ELASTIC SCALING OF SMALL STRUCTURES

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Abstract

As aerospace designers strive to build smaller systems, it is important that they understand scaling laws to take full advantage of the inherent strength of small structures.

Simple geometric scaling yields masses that scale in proportion to ℓ^3. However, in the process, the stress levels decrease, and the materials are not used to full advantage. Also, the resistance to buckling increases as the length decreases. With "elastic scaling," as the dimension parallel to the predominant load shortens, the dimension normal to the main load is thinned down even faster. This preserves a constant factor of safety with respect to the critical buckling load. The structural mass decreases even faster than ℓ^3 and the material is used more effectively than for simple geometric scaling. Examples abound in nature, from tree trunks to bones. Several such examples will be shown to illustrate this type of scaling.

Even with elastic scaling, the stress levels continue to decrease as the size is reduced. An extension of elastic scaling with more than one dimension normal to the main load-bearing direction is considered. The possibility of scaling the different lateral dimensions differently in an attempt to preserve constant stress in the material as the object shrinks is investigated. It is shown that no systematic scaling can achieve this goal, although some useful insight is developed.

A related issue is the minimum gage problem. When one attempts to use elastic scaling (or even geometric scaling), one discovers that as the size decreases, the materials required become too thin to handle. Techniques for addressing this difficulty will be discussed.

Introduction

Currently, there is a trend toward reducing the size of spacecraft, as evidenced by the existence of this conference. As we shrink the size of aerospace structures, or any other type of structures for that matter, it is helpful to understand the applicable scaling laws.

In this paper, the concepts of allometry, and elastic scaling will be defined. Examples from nature will be shown to illustrate these concepts. These ideas and illustrations are taken from a Scientific American Book, "On Size and Life[1]."

Designs created by nature seem to be well optimized. Observing trends in nature and applying them to engineering designs is a reasonable thing to do. This is especially true if we can deduce the underlying physical principles and apply them to broad classes of problems. Structural elements in nature, such as bones and tree trunks, scale elastically over many orders of magnitude in size. Bones in small animals are much more slender than in large animals. That is, the length to diameter ratio is greater for bones of small animals. This form of scaling preserves constant resistance to buckling, as will be shown. The main concept of elastic scaling is that there are two length scales which vary differently with size.

There are practical limits to the extent to which these ideas can be used in real structures. These ideas apply strictly to the main load bearing elements. In many cases, especially for very small structures, the overhead of joining the main structural elements together and of mounting other elements to the main structure consumes a significant portion of the total structural mass. Also, as the main structural elements are thinned down, they can become difficult to manufacture and to handle. This is known as the minimum gage problem. This difficulty will be discussed and some potential approaches to solving this problem will be discussed.

Isometry and Allometry

One way to compare the relative sizes of two features of an organism is to use an expression of the form

$$y = bx^a.$$

The length of a bone could be represented by x and the diameter by y. If the dimensions follow this form

of equation, then they are said to scale allometrically. If x and y are allometric, then when they are plotted on log paper for several different organisms, a straight line will be obtained with slope equal to a. Many dimensions for organisms, both plant and animal, are observed to scale this way over many orders of magnitude. It should be noted that this is merely a description of how one dimension varies with another as the overall size changes. It does not explain <u>why</u> this behavior is observed. The explanation requires further analysis.

The special case of allometric scaling for which a = 1 corresponds to pure geometric scaling. One dimension is directly proportional to the other. This special case is referred to as isometry. Adult mammals tend to scale isometrically within their own species. As an example, an adult human's armspan and height are very nearly equal, as illustrated in the famous drawing, "Vitruvian Man," from Leonardo da Vinci's notebooks. In this case, both a and b are equal to 1.

If one compares cell size, y, and the overall size, x, of an organism, one finds that the cell size is essentially independent of overall size. The value for a is very close to zero. The value for b is thus the typical cell size, which is roughly constant independent of the overall size of the animal or plant.

If one is troubled by the units of a and b, it is perhaps better to recast the allometric relation in the form of ratios:

$$\frac{x}{x_o} = b \left[\frac{y}{y_o}\right]^a$$

where x_0 and y_0 are reference values for one particular individual. Now both a and b are dimensionless.

<u>Isometry.</u>

When objects are scaled isometrically, all dimensions are proportional. This is the familiar example of scale models. Volumes are proportional to the length cubed and areas are proportional to length squared. This leads to the so called square-cube law. If the density remains the same, then the weight is proportional to volume, or cube of the length, whereas the area over which the forces are distributed is proportional to the area, or length squared. The stress, (or pressure), which is just force divided by area, is therefore proportional to volume divided by area, or simply to length. As an object grows isometrically, the stresses due to its weight increase in proportion to the length of the object. This is often quoted as the reason small children can crawl about on their hands and knees on hard surfaces without grief, whereas adults suffer tremendously if they attempt the same feat. This is also offered as the explanation why small animals such as mice can fall from great heights without being harmed. As we will see later, small animals do even better than predicted by the square-cube rule.

<u>Allometry.</u>

When one compares animals within a broad class, such as all mammals, one can find many parameters that do not follow isometry, but which are well described by the allometric expression. For example, The maximal rate of oxygen consumption is found to vary according to the 0.8 power of the body mass.

The particular examples of allometry that interest us with respect to structures are those of tree trunks and branches, and bones and muscles. These structures have a predominant load-bearing direction along their length, which we will define to be ℓ. They also have a characteristic dimension normal to the length, which we shall refer to as d.

<u>Elastic Similarity</u>

When one investigates variations of several orders of magnitude in body mass of mammals, one finds that the length of bones is not proportional to the 1/3 power as would be expected based on geometric scaling. Instead, the power is experimentally observed to be closer to 1/4. Similarly, d is proportional to the 3/8 power instead of 1/3. Equivalently, the cross-sectional area is found to scale according to the 3/4 power rather than 2/3. These variations are also observed for tree trunks. From the above relations, we can easily deduce the allometric relation between d and ℓ:

$$\ell \;\alpha\; m^{1/4} \;\rightarrow\; m \;\alpha\; \ell^4$$

$$d \;\alpha\; m^{3/8} = (\ell^4)^{3/8} = \ell^{3/2}$$

$$d \;\alpha\; \ell^{3/2}$$

This particular allometric relation is referred to as "elastic scaling." We should remember that these are merely observations of what occurs naturally. No explanation is directly given by these relations. That will come later.

Figure 1 shows the skeletons of two primates of greatly different sizes but drawn in the figure with the same height. It can readily be seen that the smaller animal, the Siamang, has relatively much more slender

bones than the larger Gorilla. This illustrates elastic scaling: the diameter increases more rapidly than the length of the bones as size increases.

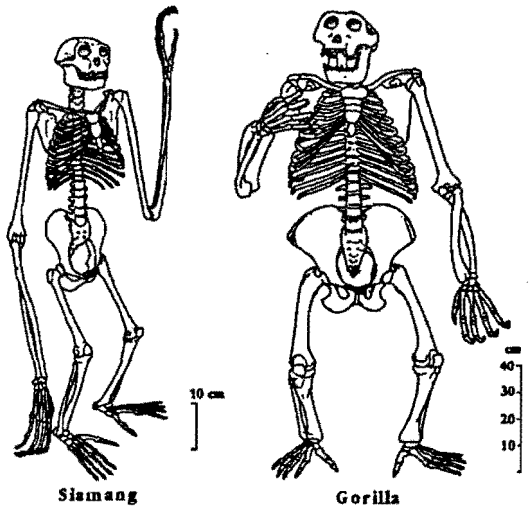

Figure 1 Elastic Scaling

The length and diameter of the humerus bone for several species of antelope are shown in Figure 2. The slope of the line of this log-log plot is 2/3. Recall that the slope of a line on log-log paper corresponds to the exponent in the relation between the two variables. Thus $\ell \propto d^{2/3}$, or equivalently, $d \propto \ell^{3/2}$. These bones exhibit elastic scaling. The figure also shows a fair amount of scatter, which is common for this type of data. Nonetheless, there is no doubt that the slope is different from the value of 1 that would be predicted by isometry.

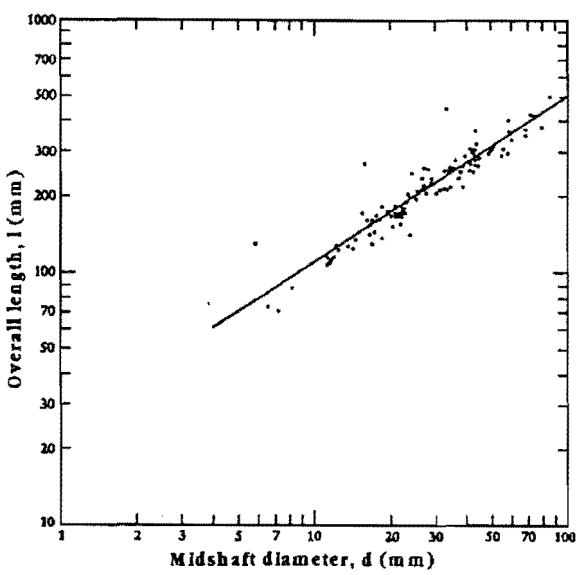

Figure 2 Antelope Humerus Bones

A similar plot is shown in Figure 3 for trees. The points plotted are for 576 record trees representing what are believed to be the tallest and broadest trees of most of the species common to the United States[1]. The diameters were measured 5 feet above the ground.

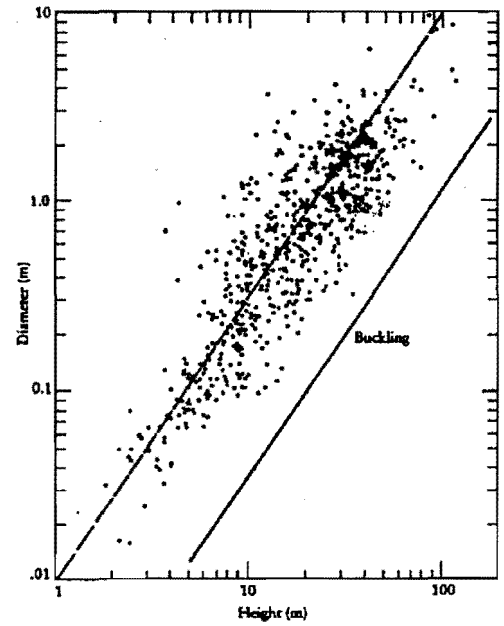

Figure 3 Tree Trunk Measurements

Buckling.

As described in many engineering texts, buckling is a phenomenon in which a slender column loaded in compression assumes a bowed shape when the load exceeds a critical value. For a column built in at the base and free at the top, the critical load is given by[2]:

$$P_{cr} = \frac{\pi^2 EI}{4\ell^2}$$

where the parameters in the equation have their usual meanings: E is the modulus of elasticity, I is the area moment of inertia of the cross-section, and ℓ is the length of the column[3]. For a circular cross-section, $I = \pi d^4/64$. Different constants multiply the buckling expression for different end conditions, but the form is always the same.

In the derivation of the expression for the critical buckling load, the weight of the column itself is neglected in comparison with the externally applied load. For a tree, the loading of the trunk is due to the weight of the tree itself. The weight of the tree is proportional to the product of its height and its cross-sectional area. Thus the load it must bear is proportional to ℓd^2.

A factor of safety is defined as the failure load divided by the expected maximum load. Let's explore the assumption that trees are "designed" with a constant factor of safety independent of size. Under this assumption, if we divide the load into the critical load, we should obtain a constant:

$$\frac{P_{cr}}{P} \alpha \frac{d^4/\ell^2}{\ell d^2} = \frac{d^2}{\ell^3} = a\ constant$$

If this is indeed constant, then d should be proportional to $\ell^{3/2}$. In Figure 3, the dotted line depicts the trend of diameters and heights for large trees. The slope of this line is 3/2 which supports our assumption. Since the tree measurements show that the diameters of the trunks (and also the branches) scale in proportion to $\ell^{3/2}$, we can see that trees of different sizes have equal resistance to buckling. This explains, at least partially, why this particular allometric ratio occurs. Almost certainly there are other factors contributing to this scaling too. Nonetheless, the evidence is quite strong that buckling resistance is a dominant aspect.

The solid line in Figure 3 is based on an analysis of buckling of wooden cylinders of constant diameter under their own weight. Poles with diameters and lengths to the right of and below this line will collapse under their own weight in the earth's gravity. The points corresponding to trees lie safely above the line, but in some cases, the factor of safety seems to be fairly small.

Elastic Scaling of Artificial Structures

The analysis of buckling in the previous section applies equally well to tubular columns designed by humans. Suppose we have a proven spacecraft design that uses tubular struts as load-bearing members and we wish to scale down the whole vehicle. How should we proceed? In order to preserve the resistance to buckling, we can apply elastic scaling. Let's assume that we can apply elastic scaling to every single part of the spacecraft. In a real example, this would not be possible. Let's also assume that the accelerations to which the vehicle will be subjected are the same for both cases. In fact, smaller structures will likely experience larger accelerations, but we'll keep it simple to keep the concept clear. Suppose we are reducing the overall size by a factor of two. Let's concentrate on the strut as a simple example. The new diameter will thus be given by:

$$\frac{d_{new}}{d_{old}} = \left[\frac{\ell_{new}}{\ell_{old}}\right]^{\frac{3}{2}} = \left[\frac{1}{2}\right]^{\frac{3}{2}} = 0.354.$$

Although the length is half of its original size, the diameter has reduced to about 35% of its original size. Clearly the length to diameter ratio has increased in the process. This may cause concern since the graphs in engineering textbooks show that the critical load decreases as ℓ/d increases. However, in this example, the load has also decreased by the same amount as the critical load since every single part of the spacecraft has been scaled elastically. Thus, the factor of safety is the same for both sizes of spacecraft. It should be noted that not only the diameter, but also the tube wall thickness should be reduced to 35% of its original value.

Continuing to assume that all parts have been scaled elastically, what can we expect the ratio of masses of the two spacecraft to be?

$$\frac{m_{new}}{m_{old}} = \frac{\rho_{new}\,\ell_{new}\,d_{new}^2}{\rho_{old}\,\ell_{old}\,d_{old}^2} = \left[\frac{\ell_{new}}{\ell_{old}}\right]^4$$

$$\frac{m_{new}}{m_{old}} = \left[\frac{1}{2}\right]^4 = \frac{1}{16} = 0.0625.$$

The half-size spacecraft is sixteen times less massive than the original. This is twice as light as would have been achieved by simple isometric scaling.

A similar analysis of honeycomb panels confirms that elastic scaling preserves resistance to buckling. In place of the diameter, we scale the thickness of the skins and the spacing between the skins in proportion to $\ell^{3/2}$. Strictly, we should also scale the other lateral dimension (the width of the panel) the same way, but the panel will never buckle that way, so we can scale that dimension as we please. Essentially, we are dealing with loading per unit span.

Extension to Elastic Scaling

Elastic scaling preserves buckling resistance. But what of the stress in the material? Are we maintaining a similar factor of safety with respect to the yield strength of the material? The answer is no. As we reduce the size, the stress goes down. As shown above, with elastic scaling, the weight scales in proportion to ℓ^4. The cross-sectional area resisting the load scales in proportion to d^2. Since $d\ \alpha\ \ell^{3/2}$, the stress goes as ℓ^4/ℓ^3. That is, the stress is directly proportional to ℓ. As the size goes down, the stress level goes down.

Is there not some way we can use our engineering knowledge to do better? Let's consider a tubular strut in isolation. Suppose we have scaled it elastically

along with the entire spacecraft just as discussed above. The cross-sectional area of a thin-walled tube is approximately πdt, where d is the diameter and t in the wall thickness. If we now increase the diameter of the tube and decrease the wall thickness in such a way that we keep the same cross-sectional area ($t \alpha 1/d$), then the moment of inertia will increase. The moment of inertia for a thin-walled tube is approximately $\frac{1}{4}Ad^2$. Since the area is constant, the moment of inertia clearly increases in proportion to d^2. With the increased moment of inertia, the buckling resistance will increase. The mass will not have changed, but somehow the design will have improved. But we don't really want to increase buckling resistance; it is already adequate. We want to decrease the cross-sectional area to get the stresses back up to the values they had in the larger spacecraft.

As an alternative approach, we could increase the diameter and decrease the wall thickness in such a way that the moment of inertia remains constant. This implies that the cross-sectional area will be proportional to $1/d^2$ and $t \alpha 1/d^3$. Since the cross-sectional area decreases and we can now attain the stress levels that were present in the original full-size structure. In the process, the mass of the strut has decreased even though the buckling resistance has not changed.

We have been considering the strut in isolation. If we could somehow apply this technique of thinning the walls and increasing the diameter to all parts of the spacecraft, then all the masses would decrease and the resulting loads would decrease. We would find that we were right back were we started: the stresses decrease as the size decreases.

In fact, for any consistent method of scaling applied to all parts of a structure, the stress will always be proportional to ℓ when the loads are due to the weight of the structure (gravitational or inertial). That is because the loads are proportional to the weight which, in turn, is proportional to the length times the cross sectional area. To obtain the stress, we divide this load by the cross-sectional area. The area then cancels and we are left with the stress being proportional to length. The key word is "consistent." If the scaling is applied to everything, then the stress will be proportional to length. Period. We cannot do magic.

We can, however, use the procedure of thinning the walls and increasing the diameter to change our design. This is no longer scaling, but it is a valuable tool. It can allow us to reduce the mass of the structure somewhat. As with many engineering decisions, there are trade-offs. With thinner walls, the tubes are more susceptible to handling forces during manufacture and assembly, and the tolerances become tighter. Also, when taken to extremes, the walls become so thin that local buckling (crumpling, or crippling) can occur. These effects are a little beyond the scope of this simplistic presentation, but should not be ignored in practice.

Mass Fraction

If all parts of a spacecraft are scaled elastically, then the masses of all parts will scale in proportion to one another. Not all of a spacecraft is structural, however. Still, if all parts including those that are not structural can be scaled elastically, then the structural mass fraction will not change as the overall size of the spacecraft changes. The only way that the mass fraction corresponding to the structure can decrease is if the rest of the spacecraft reduces in mass more slowly. For convenience, lets denote the non-structural part as "payload" even though this is not strictly accurate. Suppose, for example, that the "payload" can only be scaled geometrically. Then scaling the structural portion elastically will result in its mass being a smaller proportion of the whole than for the original larger spacecraft. But there is no guarantee that this structure will be adequate since it is now carrying a proportionately larger payload.

The analysis of this situation is not so simple since we are now adding components which scale with different powers. Some preliminary work indicates that it might be possible to reduce the structural mass fraction when the payload decreases more slowly than elastically. This result is encouraging but tentative and warrants further examination.

Bending

So far, we have considered only axial loading of structural elements. It is interesting to observe that bending of beams also scales elastically. For a cantilevered uniform beam with a square cross-section with dimension, d, and length, ℓ, deflecting under its own weight, the slope at the end is equal to[2]:

$$\theta = \frac{2\rho\ell^3}{Ed^2}$$

If one wishes to scale this case to a new length, then for the shape of the bent bar to be geometrically similar, the slope at corresponding points must be the same, including the end point for which the expression

is given above. In order to keep the slope the same, the "diameter" should scale in proportion to $\ell^{3/2}$. This is immediately recognized as the elastic scaling. It is not just this specific example of bending that scales this way. All beam bending problems have a similar form; just the constants are different. This also explains why the branches of trees scale elastically as well as the trunks.

Yield Stress

One should note that the yield stress has not appeared in any of these analyses of buckling or bending. In fact, the critical buckling load is not dependent upon the yield stress, just the modulus of elasticity. This is also the case in bending stiffness. The ultimate bending strength, however, is limited by the yield strength of the material.

The maximum stress in a beam bending problem scales in proportion to ℓ just as it did for buckling. Provided we are taking an existing successful design and making it smaller, we can safely apply elastic scaling. If we scale up, then we must be careful to check that the maximum stress does not exceed the yield stress.

Minimum Gage Problem

When one scales down a structure using either elastic scaling or geometric scaling, one eventually runs into some limits. The scaling laws can predict ideal values of thickness that cannot be achieve in practice, at least with current production methods. For example, in a recent study, honeycomb panels were to form the main structure of a launch vehicle adapter. The required thickness of the aluminum facesheets was found to be about 0.05 mm (0.002 inch). This is a factor of ten thinner than honeycomb manufacturers like to produce.

What techniques can we use to overcome such difficulties, or at least delay them to smaller vehicle sizes? If we simply use the materials available, our small structures will be much stronger than they need to be. This is acceptable if mass is not an issue. In most aerospace structures mass *is* an issue.

Currently, there is an emphasis on developing technologies that will enable very small micro-spacecraft. A study is underway at JPL to develop a 5 kg spacecraft concept that could execute a flyby of a near-earth asteroid or comet. To meet such goals, it is not adequate to simply accept existing methods. New techniques, or at least new applications of old techniques, must be identified and developed. A few of these will be discussed briefly here.

Low density materials

In all the scaling discussed above, it was implicitly assumed that the same material was used in the original large design and in the smaller derived design. We have seen that this can lead to very thin materials being specified. Another approach is to use materials with much lower densities than the metals traditionally used. If one could find a material that was ten times less dense than aluminum and with a yield stress also ten times smaller than for aluminum, then one could replace a 0.05 mm panel of aluminum by a 0.5 mm panel of the new material. The mass and load-bearing capability would be identical. The greater thickness would significantly enhance the resistance to buckling. If buckling were the dominant failure mode of the original design, then this would permit the mass to be decreased even beyond that predicted by elastic scaling.

Even if the ratio of the yield stress to density is not as high as for aluminum, it is still possible to achieve an overall weight savings using the low density material provided that buckling is the dominant failure mode. In essence, the thicker low density material provides a greater moment of inertia by spreading out the load bearing material. For a flat panel, this moment of inertia is proportional to the thickness cubed. The factor of 10 in thickness increases the buckling resistance by a factor of 1000. There is significant "gain" in this method!

Another advantage of using the low density material is that the relative tolerances are much easier to achieve. Referring back to the 0.05 mm honeycomb facesheets, a 10% variation is thickness is 5 μm (0.0002") for the aluminum, but it is 50 μm (0.002") for the low density substitute. It should be noted that composites do not fit in this category of low density materials. The densities are only about a factor of two less than aluminum. If anything, the higher strength of the composites leads one to thinner walls and less buckling resistance.

Honeycomb and Foam Core

When a single panel would buckle under its load, one can split the panel into two sheets with half the thickness each and bond honeycomb between the two. The honeycomb serves to provide shear resistance between the two panels but does not contribute

significantly to the load-bearing capability along the panel. The assembly is very stiff in bending. In a sense, the honeycomb acts like the web of an I-beam. The moment of inertia of the cross-section is proportional to the square of the spacing between the two sheets. This is another way of spreading apart the load-bearing material to increase the moment of inertia and therefore the bending stiffness.

Foam core is a similar concept. Instead of honeycomb, a lightweight foam is bonded between the two sheets. Foam core made with cardboard face sheets is widely available in art stores for mounting presentation material.

In principle, the face sheets for honeycomb or foam core could be made quite thin. Think of two pieces of shim stock with something between them. In practice, it can be difficult to achieve. Also, the mass of the material in between the two sheets begins to exceed the mass of the facesheets. Very thin sheets, while strong enough to withstand the design loads, still may be susceptible to damage by finger nails or tools during handling, assembly, or manufacturing processes. And again, realistic tolerances become a significant fraction of the total thickness.

Isogrid

Isogrid is the product of a milling machine operation in which triangular holes are cut through a thick (say 10 mm) slab of material. Thin walls are left between the triangular cells. These walls connect the nodes which occur at the points of the triangles. These nodes form a hexagonal pattern. Isogrid has been in use for about three decades and is fairly simple to analyze[3].

Over a distances bigger than a few cells, a panel of isogrid behaves very much like a solid slab with a much smaller average density than the parent material. Its effective modulus goes down more than its mean density, however. It is used in a fashion quite analogous to the low density materials discussed above. By making the cells larger, one can reduce the average density of the slab. Ultimately, one runs into the minimum gage problem with the webs, but at a lower effective density than honeycomb with its continuous facesheets.

Incidentally the triangular holes need not completely penetrate the original block from which the isogrid panel was machined. One can leave a thin skin on one side. Also, by using an undercutting milling tool, one can give the webs a T-section on the upper surface. This helps maintain the buckling resistance of the webs themselves.

A nice feature of isogrid is that the nodal points can be drilled and tapped to provide a built in set of mounting fixtures. This can significantly reduce the secondary mass by eliminating mounting brackets. This was demonstrated on Skylab, which had an isogrid floor/ceiling[4]. Given that the primary structure can be made quite light, for small structures it is especially important to pay attention to the mass of joints, fasteners, brackets, attachments, and other secondary structure.

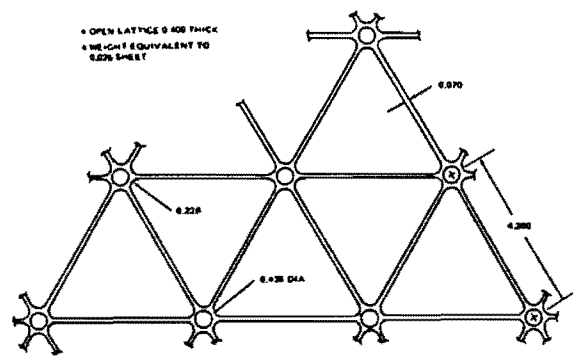

Figure 4 Isogrid Skylab Floor

Conclusions

Scaling of small structures has been discussed. The concepts of allometry and isometry have been defined. Elastic scaling is the particular allometric relation $d \propto \ell^{3/2}$. The dimension normal to the main load (diameter) scales in proportion to the 3/2 power of the length. Bones and trees are observed to scale this way. Elastic scaling can be used to scale down aerospace structures and should reduce the mass faster than simple geometric scaling would.

Elastic scaling preserves constant resistance to buckling and bending. As the size decreases, the stress levels decrease. This can be a problem when scaling up but should not be an issue when scaling down.

The square-cube law was mentioned. It is reasonably convincing and suffices to convey the concept that small objects are inherently stronger for their weight than large objects. In fact the argument is really not complete. The main load bearing bones of skeletons of mammals tend to scale elastically.

The minimum gage problem, which can be an issue for small structures, has been addressed. Some techniques for dealing with it have been suggested.

Acknowledgements

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