

2012

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## Recommended Citation

Hazboun, J. S., & Wheeler, J. T. (2012). A systematic construction of curved phase space: A gravitational gauge theory with symplectic form. *Journal of Physics: Conference Series*, 360, 012013. doi:10.1088/1742-6596/360/1/012013

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2012 J. Phys.: Conf. Ser. 360 012013

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# A systematic construction of curved phase space: A gravitational gauge theory with symplectic form

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**Abstract.** General relativity can be constructed as a gauge theory using the quotient manifold strategy of [1, 2]. We consider a conformal gauging where the geometry is far richer than normal spacetime, including a symplectic form and the necessary emergence of Lorentzian signature. The resulting  $2n$ -dim manifold constitutes a relativistic phase space, and general relativity is recovered when we demand that the momentum space is flat. However, the full geometry allows for curved phase space.

**Introduction.** We construct a generalization of space-time geometry to a fully curved phase space with general relativity on the configuration submanifold. A variant of such a geometry was first considered by Max Born, attempting to introduce complementarity into general relativity [3]. More recent variants include work on regularization of 2+1 gravity [4], where curvature on the momentum space is a consequence of the theory, and in the principle of relative locality [5], where the structure of momentum space is the starting point. Abraham and Marsden define a relativistic phase space as the spacetime co-tangent bundle with a symplectic form [6]. This implies a metric of Lorentz signature<sup>1</sup>, and a symplectic form which defines canonical conjugacy. References [3, 4, 5] add curvature of the momentum space.

In this article we examine the space endowed *naturally* with all of these characteristics from the structure of Lie groups and techniques of gauge theory. We then take the picture further by developing an action functional over the entire phase space and showing how the resulting 8-dim structure reduces to general relativity on the configuration submanifold.

In the next section we explain the quotient manifold method of gauge theory. The result is *biconformal space*, which possesses all of the necessary characteristics described above. Next, we describe how a full phase space action, together with the structure equations, leads directly to general relativity in the special case of a momentum space without curvature. Concluding, we show that a gauge change of this space curves momentum space instead while leaving spacetime flat, thereby achieving properties that we expect underlie gravitational complementarity.

**Gauging using the quotient manifold method.** Standard approaches to gauge theory begin with a matter action globally invariant under some symmetry group  $\mathcal{H}$ . This action generally fails to be locally symmetric due to the derivatives of the fields, but can be made locally invariant by introducing an  $\mathcal{H}$ -covariant derivative. The connection fields used for this derivative are called

<sup>1</sup> This often goes unstated, since it is usually imposed from the beginning. We will discuss how its appearance is non-trivial and physically interesting.

gauge fields. The final step is to make the gauge fields dynamical by constructing their field strengths, which may be thought of as curvatures of the connection.

This procedure has led to a fiber bundle: in a neighborhood of every point of space-time, the bundle is a direct product of that neighborhood with the Lie symmetry group  $\mathcal{H}$ . This gives a larger manifold with dimension equal to the dimension of space-time plus the dimension of the Lie group. There exists a projection operator which projects these copies of  $\mathcal{H}$  back down to space-time. Choosing a gauge amounts to picking a cross-section of this bundle, i.e., one point from each of these copies of  $\mathcal{H}$ . Local symmetry amounts to dynamical laws which are independent of the choice of cross-section.

Attempts to write general relativity as a gauge theory led to an inversion of the standard construction. In addition to allowing the standard model and gravity to be placed on identical mathematical footings, this inverted approach has the advantage of allowing us to specify the symmetry in advance. This newer construction uses techniques developed by Cartan [7], giving the quotient group method of Ne'emann and Regge[1, 2]. We begin by taking the quotient of a Lie group,  $\mathcal{G}$ , by a Lie subgroup,  $\mathcal{H}$ . This quotient is necessarily a manifold,  $\mathcal{M}$ , which we identify with space-time or phase space. The quotient turns the group manifold  $\mathcal{G}$  into a fiber bundle with symmetry  $\mathcal{H}$  and base manifold  $\mathcal{M}$ . The Maurer-Cartan equations of  $\mathcal{G}$  (equivalent to the Lie algebra) immediately supply a natural connection on this bundle, giving us our gauge fields. Generalizing the Maurer-Cartan connection to allow horizontal curvature is precisely the introduction of the field strengths of the gauge fields. The horizontality property guarantees that calculations on any cross-section give identical results, making the entire structure locally gauge invariant. Finally, the curvatures and any desired matter fields are composed into an action. This action will have  $\mathcal{H}$ -symmetry, with the remaining, broken, group transformations of  $\mathcal{G}$  being replaced by diffeomorphisms on  $\mathcal{M}$  [8].

Ne'emann and Regge used the quotient manifold method to get general relativity as a gauge theory by taking the quotient of the Poincaré group by the Lorentz group, but in principle, any Lie group containing the Poincaré group is interesting in terms of gravity. MacDowell and Mansouri [9] obtained general relativity by gauging the de Sitter or anti-de Sitter groups, and using a Wigner-Inonu contraction to recover Poincaré symmetry. In their 1982 papers [10, 11], Ivanov and Niederle exhaustively considered quotients of the groups  $ISO(3,1)$ ,  $SO(4,1)$ ,  $SO(3,2)$  and  $SO(4,2)$  by various subgroups containing the Lorentz group.

**Biconformal Gauging.** We concern ourselves with a specific gauge theory of general relativity leading to a curved phase space, found by taking the quotient of the conformal group by its Weyl subgroup (Lorentz  $\otimes$  Dilatations). This quotient, first considered in [10, 11] results in an 8-dim ( $n$ -dim) manifold called biconformal space. It has been shown to be a phase space [12], to have the co-tangent bundle of general relativity as torsion-free solutions, and to provide a natural background for quantum mechanics, [13].

Throughout the following, Latin indices run  $a, b, \dots = 1, \dots, n$ . Though we describe the 8-dim ( $n = 4$ ) case, all results hold in any dim  $2n$ , with  $n > 2$ . Differential forms are written in boldface. Corresponding to the standard representation for the generators of the conformal group, we have connection 1-forms: the Lorentz or Euclidean spin connection,  $\omega_b^a$  and Weyl vector,  $\omega$ . The phase space is spanned by basis forms  $\mathbf{e}^a$  and  $\mathbf{f}_a$ , deriving from the translations and special conformal transformations, respectively. Because raised and lowered indices carry different conformal weight, we generally write the metric explicitly where required.

Of all the quotients looked at by Ivanov and Niederle there are two properties unique to this gauging. First, only in this gauging can the dilatational structure equation,  $\mathbf{d}\omega = \mathbf{e}^a \wedge \mathbf{f}_a + \Omega$ , give a symplectic form. This is the case if the dilatational curvature is closed,  $\mathbf{d}\Omega = 0$ . Then, since the connection forms from special conformal transformations ( $\mathbf{f}_a$ ) and translations ( $\mathbf{e}^a$ ) span the entire quotient manifold,  $\mathbf{e}^a \wedge \mathbf{f}_a$  is a manifestly non-degenerate 2-form. With  $\mathbf{d}\Omega = 0$ ,  $\mathbf{e}^a \wedge \mathbf{f}_a$  is closed, and therefore symplectic [6].

The second unique characteristic of biconformal space is that only in this gauging is the Killing metric non-degenerate when restricted to the base manifold. Therefore, we get a metric from the group structure of the problem, rather than declaring a metric by fiat, as is the case with all the other gaugings above, and indeed standard treatments of general relativity.

We have already spoken about the physical motivation for having a canonically conjugate basis on a metric space, but in order to connect this geometry with relativistic phase space we require separate configuration and momentum metric submanifolds. Therefore, the basis forms  $\mathbf{e}^a$  and  $\mathbf{f}_a$  are separately required to be in involution. Orthogonality of these sets of basis forms is required in order for the submanifolds to be metric.

When the basis is both orthogonal and canonical [12], the signature of the submanifolds is severely limited, leading to a Lorentzian configuration space, and hence the origin of time. In the case of flat biconformal space Spencer and Wheeler proved the following theorem: *Flat 8-dim biconformal space is a phase space with canonically conjugate, orthogonal, metric submanifolds if and only if the initial 4-dim space we gauge is Euclidean or signature zero. In either of these cases the resulting configuration sub-manifold is necessarily Lorentzian* [12] Thus, when we impose the conditions necessary to make biconformal space a metric phase space, only in a restricted subclass of cases can we get a metric configuration space, and that metric *must* be Lorentzian. With a suitable choice of gauge the metric can be written as  $h_{ab} = \frac{1}{(y^2)^2} (2y_a y_b - y^2 \delta_{ab})$  where  $\delta_{ab}$  is the Euclidean metric and the signature changing character of the metric can be easily seen if the coordinates are chosen so that  $y_a/\sqrt{y^2} = (1, 0, 0, 0)$ . It is important to note that in the metric above,  $y_a = W_a$ , is the Weyl vector of this space. This points to another unique characteristic of flat biconformal space. The group structures of the conformal group, projected down to this quotient manifold and written as canonically conjugate, orthogonal, metric submanifolds, gives rise to a natural notion of time, given by the gauge field of dilatations.

As an illustration of the portent of this natural notion of time one can calculate the Riemannian curvature of the momentum space metric, even though its conformal curvature is zero. The Riemann curvature of this metric can be written in the form  $R_{abcd} = k_{bd}k_{ca} - k_{bc}k_{da}$ , where  $k_{ab} = \frac{y_a y_b}{(y^2)^2} - h_{ab}$  is a projection operator in the direction of time (the Weyl vector). This is exactly the form of the metric obtained in the ADM formalism when one chooses a time direction and constructs a projection onto the spatial dimensions. In our formulation, however, these expressions are covariant.

**General Relativistic Phase Space.** The above discussion reveals that flat biconformal space is a  $2n$ -dimensional space with symplectic form. When the connection of biconformal space is generalized to include curvature, the goal of a fully curved phase space is reached. In order to have a dynamical geometry we write down an action functional on biconformal space. Such an action was first investigated by Wehner and Wheeler [14]. Since the volume element of biconformal space is dimensionless, an action functional linear in the curvatures can be conformally invariant, unlike in Weyl gravity, where curvature squared terms lead to complications, such as higher order field equations and possibly negative normed states.

Wehner and Wheeler showed when the torsion is taken to vanish general relativity on the space-time tangent bundle is obtained. However, the basis they use is not orthogonal and the metric on configuration space is not the Killing metric. As in [14], we begin with the curvature-linear action, but we use the canonical, orthogonal basis of [12]. Then, by reducing the full structure equations and field equations, we derive general relativity. All structures used follow directly from the geometry of the conformal group.

The action, in terms of curvature 2-forms, is

$$S = \int (\alpha \Omega_b^a + \beta \delta_b^a \Omega + \gamma \mathbf{e}^a \mathbf{f}_b) \varepsilon_{ac\dots d} \varepsilon^{be\dots f} \mathbf{e}^c \dots \mathbf{e}^d \mathbf{f}_e \dots \mathbf{f}_f$$

where  $\Omega_b^a$  is the curvature associated with Euclidean rotations and  $\Omega$  is the dilatational

curvature. The torsion,  $\mathbf{T}^a$ , and co-torsion,  $\mathbf{S}_a$ , cannot appear in a linear, conformally invariant action, but do appear in the field equations.

In order to make contact with general relativity we require flatness of the momentum sector to guarantee that we have the cotangent bundle of spacetime. These flatness conditions are looked at in detail in [12]. Briefly, the flatness condition sets the maximal number of momentum curvature pieces to zero, but not the entire  $\Omega_b^{acd}$ . If  $\Omega_b^{acd} = 0$  is imposed then  $h_{ab} = \delta_{ab}$ , which is too restrictive, and does not give us involute submanifolds (see [12]). Although we impose this condition by hand, it may be automatic since it was found in [14] that the curvature on their momentum subspace is zero as a result of the field equations.

Our approach for solving these equations is similar to Spencer and Wheeler, except that the curvatures are no longer set to zero; instead they satisfy field equations. The field equations set conditions on the curvatures, including setting parts of the co-torsion to zero. Since  $\mathbf{e}^a$  is in involution, we may study the  $\mathbf{e}^a = 0$  momentum submanifolds. On these submanifolds, we solve for the momentum part of the connection by making judicious gauge choices. We then extend this form of the connection back to the full space. Using the field equations, structure equations and Bianchi identities we can write down the solution in terms of the solder form coefficients, and the Weyl vector. Of particular note, combining the field equations with the Bianchi identities on the basis form structure equations shows that the configuration and momentum space dilatational curvatures are zero generically. Together, these conditions lead to the Ricci flat version of Einstein's field equation on the configuration submanifold,  $R_{bac}^a = 0$ . Since the field equations imply metric compatibility of the connection, these restrictions lead to general relativity on the spacetime submanifold. The momentum sector contains the conformally Ricci flat equivalent.

$$R_a^{bac} - \frac{1}{2(n-1)} \delta^{bc} h_{de} R_a^{dae} - (n-2) \left( \partial^c W^b - W^b W^c + \frac{1}{2} \delta^{bc} W^2 \right) = 0$$

The gauge choice of the Weyl vector is particularly interesting, because it enables us to place the Weyl vector entirely on either the configuration or the momentum submanifold. In fact if we change this gauge, now requiring the configuration space to have zero curvature, then we have a space that describes curved momentum space. This ability to switch between a normal general relativity picture and a geometry where the roles of momentum and position are switched is exciting from the view point of complementarity and fully achieves the goal of Born to have reciprocity of the variables of gravity.

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