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Abstract

A reinterpretation of the Weyl vector in a Weyl geometry is shown to yield Dirac's 1951 classical theory of the electron, thereby avoiding the most obvious problems of measurement inherent in Weyl's original geometric formulation of electromagnetism. This paves the way for the consistency of recent metric-connection theories, for which similar questions of measurement can arise.

A number of geometric theories have been proposed which generalize the ideas of Weyl [1] concerning length change. These theories [2] allow the metric and the connection to be varied independently, and generically allow the possibility of noncompatible connections. Because of this, such metric-connection theories must answer the objections to Weyl's original theory. In this paper we show that a simple reinterpretation of the Weyl vector gives a fully consistent classical geometric theory of single-particle electromagnetism. It should be remembered that the theories of real interest may also avoid objections in other ways:

1. Lengths may not change. It is possible for the metric and connection to be noncompatible in ways that merely distort objects without altering spacetime volume. The measurement of such distortions is not considered here.

2. The scale of metric-connection theories is generally the Planck length rather than the atomic scale. If it can be shown that at atomic scales the metric and connection become compatible then there will be no obvious conflict with experiment. While the accuracy with which the compatibility must hold is rather high [3], it is actually less stringent than the cancellation of charges required to produce the known electrical neutrality of bulk matter.

In this paper, the issue is addressed on its own ground. A different physical interpretation of the Weyl vector is made which removes the immediate conflict with observation for the motion of a point particle in an electromagnetic field. The reinterpretation preserves Weyl's view of electromagnetism as a geometric phenomenon. The electromagnetic theory which emerges is identical to that formulated by Dirac in the early 50's in an attempt to realize the quantization of electric charge as a purely quantum phenomenon.

I. Weyl's theory.

For a complete discussion of Weyl geometry in the language of fibre bundles, see Audretsch, Gähler, and Straumann [4]. Here we will be content with a brief look at selected features. Weyl's theory of electromagnetism takes place in an extension of Riemannian geometry in which not only the orientation, but also the length of a vector changes under parallel transport. The additional vector required to describe the resulting non-Riemannian geometry is identified with the electromagnetic potential. The action is conformally invariant. A conformal transformation

shifts the vector potential by a gradient, thereby giving a geometric understanding of electromagnetic gauge transformation.

Briefly, the theory depends on a vector field W_μ related to the covariant derivative of the metric by:

$$D_\mu g_{\alpha\beta} = W_\mu g_{\alpha\beta} \tag{1}$$

W_μ is identified by Weyl with the electromagnetic potential, A_μ. A conformally transformed metric

$$g'_{\alpha\beta} = e^{\phi} g_{\alpha\beta} \tag{2}$$

has the derivative

$$D_\mu g'_{\alpha\beta} = (W'_\mu + \partial_\mu \phi) g'_{\alpha\beta}. \tag{3}$$

Eq.(1) remains conformally invariant if W_μ is altered at the same time by

$$W'_\mu = W_\mu - \partial_\mu \phi. \tag{4}$$

Now consider the consequence of this ansatz for the length of a vector when its components are parallel-transported around a closed path with tangent vector $w^\mu = \frac{dx^\mu}{d\tau}$. With

$$w^\mu D_\mu v^\alpha = 0 \tag{5}$$

we have

$$w^\mu D_\mu(L^2) \equiv w^\mu D_\mu(g_{\alpha\beta} v^\alpha v^\beta) = w^\mu W_\mu L^2 \tag{6}$$

so that after transport around a closed path the length-squared becomes:

$$L^2 = L_0^2 \exp[\oint W_\mu dx^\mu] \tag{7}$$

This expresses the principal difference between the Weyl and Riemannian geometries, and it was also the downfall of the theory. Einstein [5] pointed out that such a change in length is incompatible with the observed constancy of atomic frequencies. If the size of a particle depends upon the fields along its path, then atoms passing through different electromagnetic fields should have different sizes and hence different characteristic frequencies. The only way out of this

difficulty is to imagine that the integral in eq.(7) vanishes. But Stoke's theorem then implies the vanishing of the electromagnetic field strength, $F_{\mu\nu} = 2W_{[\mu,\nu]}$.

A second objection to the idea that length changes was put forward by Marzke and Wheeler [3]. It is based on the observation that masses as well as lengths should be altered in a Weyl-type theory. If this is true then electrons would become distinguishable by their different masses. The Pauli exclusion principal would no longer apply and all atomic orbitals would collapse to the lowest state. Based on rough estimates of the magnetic field near the earth's core a limit of about 1 part in 10^{60} for length change is produced.

II. An alternative interpretation of the Weyl potential, W_μ.

We will now consider an alternative version of Weyl's theory which not only avoids these problems, but turns the observations to some advantage. We will restrict the presentation to the case af a single particle moving in an electromagnetic field since our intent is to show the viability of length change theories in general, and not to develop a detailed model. Our real interest lies in establishing the consistency of metric-connection theories, in which the spacetime connection is regarded as independent of the metric.

The new aspect to be introduced to the Weyl ansatz is motivated by, but not dependent upon, the idea that all energy is equivalent in its effect on geometry. If this is the case then all forms of energy should be able to produce length change, and not just electromagnetism. Therefore, the line integral of W_μ must be generalized to include the energy-momentum vector.

In particular, consider the case of a point particle moving freely in empty, flat space. We interpret W_μ as the momentum of the particle:

$$W_\mu = p_\mu = m u_\mu \tag{8}$$

where the velocity u_μ satisfies

$$u_\mu u^\mu = -1. \tag{9}$$

With this supposition, the condition of eq.(7) that no change in size occur

$$\oint p_\mu dx^\mu = 0 \tag{10}$$

gives the requirement

$$p_{\mu,\nu} - p_{\nu,\mu} = 0 \tag{11}$$

The projection of eq.(11) along the path of the particle is:

$$0 = u^{\nu}(p_{\mu,\nu} - p_{\nu,\mu}) = \frac{dp_{\mu}}{d\tau} - \frac{1}{2}(u_{\nu}u^{\nu})_{,\mu}. \tag{12}$$

Because of eq.(9) this reduces to Newton's law for a free particle:

$$\frac{dp_{\mu}}{d\tau} = 0. \tag{13}$$

The spacelike part of eq.(11) only makes sense if the particle velocity is 'smeared out' into a velocity field. In this case it gives:

$$\text{curl } \mathbf{v} = 0. \tag{14}$$

The particle therefore is not spinning. In its integral form, $\oint p_i\, dx^i = 0$, this consequence of eq.(14) is easy to see. It implies the vanishing of angular momentum in a local sense. In general one might have counterrotating parts of a fluid with zero net angular momentum, but such a situation is ruled out here.

When the particle has charge q and an external electromagnetic field with potential A_{μ} is applied then W_{μ} must also include the electromagnetic contribution:

$$W_{\mu} = p_{\mu} - qA_{\mu} \tag{15}$$

We then have the condition

$$p_{\mu,\nu} - p_{\mu,\nu} - q(A_{\mu,\nu} - A_{\mu,\nu}) = 0. \tag{16}$$

These are precisely the equations of motion postulated by Dirac[6] in his 1951 classical theory of the electron. Dirac introduced his new formulation in an attempt to make the quantization of charge a quantum property of electromagnetism.

The implications of eq.(16) are easy to find. When contracted with u^{ν} this gives the Lorentz force law:

$$u^{\nu}(p_{\mu,\nu} - p_{\mu,\nu} - qF_{\mu\nu}) = \frac{dp_{\mu}}{d\tau} - qu^{\nu}F_{\mu\nu} = 0. \tag{17}$$

$$\frac{dp_{\mu}}{d\tau} = qu^{\nu}F_{\mu\nu}. \tag{18}$$

Again considering the particle to be composed of a velocity field so that we can take the spacelike part we find a relation between the local spin and the applied magnetic field.

$$\text{curl } \mathbf{v} = \frac{e}{mc}\mathbf{B}. \tag{19}$$

In the integral form:

$$\oint p_i\, dx^i = \frac{e}{c}\Phi, \tag{20}$$

where Φ is the magnetic flux through any surface bounded by the line integral. One can find distributions of material consistent with this law.

It should also be observed that eq.(19) is the London equation for superconductivity, so that there is the possibility here for a nonstandard theory of superconducting matter. Perhaps one can think of the electron as being composed of some fundamental superconducting fluid which is measurable only in discrete amounts when quantized.

It is important to note that the equation of motion, eq.(16), has a well-posed initial value formulation. To see this, note that eq.(16) implies that the Weyl potential is a pure gradient,

$$W_{\mu} = \phi_{,\mu} \tag{21}$$

so that eq.(15) becomes

$$p_{\mu} - qA_{\mu} = \phi_{,\mu} \tag{22}$$

This has a solution for the momentum field p_{μ} whenever there exists a gauge transformation such that the potential has norm $\hat{A}^2 = -m^2$. But classically such a gauge always exists because this condition

$$\hat{A}^2 = (qA_{\mu} = \phi_{,\mu})^2 = -m^2 \tag{23}$$

is the Hamilton-Jacobi equation for electromagnetism when ϕ is taken as the action evaluated along the classical path. This gauge is called the Dirac gauge or the nonlinear gauge, and has been used by Dirac[6] and Nambu[7].

In general what we find with this reinterpretation of W_μ is that in order for the system to move without scale change it must follow the classical equations of motion, with a subsidiary condition on the spin.

The subsidiary condition may be avoided if we introduce an additional term in our expression for W_μ. Taking our que from Dirac[8], eq.(15) is replaced by:

$$W_\mu = p_\mu - qA_\mu - \xi\partial_\mu\eta \qquad (24)$$

where ξ and η are functions such that

$$p^\mu\partial_\mu\xi = p^\mu\partial_\mu\eta = 0. \qquad (25)$$

Then the vanishing of the field strength of W_μ implies the existence of a gauge for A_μ such that:

$$p_\mu - qA_\mu - \xi\partial_\mu\eta = 0. \qquad (26)$$

The equations of motion become:

$$\frac{dp_\mu}{d\tau} = qu^\nu F_{\mu\nu}. \qquad (18)$$

and

$$\text{curl } v = \frac{e}{mc} B + \nabla\xi \times \nabla\eta. \qquad (19)$$

As shown by Dirac[8], the functions ξ and η can be chosen to allow fully general electromagnetic fields.

Therefore, with the identification of W_μ as in eq.(24), *a Weyl geometry provides a consistent geometric theory of electromagnetism which predicts no change in length measurements for a charged particle obeying the classical equations of motion*.

III. Conclusions and some further observations.

We have briefly considered an alternate interpretation of Weyl's theory. While many of the relations following from the change are suggestive, what one really needs is a complete theory of measurement in a Weyl geometry. Rather, the claim to be stressed here is that even such a slight reinterpretation of Weyl's theory as we have made can rescue the theory from its major criticisms. The condition that no change in length be observed for a particle moving in an electromagnetic field is now satisfied by requiring the particle to obey the Lorentz force law. Rather than forcing us to abandon the theory because length changes would conflict with observation, we now find that there will be no conflict with our measurements as long as a charged particle follows its classical trajectory. The interpretation suggests that classical physics may be regarded as emerging from a theory of length change in the limit of negligible length change.

The objection raised by Marzke and Wheeler, that the resulting change in masses will cause a breakdown in the exclusion principle, is also overcome. Since the Weyl geometry is conformally invariant, the notion of 'identical particles' must be modified to include particles which are identical modulo conformal transformation. Then even for nonclassical paths the exclusion principle still applies. No physical meaning can be ascribed to any differences which are related by local scale changes.

If we try to extend this model to the quantum realm, there is a dramatic difference from Weyl's theory. In Weyl's theory the magnitude of the deviation from identical masses depends on the magnitude of the electromagnetic field. To produce large differences in atomic spectra requires only moderate field strengths. But in the theory presented here, any deviation is on the order of Planck's constant, since the deviation of electrons from the classical path is scaled by $\hbar$. Planck's constant will therefore scale any spreading of spectral lines. In Weyl's original theory the scale for length change is set by the scale of electromagnetism. Here, the scale is still arbitrary since the classical motion is scale invariant. A constant multiplier for W_μ does not alter the motion.

The importance the observation by Marzke and Wheeler that mass should alter with length can be appreciated by considering what would happen in a scaling theory if masses did not change. Fixed orbits would not exist, because the radius of the orbit would change but the momentum of the orbiting body would not. Unless all of the equations required to specify the motion of the orbiting particle are invariant under scalings of the length, the orbit could not survive in a region of nonzero field. We conclude that the length dependence of mass is necessary in order to make any sense of classical measurement at all. In order to have a classical, length-preserving realm, mass must scale as an inverse length in the length-changing realm.

Finally, recall that the models we really wish to motivate are spacetime theories in which the connection and the metric are varied independently. In such theories, the metric is in general not compatible with the connection and lengths can change. What we conclude from the examples above is that these theories' failure to preserve lengths does not doom them at the outset. While they may require significant rethinking of measurement theory, it is at least conceivable that a metric-connection theory can describe the physical world.

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