone. The nonzero components of the curvature tensor are:

$$R_{ij}{}^{km} = r^{-2}(\ 1 - h^2\ )\ (\ \delta_i{}^m\delta_j{}^k - \delta_i{}^k\delta_j{}^m)$$

$$R_{ir}{}^{jr} = r^{-1}(hh')\ \delta_i{}^j$$

$$R_{or}{}^{or} = f^{-1}h(f'h)'$$

$$R_{io}{}^{jo} = r^{-1}(f^{-1}h^2f')\ \delta_i{}^j$$

After integrating over the x coordinates, the lagrangian density becomes:

$$\begin{aligned} \pounds'_{1+2} = {} & d(d-1)(d-2)(d-3)\,c_2 r^{d-4} f h^{-1}(1-h^2)^2 + 8d(d-1)c_2 r^{d-2} h^2 f'h' \\ & - 4d(d-1)(d-2)\,c_2 r^{d-3}(fh)'(1-h^2) - 4d(d-1)c_2 r^{d-2}(f'h)'(1-h^2) \\ & + 2c_1[r^d(f'h)' + d r^{d-1}(fh)'] - d(d-1)c_1 r^{d-2} f h^{-1}(1-h^2) \end{aligned}$$

Varying $\pounds'_{1+2}$ with respect to f and h and collecting terms we find

$$\begin{aligned} \delta f: \quad 0 = {} & d(d-1)(d-2)\,c_2[(d-3)r^{d-4}h^{-1}(1-h^2)^2 - 4r^{d-3}h'(1-h^2)] \\ & - dc_1[(d-1)r^{d-2}h^{-1}(1-h^2) + 2r^{d-1}h'] \end{aligned} \qquad (6)$$

$$\begin{aligned} \delta h: \quad 0 = {} & - d(d-1)(d-2)\,c_2[(d-3)r^{d-4}f(1-h^2)^2 - 4r^{d-3}f'h^2(1-h^2)] \\ & + dc_1[(d-1)r^{d-2}f(1-h^2) - 2r^{d-1}f'h^2]. \end{aligned} \qquad (7)$$

Adding h times equation (6) to f^{-1} times equation (7) gives:

$$(\beta r^{d-1} - r^{d-3}(1-h^2))(hh' - h^2f^{-1}f') = 0$$

where

$$\beta \equiv c_1 / 2c_2(d-1)(d-2).$$

Requiring $(\beta r^{d-1} - r^{d-3}(1-h^2)) = 0$ forces $h = 1 - r^2$; equation (7) then requires $g = 0$. This is impossible, so the second factor must vanish instead:

$$h' - hf^{-1}f' = 0.$$

Therefore, f is proportional to h, and may be made equal by rescaling the time by a constant factor.

To solve for h^2, note that the expression in equation (6) is a total derivative:

$$[r^{d-3}((1 - h^2)^2 - 2\beta r^2(1 - h^2))]' = 0.$$

With constant of integration s, this gives a quadratic equation for $(1 - h^2)$. The end result is

$$h^2 = f^2 = 1 - \beta r^2 \pm \beta r^2 (1 + 2s/\beta c_1 d r^{d+1})^{1/2} \qquad (8a)$$

where

$$\beta \equiv c_1 / 2c_2(d-1)(d-2). \qquad (8b)$$

The metric given by equations (5) and (8) solves the generalized Schwarzschild problem for the lagrangian $\pounds_{1+2}$. This is the central result of this section.

We now consider a few special cases. In four dimensions, the quadratic terms in $\pounds_{1+2}$ assemble to an exact divergence, and thus do not affect the field equations. Therefore the solution must exactly reproduce the ordinary Schwarzschild result. This follows since when $D = 4$, $d = 2$ and β becomes infinite. Expanding the square root in equation (8a) in this limit gives the ordinary Schwarzschild solution provided the upper sign is used and the constant s is chosen to be

$$s = -4mc_1$$

with m the usual Schwarzschild mass.

For $D > 4$ $(d > 2)$, ordinary Schwarzschild solutions are obtained from the generalized Schwarzschild solutions (5), (8) only as limiting cases for $c_1 \gg c_2$. Choosing the positive sign in (8a) again, and expanding the square root for $\beta \gg 1$ gives the simple result:

$$h^2 = f^2 = 1 + s/c_1 d r^{d-1} + O(\beta^{-1}).$$

Another important case is the asymptotic ($r \to \infty$) limit. It is essentially the same as the $\beta \to \infty$ limit, that is:

$$h^2 = f^2 = 1 + s/c_1 d r^{d-1} + O(r^{-2d})$$

Notice that $s/c_1 < 0$ is required for the potential to be attractive at large distances from the origin. This will be assumed in the following arguments.

We end this section with a few observations concerning the general case. Define $(r_s)^{d+1} \equiv |2s/\beta c_1 d|$. Then a brief calculation shows that the scalar curvature is given by

$$R = -\beta(d+1)(d+2)\{1-(1\pm(r_s/r)^{d+1})^{-3/2}[1 \pm {}^3/_2(r_s/r)^{d+1}+(d+3)(r_s/r)^{2(d+1)}/4(d+2)]\}.$$

where the top sign is for $\beta < 0$ and the bottom sign is for $\beta > 0$ (with $s/c_1<0$). This expression diverges at $r = 0$ for $\beta < 0$, and at $r = r_s$ for $\beta > 0$. We are interested in whether or not these singularities are surrounded by event horizons.

For a metric of the form (5) with $f^2 = h^2$, an event horizon can only occur at r_0 if $f^2(r_0) = h^2(r_0) = 0$. Such a pair of zeros of metric components will fail to signify an event horizon only if there is a singularity in the curvature at r_0. Rather than calculating the curvature at r_0 it is often simpler to find a coordinate transformation which makes the geometry transparent at the point in question (eg., Kruskal coordinates for the Schwarzschild solution). In particular, it is shown in Appendix I that given a spacetime with line element of the form

$$ds^2 = -g(r)\,dt^2 + g^{-1}(r)\,dr^2 + r^2\,d\Omega^2$$

where $g(r_0) = 0$, $g'(r_0) = k_1 \neq 0$ and $g''(r_0) = k_2$ there exists a local coordinate transformation in a neighborhood about r_0 which puts the line element in the form

$$ds^2 = -e^{k_2\eta/k_1}dUdV + r(U,V)^2 d\Omega^2.$$

In this expression, $r = r_0 + \eta$. At $r = r_0$, $\eta = 0$, $UV = 0$, so the spacetime is completely regular at $r = r_0$.

Now, consider the solution (8a) above for $\beta < 0$. $f^2 = h^2$ has only one real zero and $(f^2)' > 0$ everywhere, so there is exactly one horizon, given by the vanishing of the expression (8a). For $|\beta r_{Sch}^2| >> 1$, r_0 is given by

$$(r_0/r_{Sch})^{d-1} = 1 + (2\beta r_{Sch}^2)^{-1} + [{}^{(d-3)}/_{(d-1)}]\,(2\beta r_{Sch}^2)^{-2} + \cdots$$

where $r_{Sch} = -s/c_1 d$ is the ordinary Schwarzschild radius in $D = d + 2$ dimensions. In this approximation, $r_0 < r_{Sch}$. In string models β is inversely proportional to the slope parameter.

For $\beta > 0$ the solution given by equations (5) and (8a) fails whenever $r^{d+1} < (r_s)^{d+1} \equiv -2s/\beta c_1 d$, and the curvature is singular at r_s. The horizon position is determined by

$$f^2(r_0) = 1 - \beta r_0^2 + \beta r_0^2 (1-(r_s/r_0)^{d+1})^{1/2} = 0$$

The necessary and sufficient condition for the event horizon to lie outside of the singularity is that $r_0 > r_s$, which will in turn be the case only if

$$f^2(r_s) < 0$$

since $f^2(r)$ is monotone increasing for $r > r_s$. Thus, to have an event horizon requires

$$r_s^2 = (-2s/\beta c_1 d)^{2/(d+1)} > \beta^{-1}$$

or

$$-2s/c_1 d = 2r_{Sch} > \beta^{-(d-1)/2}.$$

For $D = 4$, in view of the definition (8b), this is satisfied for all $s/c_1 < 0$ (all positive mass solutions), but when $D > 4$ no horizon exists for sufficiently small values of $|s/c_1|$.

In the Virasoro-Shapiro string model, $\beta < 0$, so the string serves as a cosmic censor.

III. Cosmological Solution

For an isotropic, homogeneous spacetime, the metric may be put into the form:

$$g_{ab} = \begin{pmatrix} -1 & \\ & g^2[\delta_i{}^j + kx_i x^j/(1-kx^2)] \end{pmatrix} \qquad i,j = 1, 2, \ldots, d \tag{9}$$

where the x_i are dimensionless coordinates of the maximally symmetric $d = (D-1)$-dimensional subspace, $x^i = x_i$, and g is a function of t only. $k = 0, \pm 1$, depending on the sign of the curvature of the spacelike hypersurfaces. The stress-energy tensor following from the variation of the matter lagrangian $\pounds_M$ is assumed to take the ideal fluid form:

$$T_{ab} = p\, g_{ab} + (\rho + p)\, u_a u_b$$

where u_a is the comoving D-velocity of the fluid.

The calculation is essentially the same as for the static, spherically symmetric case in the previous section. The variation of the action leads to

$$\begin{aligned} 16\pi G c^2 d g^{d-1} p = {} & a(d-4) g^{d-5} (k + g'^2)^2 + 4a g^{d-4} g''(k + g'^2) \\ & -(d-2) b\, g^{d-3} (k + g'^2) - 2b g^{d-2} g'' \end{aligned} \tag{10a}$$

where a and b are defined as

$$\begin{aligned} a &= d(d-1)(d-2)(d-3)c_2 \\ b &= d(d-1)c_1 \,. \end{aligned} \tag{10b}$$

Both sides of this equation are total derivatives. The right side is

$$[a g^{d-4}(k + g'^2)^2 - b g^{d-2}(k + g'^2)]'$$

while conservation of energy ($T^{ab}{}_{;b} = 0$) gives on the left:

$$-d g^{d-1} g' p = (\rho g^d)'$$

Integrating, with constant of integration s, equation (10a) becomes a quadratic equation for $(k + g'^2)$.

The final solution is

$$k + g'^2 \equiv W(g) \equiv (bg^2/2a)[\,1 \pm (1 - 64\pi G(\rho + sg^{-d})ac_1/c_2b^2)^{1/2}] \qquad (11a)$$

where g may be found by inverting

$$t - t_0 = \int dg \,/\, (W(g) - k)^{1/2} \,. \qquad (11b)$$

The metric given by equations (9), (10b) and (11) is the homogeneous, isotropic solution for the lagrangian $£_{1+2} + £_M$.

Now we examine a few special cases.

Of course, the Robertson-Walker solution emerges when D = 4. In this case, d = 3, so that the constant a vanishes. Choosing the minus sign and expanding the square-root then gives

$$k + g'^2 = 8\pi G(\rho + sg^{-d})g^2/3 \,.$$

which clearly coincides with the ordinary Robertson-Walker solution when s = 0 (eg., [6]). This choice of s corresponds to the initial conditions

$$g(0) = \lambda$$
$$g'(0) = (b\lambda^2/a - k)^{1/2}$$

for an arbitrary constant λ.

In an arbitrary number of dimensions equation (11b) cannot be integrated in closed form for nonvanishing ρ. However, it can be solved when $\rho = s = 0$. Defining

$$\alpha \equiv b/2a$$

equation (11a) becomes

$$k + g'^2 = \alpha g^2 \,(1 \pm 1)$$

If the bottom (-) sign is chosen then $g=(-k)^{1/2}t$ and the spatial curvatures $R_{ij}{}^{kl}$ vanish. For the upper (+) sign there are two cases depending on the sign of α, with subcases depending on the value of k. Let $\chi \equiv |c/a|^{1/2}$.

Case 1: $\alpha > 0$

$k = 0 \quad (g^2 > 0) \quad g = g_0 e^{\pm\kappa t}$

$k = 1 \quad (g^2 > k/\alpha) \quad g = (4a\kappa)^{-1}[(a^2+1)\cosh\kappa t \pm (a^2-1)\sinh\kappa t]$

$k = -1 \quad (g^2 > 0) \quad g = (4a\kappa)^{-1}[(a^2-1)\cosh\kappa t \pm (a^2+1)\sinh\kappa t]$

where $a > 1$ is a constant.

Case 2: $\alpha < 0$ (No $k = 0$ or $k = 1$ solutions exist)

$k = -1 \quad (g^2 < k/\alpha) \quad g = \kappa^{-1}\sin\kappa(t-t_o)$

Therefore, both big bang solutions and bounce solutions are possible even with no matter present.

Naturally there are many other solutions for which the d dimensional subspace is no longer maximally symmetric. Of these, the most interesting are dimensional reductions with spatial manifolds $M_d = M_3 \times M_{d_1} \times M_{d_2} \times \cdots$, where M_3 is the spatial part of an ordinary $D = 4$ cosmological solution, and M_{d_1}, M_{d_2}, etc. are compact or small-volume manifolds. For example, Müller-Hoissen [9] has found that spontaneous compactification to $R^{(1,3)} \times S^n$ occurs when $\mathcal{L}_{1+2}$ is supplemented by either a cosmological constant or the curvature-cubic lagrangian $\mathcal{L}_3$.

IV. Summary and Discussion

We have found solutions to the system described by the lagrangian $\mathcal{L}_{1+2}$ for a static, spherically symmetric vacuum spacetime and for an isotropic, homogeneous, spacetime with an ideal fluid source.

For $D \leq 4$, $\mathcal{L}_2$ is a pure divergence, so that the static, spherically symmetric solution is the ordinary Schwarzschild solution, and the cosmological solution is the ordinary Robertson-Walker spacetime.

When $D > 4$, the solutions come in pairs because of the quadratic nature of the lagrangian.

For the static, sphericallly symmetric case, one of the solutions is asymptotically flat, while the second is not. For the asymptotically flat solution there are two cases, depending on the sign of the coupling β between $\mathcal{L}_1$ and $\mathcal{L}_2$. If $\beta < 0$ the solution is qualitatively similar to the ordinary D-dimensional Schwarzschild solution, in that it has a curvature singularity at $r = 0$ which is surrounded by an event horizon. However, the strength of the singularity is altered and the position of the horizon is shifted. If $\beta > 0$ then the solution has a curvature singularity at a finite value of the r-coordinate. While there may be an event horizon about this singularity, the $\beta > 0$ solution always permits naked singularities for sufficiently small masses. For the Virasoro-Shapiro string model, $\beta < 0$, so cosmic censorship is automatically enforced.

For $D > 4$, one of the pair of homogeneous, isotropic solutions is a modified version of the Robertson-Walker solution. The second solution includes both big bang and bounce cases in the absence of matter.

$\mathcal{L}_{1+2}$ contains only the first order correction to the Einstein lagrangian, and it is desirable to know how further corrections (according to the Zwiebach conjecture) affect the results. In particular, since the presence of $\mathcal{L}_2$ can reduce the radius of the horizon and alter the strength of the singularity, one wonders if further terms can remove the singularity entirely. However, the lagrangian and field equations quickly become unwieldy. The next order correction, $\mathcal{L}_3$, which is cubic in the curvature, contains 8 terms, while $\mathcal{L}_4$ contains 25 terms. These are listed in the appendix.

Nonetheless, for systems with high symmetry it is possible to obtain the solution to equations of motion containing arbitrarily high powers of the curvature. Such extended solutions will appear in a separate paper [7]. There are indications that, if the Zwiebach conjecture holds, there will be spacetime singularities.

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APPENDIX I: The event horizon in spherically symmetric spacetimes

Given a space with line element of the form

$$ds^2 = -g(r)\,dt^2 + g^{-1}(r)\,dr^2 + r^2\,d\Omega^2$$

where $g(r_0) = 0$, $g'(r_0) = k_1 \neq 0$ and $g''(r_0) = k_2$ there exists a local coordinate transformation in a neighborhood $r = r_0 + \eta$ about r_0 which puts the line element in the form

$$ds^2 = -e^{k_2\eta/k_1} dUdV + r(U,V)^2 d\Omega^2$$

which is nonsingular at $r = r_0$.

The proof follows the corresponding calculation for deriving Kruskal coordinates from the ordinary Schwarzschild solution [8]. First, define null coordinates u and v:

$$u = t - r_*$$
$$v = t + r_*$$

where

$$r_* = \int dr/g\,.$$

Then

$$ds^2 = -g\,du\,dv + r(u,v)^2\,d\Omega^2\,.$$

Now expand g(r) about r_0 by letting $r = r_0 + \eta$. Then

$$g(r) = k_1\eta + k_2\eta^2 + O(\eta^3)$$

and

$$r_* = (v-u)/2 = \int d\eta/\left[k_1\eta + k_2\eta^2 + O(\eta^3)\right]$$

$$= (1/k_1)\ln\eta - (k_2/k_1^2)\,\eta + O(\eta^2)\,.$$

Dropping terms of order η^2 or higher and exponentiating

$$\eta \approx e^{-k_1(u-v)} e^{k_2 \eta / k_1}$$

gives an implicit solution for η, so that the line element becomes:

$$ds^2 \approx -e^{-k_1(u-v)} e^{k_2 \eta / k_1} du dv + r(u,v)^2 d\Omega^2$$

Here, the potential horizon corresponds to $u \to \infty$, and/or $v \to -\infty$. Finally, defining new coordinates by

$$U = (1/k_1) e^{-k_1 u}$$
$$V = (1/k_1) e^{k_1 v}$$

puts the line element in the form

$$ds^2 = -e^{k_2 \eta / k_1} dU dV + r(U,V)^2 d\Omega^2$$

where now the horizon is given by $\eta = 0$, $UV = 0$ so that the spacetime is completely regular at $r(U,V)\big|_{UV=0} \equiv r_o$.

Appendix II: Third and fourth order extended Euler characteristics.

As mentioned in the main text, the expressions for $\mathcal{L}_k$ in terms of the curvature become increasingly complex as k increases. We give here the full expressions for $\mathcal{L}_3$ and $\mathcal{L}_4$. $\mathcal{L}_3$ has been derived independently by Müller-Hoissen [9].

$$
\begin{aligned}
\mathcal{L}_3 = (-g)^{1/2} \{ & 8R^3 - 96\, RR_a{}^bR_b{}^a + 24\, RR_{ab}{}^{cd}R_{cd}{}^{ab} + 192\, R_a{}^bR_c{}^dR_{bd}{}^{ac} \\
& + 128\, R_a{}^bR_b{}^cR_c{}^a - 192\, R_a{}^bR_{bc}{}^{de}R_{de}{}^{ac} + 16 R_{ab}{}^{cd}R_{cd}{}^{ef}R_{ef}{}^{ab} \\
& - 64\, R_{ce}{}^{ab}R_{af}{}^{cd}R_{bd}{}^{ef} \}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_4 = (-g)^{1/2} \{ & 16\, R^4 + 96\, R^2R_{ab}{}^{cd}R_{cd}{}^{ab} - 384\, R^2R_a{}^bR_b{}^a \\
& + 1344\, RR_a{}^bR_c{}^dR_{bd}{}^{ac} + 1024\, RR_a{}^bR_b{}^cR_c{}^a - 1536\, RR_a{}^bR_{bc}{}^{de}R_{de}{}^{ac} \\
& + 128\, RR_{ab}{}^{cd}R_{cd}{}^{ef}R_{ef}{}^{ab} - 704\, RR_{ce}{}^{ab}R_{af}{}^{cd}R_{bd}{}^{ef} + 768\, (R_a{}^bR_b{}^a)^2 \\
& - 384\, (R_a{}^bR_b{}^a)(R_{cd}{}^{ef}R_{ef}{}^{cd}) + 3072\, R_a{}^bR_{bc}{}^{ad}R_{de}{}^{fg}R_{fg}{}^{ce} \\
& - 1536\, R_a{}^bR_{bc}{}^{de}R_{de}{}^{fg}R_{fg}{}^{ac} + 6912\, R_a{}^bR_{bc}{}^{de}R_{df}{}^{ag}R_{eg}{}^{cf} \\
& - 5376\, R_a{}^bR_{bc}{}^{ad}R_d{}^eR_e{}^c - 1536\, R_a{}^bR_b{}^cR_c{}^dR_d{}^a - 2688\, R_a{}^bR_{bc}{}^{ad}R_{de}{}^{cf}R_f{}^e \\
& + 2880\, R_a{}^bR_b{}^cR_{cd}{}^{ef}R_{ef}{}^{ad} + 1344\, R_a{}^bR_{bc}{}^{de}R_{de}{}^{af}R_f{}^c + 3072\, R_a{}^bR_{fc}{}^{ae}R_{eb}{}^{cg}R_g{}^f \\
& + 48\, (R_{ab}{}^{cd}R_{cd}{}^{ab})^2 - 768\, R_{ab}{}^{cd}R_{cd}{}^{ae}R_{ef}{}^{gh}R_{gh}{}^{bf} + 96\, R_{gh}{}^{ab}R_{ab}{}^{cd}R_{cd}{}^{ef}R_{ef}{}^{gh} \\
& - 1920\, R_{eg}{}^{ab}R_{ab}{}^{cd}R_{ch}{}^{ef}R_{df}{}^{gh} + 768\, R_{ce}{}^{ab}R_{ag}{}^{cd}R_{bh}{}^{ef}R_{df}{}^{gh} \\
& - 1920\, R_{cg}{}^{ab}R_{ae}{}^{cd}R_{bh}{}^{ef}R_{df}{}^{gh} \}
\end{aligned}
$$

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