2013

Weyl Gravity as a Gauge Theory

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WEYL GRAVITY AS A GAUGE THEORY

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree

of

DOCTOR OF PHILOSOPHY

in

Physics

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Logan, Utah

2013

ABSTRACT

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Utah State University, 2013

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In 1920, Rudolf Bach proposed an action based on the square of the Weyl tensor or $C^{abcd}C_{abcd}$ where the Weyl tensor is an invariant under a scaling of the metric. A variation of the metric leads to the field equation known as the Bach equation. In this dissertation, the same action is analyzed, but as a conformal gauge theory. It is shown that this action is a result of a particular gauging of this group. By treating it as a gauge theory, it is natural to vary all of the gauge fields independently, rather than performing the usual fourth-order metric variation only. We show that solutions of the resulting vacuum field equations are all solutions to the vacuum Einstein equation, up to a conformal factor – a result consistent with local scale freedom. We also show how solutions for the gauge fields imply there is no gravitational self energy.

(109 pages)

PUBLIC ABSTRACT

Weyl Gravity as a Gauge Theory

Juan Teancum Trujillo

A gauge theory is a theory in which the governing functional, known as the action, remains invariant under a continuous group of local transformations that form its symmetry. Each of the known fundamental interactions in the universe, such as electricity and magnetism, can be explained as arising from a particular gauge theory. Gravitation is no exception. Just as calculus can be used to find the value of a variable that maximizes or minimizes a function, calculus of variations can be used to find the equations, known as the field equations, that extremize the action, and these are the main equations of interest. Solving, or finding solutions to these equations, provides the physical predictions or describes the expected physical results from a particular theory. Different actions with different symmetries may or may not be equivalent. In this work, we consider a theory of gravity whose action is invariant under local scale transformations, but as a gauge theory under the microscope of the entire range of such transformations. We show what the implications are and how this might give a better and fuller description of reality.

ACKNOWLEDGMENTS

Many thanks to Dr. James T. Wheeler, without whose assistance and availability, this work would not be possible. Many thanks also to my wife, Ana M. Trujillo, for her patience and support.

Juan Teancum Trujillo

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INDEX OF SYMBOLS

The following list contains all uncommon mathematical symbols used throughout this work with brief descriptions and the page numbers of their first appearance.

CHAPTER 1

INTRODUCTION

1.1. History

Beginning with Einstein's theory of general relativity in 1915 [1], gravity as a gauge theory emerged gradually. Although not strictly a gauge theory, it utilized some important concepts. For example, herein lay the idea of the covariant derivative, and from that, the connection. Generalizations were soon to follow. Much to Einstein's delight, Hermann Weyl introduced the concept of parallel transfer [2, 3] in which vectors associated with different points on a manifold could be compared. Weyl stated, "All physical quantities have a world-geometrical meaning," [3] in his attempt to unify electromagnetism with gravitation. As a starting point for a physical theory, he wrote down the quadratic curvature action, $S = \int R^{i}{}_{jkl} R_{i}^{jkl} \sqrt{g} dx$, although he created a nonintegrable theory in trying to identify the electromagnetic potential with the gauge field of dilatations, or local scale transformations.[1] In fact, it was Weyl himself who was the first to apply the word gauge[2] to a physical theory [4]. Rudolph Bach[3] [6] introduced the concept of the Weyl curvature tensor, which is invariant under these scale transformations. In keeping with ideas of Weyl, he, too, constructed a curvature-squared action from this tensor 4.1. A variation of this action, with respect to the metric, leads to the celebrated Bach equation (4.25) which will be discussed in further detail in this work. With the advent of quantum mechanics, London [7] realized that Weyl's nonintegrable factor could be be purely imaginary, and in the presence of an electromagnetic field, become the phase factor associated with wavefunctions. Upon seeing this, Weyl realized the phase factor could be made local and formalized his findings with the U(1) formulation of electromagnetism. In his paper [8], he introduced the concept

[1] It was Einstein himself who noted these size changes would be unphysical since in the case of atoms, their spectra depends only on their chemical classification and not on their histories [4].

[2] The term was originally used to denote the distance between the rails of a railroad track. Since tracks of various gauges, or widths, were in use, it seems only appropriate that the same terminology be used to denote the scale factor for the metric.

[3] Bach was a pseudonym. His given last name was Förster [5].

of the tetrad and generalized the covariant derivative to include the connection for abelian groups.

Building on the work of Weyl, Yang and Mills[4] were able to create an SU(2) invariant theory for nuclear interactions, and thus, were able to extend the concept of the curvature, or a field strength, to a nonabelian group [10]. Without realizing that Yang and Mills were also concurrently generalizing gauge theory, Ryoyu Utiyama was able to do so for Lie groups [11], and in particular, to the Lorentz group. The first gauge theory of gravity can be attributed to him. Simplifying the discussion of Utiyama, Kibble [12] stated the Lorentz transformations “become arbitrary functions of position” and “they may be interpreted as general coordinate transformations and rotations of the vierbein system.” Utiyama and Kibble helped pave the way by requiring gauge theories of gravity to be Lorentz invariant. Ne’eman and Regge [13] formalized a gauge theory construction of gravity by using the techniques developed by Cartan, Kobayashi and Nomizu [14, 15]. Ivanov and Niederle [16] considered a gauge theory of gravity based explicitly on the Poincaré group. Kaku, Townsend, and Nieuwenhuizen wrote a quadratic Lorentz curvature action based on the conformal group with Lorentz, dilatational, and special conformal symmetry [17] and showed how it reduced to Bach’s action (4.1). In fact, the action they consider is (5.67) without the addition of quadratic dilatational curvature term. They specifically refer to Bach’s action as Weyl gravity, [5] a terminology which we use in the remainder of this work.

1.2. Recent work

Some recent work has been completed on Weyl gravity. Among the motivation for studying theories based on the work of Weyl are because the “fourth-order terms can prevent the big bang singularity of GRT [General Relativity Theory]; the gravitational potential of a point mass is bounded in the linearized case; the inflationary cosmological model

[4]C.N. Yang did, on occasion, meet with Weyl, but their interaction never led to a discussion on physics, or even mathematics [9].

[5]From their paper [17], it is apparent they believe (4.1) was written by Weyl. Unmistakably, both Weyl and Bach were students of David Hilbert at Göttingen [5].

is a natural outcome of this theory" and that they tend to be "renormalizable at the one-loop quantum level" [18], which is a property not shared by standard general relativity. In an attempt to extend these results, Mannheim and O'brien [19] claim a spherically symmetric solution to the Bach equation [20], reducible to the Schwarzchild metric, introduces a linear term to the gravitational potential, which can then be used to reproduce galactic rotation curves without having to resort to dark matter. Sultana et al. [21] claim fitting parameters in the same metric to perihelion procession in the case of Mercury agree with those obtained from the galactic rotation curves, as well as with the observed perihelion shift.

Likewise, Edery and Paranjabe have proposed tests in Weyl gravity measuring "the deflection of light and time delay in the exterior of a static spherically symmetric source" in which the extra parameter "imitates the effect of dark matter" [22], and the results seem to agree with what has been determined experimentally. Lobo [23] considered the conditions for traversable wormholes in this theory. Klemm [24] discussed how certain exact solutions to the Bach equation can be interpreted as topological black holes.

1.3. Overview

In this work, we present the more generally known results and techniques first and culminate with our findings. We begin with the standard construction of general relativity in Chapter 2 in which a covariant derivative, with its properties, is defined. We then show how it transforms under changes in coordinates, how the covariant derivative acts on forms, and how parallel transport leads to metric compatibility. Then we use the covariant derivative to define the Riemann curvature tensor, the Ricci tensor, the Ricci scalar, and show their transformation properties. From the transformation properties, an action is constructed and is varied with respect to the dynamical fields to arrive at the field equations, from which we get the Einstein equation.

In Chapter 3, we present general relativity as a gauge theory of the Poincaré group. This is accomplished by first defining the transformations of this group, their infinitesimal transformations, and then their commutators. The differential relationship between the forms dual to the infinitesimal transformations, via the structure equations, is shown, as well as the corresponding relationships through the Bianchi identities. The quotient method is presented and tensors are then identified, from which an action functional with the desired symmetry is formed. We show how varying the action leads to field equations that are in agreement with the more standard construction of general relativity.

Chapter 4 shows the construction of Weyl gravity by presenting the Weyl curvature tensor and a quadratic action formed from it. We then show how the Gauss-Bonnet identity can be used to rewrite the action and the field equation that comes from varying the metric, and the connection between this theory and general relativity. Chapter 5 presents Weyl gravity as a gauge theory of the conformal group in the same way general relativity was presented as a gauge theory of the Poincaré group in Chapter 3. Chapter 6 shows solutions to the gauge theory solve the Bach equation. In Chapter 7, we show all solutions to this gauge theory are conformal transformations of solutions to the vacuum Einstein equation. Lastly, in Chapter 8, we discuss how the field equations show no gravitational self energy exists and that even with an arbitrary conformal factor, no length changes are possible. We also summarize our results.

CHAPTER 2

GENERAL RELATIVITY

In this chapter, we present the standard formulation of general relativity, beginning with the covariant derivative and how it transforms under general coordinate transformations. From this, we find the transformation of the Christoffel connection within the covariant derivative and define metric compatibility, from which we arrive at a definition of the Christoffel connection. We then show how to arrive at the standard tensors starting with the Riemann curvature tensor and its contractions. Lastly, we present an action and vary it to arrive at the Einstein equation.

2.1. The covariant derivative and the Christoffel connection

General relativity embraces the concept of general coordinate invariance, i.e. a theory that is independent of the choice of coordinates that one chooses to express it in. One of the fundamental concepts of this theory is the notion of the covariant derivative , D_α, which expresses how vectors on a manifold are differentiated. Its utility is manifest in that it can be used to compare vectors, which reside on different tangent spaces (or co-tangent spaces) of the manifold. In this section, we present the properties of the covariant derivative and show how the Christoffel connection, as a fundamental part of this derivative, transforms under a change of coordinates. Using the definition of the covariant derivative on contravariant vectors, we derive the form of the covariant derivative on covariant vectors, or forms. We show what the implications of parallel transfer are for the covariant derivative of the metric. From the results of parallel transfer (metric compatibility) and the covariant derivative of forms, we solve for the Christoffel connection in terms of the metric and its derivatives.

2.1.1. Properties of the covariant derivative

The covariant derivative is an object, which must be linear in the sense that

$$D_\alpha\left(\alpha f+\beta g\right)=\alpha D_\alpha f+\beta D_\alpha g \tag{2.1}$$

for constants α and β and tensors f and g of the same rank. It must also follow the Leibniz rule,

$$D_\alpha(fg)=fD_\alpha g+gD_\alpha f. \tag{2.2}$$

The covariant derivative of a contravariant vector is defined as

$$D_\alpha v^\beta \equiv \partial_\alpha v^\beta+\Gamma^\beta_{\mu\alpha}v^\mu, \tag{2.3}$$

where $\Gamma^\beta_{\mu\alpha}$ is a Christoffel symbol or simply the connection. With a coordinate transformation matrix for the vector field from coordinates x^β to y^α or Jacobian matrix given by

$$J^\alpha{}_\beta=\frac{\partial y^\alpha}{\partial x^\beta}$$

and the inverse transformation matrix from coordinates y^α to x^β or inverse Jacobian matrix given by

$$\bar{J}^\beta{}_\alpha=\frac{\partial x^\beta}{\partial y^\alpha},$$

covariance is defined as

$$D_\alpha^{'}v'^\beta=\left(D_\alpha v^\beta\right)',\tag{2.4}$$

where

$$v'^\beta\equiv J^\beta{}_\mu v^\mu,\tag{2.5}$$

$$D_\alpha^{'}\left(\partial,\Gamma\right)=D_\alpha\left(\partial',\Gamma'\right),\tag{2.6}$$

and

$$\left(D_\alpha v^\beta\right)' = \bar{J}^\mu{}_\alpha J^\beta_\nu D_\mu v^\nu. \tag{2.7}$$

From conditions (2.5), (2.6), and (2.7), we get

$$\left(\bar{J}^\mu{}_\alpha \partial_\mu\right)\left(J^\beta_\nu v^\nu\right) + \Gamma'^\beta_{\mu\alpha}\left(J^\mu_\nu v^\nu\right) = \bar{J}^\mu{}_\alpha J^\beta_\nu\left(\partial_\mu v^\nu + \Gamma^\nu_{\rho\mu} v^\rho\right). \tag{2.8}$$

Renaming indices so the arbitrary vectors drop out and multiplying both sides by $\bar{J}^\nu{}_\sigma$, we find

$$\Gamma'^\beta{}_{\sigma\alpha} = \bar{J}^\nu{}_\sigma \bar{J}^\mu{}_\alpha J^\beta_\rho \Gamma^\rho_{\nu\mu} - \bar{J}^\nu{}_\sigma \bar{J}^\mu{}_\alpha \partial_\mu\left(J^\beta_\nu\right). \tag{2.9}$$

This gives the transformation of the Christoffel connection under general coordinate transformations.

The Christoffel connection is defined to be symmetric on the final two indices. If we define a more general connection, we introduce a new tensor. Anti-symmetrizing on the σ and α indices, we find

$$\Gamma'^\beta{}_{[\sigma\alpha]} = \bar{J}^\nu{}_\sigma \bar{J}^\mu{}_\alpha J^\beta_\rho \Gamma^\rho_{[\nu\mu]}. \tag{2.10}$$

With the inhomogeneous part now gone, the anti-symmetric part transforms linearly and homogeneously under general coordinate transformations, and we identify this part as a tensor called the torsion.

2.1.2. Covariant derivative of a form

From the Leibniz rule (2.2), we have the covariant derivative of the contraction of a vector with a 1-form is given by

$$D_\alpha\left(w_\beta v^\beta\right) = w_\beta D\left(v^\beta\right) + D(w_\beta)v^\beta. \tag{2.11}$$

Since the left side has no free indices, we have $D_\alpha\left(w_\beta v^\beta\right) = \partial_\alpha\left(w_\beta v^\beta\right)$; therefore,

$$D_\alpha\left(w_\beta v^\beta\right) = w_\beta \partial_\alpha\left(v^\beta\right) + \partial_\alpha\left(w_\beta\right) v^\beta, \tag{2.12}$$

and consequently,

$$\partial_\alpha\left(w_\beta\right) v^\beta = w_\beta \Gamma^\beta_{\ \mu\alpha} v^\mu + D_\alpha(w_\beta) v^\beta. \tag{2.13}$$

Solving for the covariant derivative from the right side and stripping the arbitrary vector gives

$$D_\alpha(w_\beta) = \partial_\alpha\left(w_\beta\right) - w_\mu \Gamma^\mu_{\ \beta\alpha} \tag{2.14}$$

as the covariant derivative of a $\binom{0}{1}$ tensor.

2.1.3. Metric compatibility

Suppose that v^α and w^α are two vector fields that are covariantly constant along a curve. That is

$$u^\alpha D_\alpha v^\beta = 0 \tag{2.15}$$

and

$$u^\alpha D_\alpha w^\beta = 0 \tag{2.16}$$

for all points on a curve with tangent vector u^a. Put differently, moving along a curve, the vectors in each vector field do not change and maintain their orientation.

Since the orientation does not change, we demand the inner product, $v_\beta w^\beta$, must also not change either, i.e.,

$$u^\alpha D_\alpha\left(v_\beta w^\beta\right) = 0, \tag{2.17}$$

which is the same as

$$u^\alpha D_\alpha\left(g_{\mu\nu} v^\mu w^\nu\right) = 0. \tag{2.18}$$

Expanding (2.18) gives

$$u^{\alpha} D_{\alpha}\left(g_{\mu\nu}\right) v^{\mu} w^{\nu} + g_{\mu\nu} u^{\alpha} D_{\alpha}\left(v^{\mu}\right) w^{\nu} + g_{\mu\nu} v^{\mu} u^{\alpha} D_{\alpha}(w^{\nu}) = 0, \tag{2.19}$$

which becomes

$$u^{\alpha} D_{\alpha}\left(g_{\mu\nu}\right) v^{\mu} w^{\nu} = 0. \tag{2.20}$$

Since this must hold for arbitrary vectors u^{α}, v^{μ}, and w^{ν}, this becomes

$$D_{\alpha}\left(g_{\mu\nu}\right) = 0, \tag{2.21}$$

which is the condition for metric compatibility.

2.1.4. Solution for the Christoffel Connection

Because of metric compatibility (2.21), we may write a sum-sum-difference rule given by

$$0 = D_{\alpha} g_{\mu\nu} + D_{\mu} g_{\nu\alpha} - D_{\nu} g_{\alpha\mu}. \tag{2.22}$$

Writing out the covariant derivative of forms as given by (2.14) and simplifying, we arrive at

$$0 = \partial_{\alpha} g_{\mu\nu} + \partial_{\mu} g_{\nu\alpha} - \partial_{\nu} g_{\alpha\mu} - 2 g_{\nu\beta} \Gamma^{\beta}{}_{\alpha\mu}. \tag{2.23}$$

Solving for the term with the Christoffel symbol and contracting with an inverse metric (and renaming indices) we find

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}\left(g_{\nu\beta,\mu} + g_{\beta\mu,\nu} - g_{\mu\nu,\beta}\right), \tag{2.24}$$

where a comma defines a partial derivative $\left(g_{\mu\nu,\beta} \equiv \partial_{\beta} g_{\mu\nu}\right)$. This gives the Christoffel connection in terms of the metric and its derivatives.

2.2. Tensors in general relativity

Tensors are important for forming actions that are invariant under a particular group of transformations. In the previous section, we arrived at one tensor, the torsion, which transformed linearly and homogeneously under general coordinate transformations. Using the covariant derivative, we arrive at one more, the Riemann curvature tensor. Taking two contractions of this tensor gives us two more, the Ricci tensor, and the Ricci scalar.

2.2.1. Riemann Curvature Tensor

The Riemann curvature tensor comes from a commutator of two covariant derivatives acting on a vector, that is

$$\left[D_\mu, D_\nu\right] w^\alpha = D_\mu D_\nu w^\alpha - D_\nu D_\mu w^\alpha. \tag{2.25}$$

Using the definition of the covariant derivative (2.3), we arrive at

$$\left[D_\mu, D_\nu\right] w^\alpha = \left[\Gamma^\alpha_{\beta\nu,\mu} - \Gamma^\alpha_{\beta\mu,\nu} + \Gamma^\rho_{\beta\nu}\Gamma^\alpha_{\mu\rho} - \Gamma^\rho_{\beta\mu}\Gamma^\alpha_{\rho\nu}\right] w^\beta. \tag{2.26}$$

We then define

$$R^\alpha_{\beta\mu\nu} \equiv \Gamma^\alpha_{\beta\nu,\mu} - \Gamma^\alpha_{\beta\mu,\nu} + \Gamma^\rho_{\beta\nu}\Gamma^\alpha_{\mu\rho} - \Gamma^\rho_{\beta\mu}\Gamma^\alpha_{\rho\nu} \tag{2.27}$$

as the Riemann curvature tensor so

$$\left[D_\mu, D_\nu\right] w^\alpha = R^\alpha_{\beta\mu\nu} w^\beta. \tag{2.28}$$

Since the covariant derivative transforms as a tensor via (2.6) and w^α is a tensor, we conclude $R^\alpha_{\beta\mu\nu}$ is a tensor with transformation property given by

$$\tilde{R}^\alpha_{\beta\mu\nu} = \bar{J}^\rho_\nu \bar{J}^\sigma_\beta \bar{J}^\gamma_\mu J^\alpha_\delta R^\delta_{\sigma\gamma\rho}. \tag{2.29}$$

2.2.2. Ricci tensor and Ricci scalar

The Ricci tensor and the Ricci scalar are contractions of the Riemann curvature tensor. That is,

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}. \tag{2.30}$$

It transforms as

$$\tilde{R}_{\mu\nu} = \bar{J}^{\alpha}_{\mu}\bar{J}^{\beta}_{\nu}R_{\alpha\beta}. \tag{2.31}$$

Lastly, the Ricci scalar is defined as

$$R = g^{\mu\nu}R_{\mu\nu}, \tag{2.32}$$

and it transforms as

$$\begin{aligned}\tilde{R} &= \tilde{g}^{\mu\nu}\tilde{R}_{\mu\nu} \\ &= J^{\mu}_{\alpha}J^{\nu}_{\beta}g^{\alpha\beta}\bar{J}^{\rho}_{\mu}\bar{J}^{\sigma}_{\nu}R_{\rho\sigma} \\ &= g^{\mu\nu}R_{\mu\nu},\end{aligned} \tag{2.33}$$

and therefore,

$$\tilde{R} = R \tag{2.34}$$

so the Ricci scalar is invariant under general coordinate transformations.

2.2.3. Transformation of the metric and volume element

The metric transforms as

$$g'_{\mu\nu} = \bar{J}^{\rho}_{\mu}\bar{J}^{\sigma}_{\nu}g_{\rho\sigma}. \tag{2.35}$$

Using g as the determinant of the metric and $\bar{J}$ as the determinant of the inverse Jacobian metric, the determinant of (2.35) is

$$g' = \bar{J}^2 g. \tag{2.36}$$

Taking a square root of both sides yields

$$\sqrt{g}' = \bar{J}\sqrt{g}. \tag{2.37}$$

Since

$$J^a_b \bar{J}^b_c = \delta^a_c, \tag{2.38}$$

taking the determinant of both sides yields

$$J\bar{J} = 1, \tag{2.39}$$

where J is the Jacobian so (2.37) can be written as

$$\sqrt{g}' = \frac{1}{J}\sqrt{g}. \tag{2.40}$$

The relation between differential forms and ordinary coordinate differentials is

$$d^4x \leftrightarrow \frac{1}{4!}\varepsilon_{\mu\nu\alpha\beta}\mathbf{d}x^\mu \wedge \mathbf{d}x^\nu \wedge \mathbf{d}x^a \wedge \mathbf{d}x^\beta, \tag{2.41}$$

where $\varepsilon_{\mu\nu\alpha\beta}$ is the Levi-Civita symbol. Changing coordinates,

$$d^4x' \quad \leftrightarrow \quad \frac{1}{4!}\varepsilon_{\mu\nu\alpha\beta}J^\mu_\rho J^\nu_\sigma J^\alpha_\lambda J^\beta_\tau \mathbf{d}x^\rho \wedge \mathbf{d}x^\sigma \wedge \mathbf{d}x^\lambda \wedge \mathbf{d}x^\tau, \tag{2.42}$$

which gives

$$d^4x' = \frac{1}{4!}J\varepsilon_{\rho\sigma\lambda\tau}\mathbf{d}x^\rho \wedge \mathbf{d}x^\sigma \wedge \mathbf{d}x^\lambda \wedge \mathbf{d}x^\tau \tag{2.43}$$

so

$$d^4x' = Jd^4x. \tag{2.44}$$

Defining the volume form as

$$dV = \sqrt{-g}d^4x \tag{2.45}$$

then in different coordinates and using (2.40) and (2.44), we have

$$dV' = \left(\frac{1}{J}\sqrt{-g}\right)\left(Jd^4x\right) \tag{2.46}$$

and $dV' = dV$ so the volume form remains unchanged by general coordinate transformations.

2.2.4. Summary

Having obtained the Riemann curvature tensor, $R^{\alpha}_{\beta\mu\nu}$, as a commutator of two covariant derivatives, we contract on the first and third index to get the Ricci tensor, $R_{\mu\nu}$, and then contract with the metric to arrive at the Ricci scalar, R. The transformation properties of these tensors is summarized below:

$$\begin{aligned}
\tilde{R}^{\alpha}_{\beta\mu\nu} &= \bar{J}^{\rho}_{\nu}\bar{J}^{\sigma}_{\beta}\bar{J}^{\gamma}_{\mu}J^{\alpha}_{\delta}R^{\delta}_{\sigma\gamma\rho}, && (2.47)\\
\tilde{R}_{\mu\nu} &= \bar{J}^{\alpha}_{\mu}\bar{J}^{\beta}_{\nu}R_{\alpha\beta}, && (2.48)\\
\tilde{R} &= R, \text{ and} && (2.49)\\
dV' &= dV. && (2.50)
\end{aligned}$$

These can be combined to form invariant scalar actions.

2.3. Action and field equations

In this section, we use our set of tensors to construct an action. Using calculus of variations, we vary the action with respect to its independent fields, the metric and the

Christoffel connection, to arrive at the field equations. That is, we find the equations that extremize this particular functional. We consider their implications.

2.3.1. Action

There are many possible actions that could be constructed that would be invariant under general coordinate transformations. The simplest one is one constructed linearly from the Ricci scalar,

$$S = \int R\sqrt{-g}d^4x. \tag{2.51}$$

This is known as the Einstein-Hilbert action. We consider two methods of the variation of S, the Palatini variation and the second-order variation.

2.3.2. Palatini variation of the metric

In the Palatini variation, we vary the metric and the connection independently. Using (2.32), we can expand (2.51) as

$$S = \int g^{\mu\nu}R_{\mu\nu}\sqrt{-g}d^4x. \tag{2.52}$$

Variation of this action with respect to the metric gives

$$\delta_g S = \int \left[\delta\left(g^{\mu\nu}\right)R_{\mu\nu}\sqrt{-g} + R\delta\left(\sqrt{-g}\right)\right]d^4x. \tag{2.53}$$

The variation of the determinant is given by

$$\delta\left(\sqrt{-g}\right) = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu}. \tag{2.54}$$

Given

$$g^{\mu\nu}\delta\left(g_{\mu\nu}\right) = -\delta\left(g^{\mu\nu}\right)g_{\mu\nu}, \tag{2.55}$$

we may write (2.54) as

$$\delta\left(\sqrt{-g}\right) = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta\left(g^{\mu\nu}\right). \tag{2.56}$$

Substitution of (2.56) into (2.53) yields

$$0 = \int\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right]\delta\left(g^{\mu\nu}\right)d^4x, \tag{2.57}$$

from which the field equation can be extracted for an arbitrary variation of the metric:

$$0 = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R. \tag{2.58}$$

The Einstein tensor is then defined as

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R, \tag{2.59}$$

so the field equation from the variation of the metric can also be written as simply

$$G_{\mu\nu} = 0. \tag{2.60}$$

This is the vacuum Einstein equation.

2.3.3. Variation of the connection

The Palatini variation also requires us to vary the connection, ${\Gamma^{\alpha}}_{\mu\nu}$, which is assumed to be symmetric, but not necessarily Christoffel. From (2.52), a variation of the connection, gives

$$\delta_{\Gamma}S = \int\delta_{\Gamma}\left(R_{\mu\nu}\right)g^{\mu\nu}\sqrt{-g}d^4x. \tag{2.61}$$

Contracting on an index from the definition of the Riemann curvature tensor (2.27) to get the Ricci tensor and then taking a variation of the connection, we find

$$\delta R_{\mu\nu} = D_\nu \left(\delta\Gamma^\alpha_{\ \mu\alpha}\right) - D_\alpha \left(\delta\Gamma^\alpha_{\ \mu\nu}\right). \tag{2.62}$$

Substituting (2.62) into (2.61) and integrating by parts, we find

$$\int \left[D_\alpha \left(g^{\mu\nu}\sqrt{-g}\right)\delta^\beta_\nu - D_\nu \left(g^{\mu\nu}\sqrt{-g}\right)\delta^\beta_\alpha\right]\delta\Gamma^\alpha_{\ \mu\beta} d^4x = 0, \tag{2.63}$$

and the field equation becomes

$$D_\alpha \left(g^{\mu\nu}\sqrt{-g}\delta^\beta_\nu\right) - D_\nu \left(g^{\mu\nu}\sqrt{-g}\delta^\beta_\alpha\right) = 0. \tag{2.64}$$

Contracting on β and α (and with $n = 4$), we find

$$D_\nu \left(g^{\mu\nu}\sqrt{-g}\right) = 0. \tag{2.65}$$

Substituting this back into (2.64) we are left with

$$D_\alpha \left(g^{\mu\beta}\sqrt{-g}\right) = 0. \tag{2.66}$$

Finally, assuming covariant constancy of the volume element, $D_\alpha\sqrt{-g} = 0$, we have

$$D_\alpha g^{\mu\beta} = 0, \tag{2.67}$$

which is the condition for metric compatibility (2.21).

2.3.4. Second-order variation of the metric

In the Palatini variation of the metric, only the metric appearing explicitly was varied. However, in a second-order variation, the metric dependence of the Ricci tensor must also be considered. In that case, the initial variation gives three terms instead of two:

$$\delta_g S = \int \left[\delta_g \left(g^{cd} \right) R_{cd} \sqrt{-g} + g^{ab} R_{ab} \delta_g \left(\sqrt{-g} \right) + g^{ab} \delta_g \left(R_{ab} \right) \sqrt{-g} \right] d^4 x.$$

Because in the second-order variation the Ricci scalar is assumed to depend on the Christoffel connection, its variation can be written as

$$\delta R_{ab} = D_b \left(\delta \Gamma^c{}_{ac} \right) - D_c \left(\delta \Gamma^c{}_{ab} \right). \tag{2.68}$$

The variation of the Christoffel connection with respect to the metric is given by

$$\delta \Gamma^a_{\ bc} = \delta g^{ae} g_{ef} \Gamma^f_{\ bc} + g^{ad} \delta g_{ed} \Gamma^e{}_{cb} + \frac{1}{2} g^{ad} \left(\delta g_{cd;b} + \delta g_{db;c} - \delta g_{bc;d} \right). \tag{2.69}$$

Substituting this into (2.68) gives

$$\begin{aligned} \delta R_{ab} \quad = \quad & \frac{1}{2} g^{cd} D_b D_a \left(\delta g_{cd} \right) + \frac{1}{2} g^{cd} D_b D_c \left(\delta g_{da} \right) - \frac{1}{2} g^{cd} D_b D_d \left(\delta g_{ac} \right) \\ & - \frac{1}{2} g^{cd} D_c D_a \delta g_{bd} - \frac{1}{2} g^{cd} D_c D_b \left(\delta g_{da} \right) + \frac{1}{2} g^{cd} D_c D_d \left(\delta g_{ab;dc} \right). \end{aligned} \tag{2.70}$$

Contracting the variation of the Ricci tensor with respect to the metric to give the third term in the original variation, then integrating by parts and using metric compatibility gives

$$g^{ab} \sqrt{-g} \delta_g \left(R_{ab} \right) = -D_e D^e \left(\sqrt{-g} \right) \delta \left(g^{ab} \right) g_{ab} - D_a D_b \left(\sqrt{-g} \right) \delta g^{ab}, \tag{2.71}$$

and since the volume form is covariantly constant, i.e. $D\left(\sqrt{-g}\right)=0$, we find

$$g^{ab}\sqrt{-g}\delta_g\left(R_{ab}\right)=0, \tag{2.72}$$

and the remaining terms in the variation give the vacuum Einstein equation (4.26). In this case, the Palatini and second-order variation result in the same field equation, but in general, this is not the case. For Weyl gravity, the two variations give different results.

2.3.5. Summary

Having formed an action entirely from the Ricci scalar and the volume element, we varied it to find the field equations. The Palatini variation of the metric gave the vacuum Einstein equation. In this particular case, the second-order variation also gave the vacuum Einstein equation. Lastly, assuming a covariantly constant metric determinant, the variation of the Christoffel connection implied a covariantly constant metric.

CHAPTER 3

GENERAL RELATIVITY AS A GAUGE THEORY

In this chapter, we review general relativity as a gauge theory of the Poincaré group. We begin by defining the transformations of this group and then find the corresponding infinitesimal transformations, or generators. We take commutators of these generators to determine its Lie algebra. From the Lie algebra, the relations between the forms dual to the generators are found, otherwise known as the Maurer-Cartan structure equations. The integrability condition for the structure equations, or Bianchi identities, are also determined.

Once the basic information information of the group is extracted, we turn to the quotient method, as developed by Cartan, Kobayashi, and Nomizu [14, 15] and used by Ne'eman and Regge [13]. The general basic steps to forming a gauge theory by the quotient method are fivefold. First, given a Lie group, $\mathscr{G}$, we determine a Lie subgroup, $\mathscr{H} \subset \mathscr{G}$, which will act as our symmetry group. Second, we form the group quotient, $\mathscr{G}/\mathscr{H}$, and generalize the manifold and the connections from the structure equations to form the field strengths or curvatures. In this particular case, the Lie group, $\mathscr{G}$, is the Poincaré group and the symmetry group, $\mathscr{H}$, is the group of Lorentz transformations. Third, we identify the tensors from the available symmetries, objects that transform linearly and homogeneously. Fourth, we form an action from the available tensors. The action should be invariant under the local symmetry transformations. Once the action is formed, the final step is to vary the action with respect to its independent forms, and together with the structure equations and the Bianchi identities, we try to find solutions for all the connection forms.

3.1. Group generators for the Poincaré group

Since general relativity as a gauge theory is based on the Poincaré group, a description of this group and its properties are in order. The Poincaré group is a semidirect product of two Lie groups: the group of Lorentz transformations and the group of translations. In this section, we present the mathematical definitions of the Lorentz transformations and translations, and find their infinitesimal transformations.

3.1.1. Lorentz transformations

The Lorentz transformations, represented by Λ^a_b are transformations that preserve the Minkowski metric, given by

$$\eta_{ab} = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}. \tag{3.1}$$

This is represented by

$$\eta_{ab}\tilde{x}^a\tilde{x}^b = \eta_{ab}x^a x^b, \tag{3.2}$$

where

$$\tilde{x}^a = \Lambda^a{}_b x^b \tag{3.3}$$

and x^b is a space-time vector. Transformations close to the identity can be represented by

$$\Lambda^a{}_b = \delta^a{}_b + \varepsilon^a{}_b, \tag{3.4}$$

where ε^a_b is the generator of the Lorentz transformations. Substituting (3.4) and (3.3) into (3.2) and keeping only terms that are first order in ε^a_b, we find

$$\varepsilon_{dc} = -\varepsilon_{cd}, \tag{3.5}$$

and so the generators are antisymmetric. To get a function space representation of this metric we write

$$x^a \simeq x^a + \varepsilon^a_b x^b. \tag{3.6}$$

Using

$$\delta^b_d = \eta^{bc}\eta_{cd} \tag{3.7}$$

and

$$\delta^a_e = \frac{\partial}{\partial x^e} x^a \tag{3.8}$$

it is possible to pull out an arbitrary vector, i.e.,

$$x^a \simeq \left(1 + \varepsilon^{ec}\eta_{cd}x^d \frac{\partial}{\partial x^e}\right) x^a. \tag{3.9}$$

Using the express antisymmetry of ε^a_b, the generator of the Lorentz transformations can then be represented by

$$M_{ab} = \frac{1}{2}\left(x_a\partial_b - x_b\partial_a\right). \tag{3.10}$$

Raising an index with the metric and defining the antisymmetric projection operator,

$$\Delta^{ac}_{db} \equiv \frac{1}{2}\left(\delta^a_d\delta^c_b - \eta^{ac}\eta_{db}\right), \tag{3.11}$$

the generator of the Lorentz transformations can now be expressed as

$$M^a_b = \Delta^{ac}_{db}x^d\partial_c. \tag{3.12}$$

3.1.2. Translations

Translations are given by

$$\tilde{x}^a = x^a + a^a, \tag{3.13}$$

which can be written as

$$\tilde{x}^a = \left(1 + a^c\partial_c\right)x^a. \tag{3.14}$$

Factoring out the arbitrary vector and constant, the generator is given by

$$P_b = \frac{\partial}{\partial x^b} = \partial_b. \tag{3.15}$$

3.2. Lie algebra for the Poincaré group

The Lie algebra of the Poincaré group is the set of all possible commutators from the generators, i.e.,

$$[G_A, G_B] = c_{AB}{}^C G_C, \tag{3.16}$$

where G_A is a group generator, and $c_{AB}{}^C$ is what is known as a structure constant. We have

$$[M^a_b, M^c_d] = \frac{1}{2}\left(\delta^c_b \delta^a_f \delta^h_d - \delta^a_d \delta^h_b \delta^c_f - \eta_{bd}\eta^{ch}\delta^a_f - \eta^{ac}\eta_{bf}\delta^h_d\right) M^f_h, \tag{3.17}$$

$$[M^a_b, P_c] = -\Delta^{af}_{cb} P_f. \tag{3.18}$$

A commutator of the generator of translations with itself is zero since partial derivatives commute (order of differentiation does not matter). Since there are only two sets of generators in this group, no other commutators are possible. Using $G_{\binom{a}{b}}$ to represent the generator of Lorentz transformations with indices a and b and $G_{\binom{\cdot}{c}}$ to represent the generator of the translations with index c, (3.17) and (3.18) can be represented as

$$\left[G_{\binom{a}{b}}, G_{\binom{c}{d}}\right] = c_{\binom{a}{b}\binom{c}{d}}{}^{\binom{f}{h}} G_{\binom{f}{h}}, \tag{3.19}$$

$$\left[G_{\binom{a}{b}}, G_{\binom{\cdot}{c}}\right] = c_{\binom{a}{b}\binom{\cdot}{c}}{}^{\binom{\cdot}{f}} G_{\binom{\cdot}{f}}. \tag{3.20}$$

Straightaway, we have

$$-c_{\binom{c}{d}\binom{a}{b}}{}^{\binom{f}{h}} = c_{\binom{a}{b}\binom{c}{d}}{}^{\binom{f}{h}} = \frac{1}{2}\left(\delta^c_b \delta^a_f \delta^h_d - \delta^a_d \delta^h_b \delta^c_f - \eta_{bd}\eta^{ch}\delta^a_f - \eta^{ac}\eta_{bf}\delta^h_d\right), \tag{3.21}$$

and

$$c_{\binom{a}{b}\binom{\cdot}{c}}^{\binom{\cdot}{f}} = -\Delta^{af}_{cb}, \tag{3.22}$$

which are the nonvanishing structure constants for the Poincaré group.

3.3. Maurer-Cartan structure equations of the Poincaré group

For any Lie group, there exists a one-to-one correspondence between contravariant vectors and covariant vectors, or between the generators and forms, expressed by

$$\left\langle \omega^A, G_B \right\rangle = \delta^A_B. \tag{3.23}$$

Expanded out,

$$\omega^A = \omega^A_M \mathbf{d}x^M \tag{3.24}$$

and

$$G_A = G^M_A \frac{\partial}{\partial x^M}, \tag{3.25}$$

where x^M are coordinates on the group manifold. Likewise, there is a one-to-one correspondence in a coordinate basis,

$$\left\langle \mathbf{d}x^M, \frac{\partial}{\partial x^N} \right\rangle = \delta^M_N. \tag{3.26}$$

Combining (3.23) with (3.26) gives

$$\omega^A_N G^N_B = \delta^A_B, \tag{3.27}$$

which implies

$$\omega^A_N = \bar{G}^A_N = \left[G^A_N\right]^{-1}. \tag{3.28}$$

Substitution of (3.25) into the commutator relation for generators (3.16) and utilizing (3.28) gives the Maurer-Cartan equation for a Lie group given by

$$\mathbf{d}\omega^{C} = -\frac{1}{2}c_{AB}{}^{C}\omega^{A} \wedge \omega^{B}, \tag{3.29}$$

where $\mathbf{d}$ is the exterior derivative and where $\mathbf{d}\omega^{A} = \partial_{M}\left(\omega^{A}_{N}\right)\mathbf{d}x^{M} \wedge \mathbf{d}x^{N}$. For the Poincaré group, the possible indices are

$$A \in \left\{\binom{a}{b}, \binom{a}{\cdot}\right\}, \tag{3.30}$$

and the possible forms then become

$$\begin{aligned} \omega^{A} &\in \left\{\omega^{\binom{a}{b}}, \omega^{\binom{a}{\cdot}}\right\} \\ &\equiv \left\{\omega^{a}{}_{b}, \mathbf{e}^{a}\right\}. \end{aligned} \tag{3.31}$$

Here, $\omega^{a}{}_{b}$ is the gauge field or the form that is dual to the generator of the Lorentz transformations. Another name for this particular form is the spin connection. Likewise, $\mathbf{e}^{a}$ is the gauge field of the translations. Another name for this form is the solder form.

3.3.1. Structure equation for Lorentz transformations

For the Lorentz transformations, (3.29) becomes

$$\mathbf{d}\omega^{\binom{a}{b}} = -\frac{1}{2}c_{\binom{c}{d}\binom{e}{f}}{}^{\binom{a}{b}}\omega^{\binom{c}{d}} \wedge \omega^{\binom{e}{f}}. \tag{3.32}$$

Using (3.31) and the structure constant from the commutator of two Lorentz transformations (3.17), (3.32) becomes

$$\mathbf{d}\omega^{a}{}_{b} = \omega^{c}{}_{b} \wedge \omega^{a}{}_{c}. \tag{3.33}$$

3.3.2. Structure equation for translations

For the translations, we have

$$\mathbf{d}\omega^{\binom{\cdot}{c}} = -\frac{1}{2}c_{\binom{a}{b}\binom{\cdot}{c}}{}^{\binom{\cdot}{f}}\boldsymbol{\omega}^{\binom{a}{b}} \wedge \boldsymbol{\omega}^{\binom{\cdot}{c}} - \frac{1}{2}c_{\binom{\cdot}{c}\binom{a}{b}}{}^{\binom{\cdot}{f}}\boldsymbol{\omega}^{\binom{\cdot}{c}} \wedge \boldsymbol{\omega}^{\binom{a}{b}}. \tag{3.34}$$

Using the structure constant from the commutator of the Lorentz transformations and the translations (3.22), we find

$$\mathbf{de}^a = \mathbf{e}^b \wedge \omega^a_{\ b}. \tag{3.35}$$

3.3.3. Poincaré structure equations with curvature

The connection may be generalized, thereby adding curvature. The two equations (3.33) and (3.35) become

$$\mathbf{d}\omega^a_{\ b} = \omega^c_{\ b} \wedge \omega^a_{\ c} + \mathbf{R}^a_{\ b}, \text{ and} \tag{3.36}$$

$$\mathbf{de}^a = \mathbf{e}^b \wedge \omega^a_{\ b} + \mathbf{T}^a, \tag{3.37}$$

where $\mathbf{R}^a_{\ b}$ is the Riemann curvature tensor and $\mathbf{T}^a$ is the torsion. Notice (3.36) can be rewritten to give

$$\mathbf{R}^a_{\ b} = \mathbf{d}\omega^a_{\ b} - \omega^c_{\ b} \wedge \omega^a_{\ c} \tag{3.38}$$

as a definition for $\mathbf{R}^a_{\ b}$.

3.3.4. Independence of lifting

In a particular gauging, all of the forms and curvatures must be expanded in the forms that do not span the fibers. That is, we want them to be independent of gauge or of the forms in the vertical direction [25]. Since the symmetry group in this theory is the Lorentz group, we have

$$\begin{aligned} \omega^a_{\ b} &= \omega^a_{\ bc}\mathbf{e}^c, &(3.39)\\ \mathbf{R}^a_{\ b} &= \frac{1}{2}R^a_{\ bcd}\mathbf{e}^c\wedge\mathbf{e}^d, \text{ and} &(3.40)\\ \mathbf{T}^a &= \frac{1}{2}T^a_{\ bc}\mathbf{e}^b\wedge\mathbf{e}^c. &(3.41)\end{aligned}$$

3.4. Bianchi identities for the Poincaré group

Taking the exterior derivative of another exterior derivative yields zero; that is, $\mathbf{d}^2\omega = 0$ for any p-form ω. Thus, taking the exterior derivative of each of our structure equations with curvature gives differential relations between the curvatures, called the Bianchi identities. In this section, we find the Bianchi identities for both the Lorentz transformations and the translations. These are the integrability conditions for equations (3.36) and (3.37).

3.4.1. Lorentz Transformations

Taking the exterior derivative of both sides of (3.36) gives

$$0 = \mathbf{d}\omega^c_{\ b}\wedge\omega^a_{\ c} - \omega^c_{\ b}\wedge\mathbf{d}\omega^a_{\ c} + \mathbf{d}\mathbf{R}^a_{\ b}. \tag{3.42}$$

Substituting (3.36) for $\mathbf{d}\omega^c_{\ b}$ and $\mathbf{d}\omega^a_{\ c}$ gives

$$0 = \mathbf{d}\mathbf{R}^a_{\ b} + \mathbf{R}^c_{\ b}\wedge\omega^a_{\ c} - \omega^c_{\ b}\wedge\mathbf{R}^a_{\ c}, \tag{3.43}$$

which is the covariant derivative as given by (2.3) where the Christoffel connection is replaced by the spin connection. This is the covariant exterior derivative and (3.43) can be written as

$$\mathbf{D}\mathbf{R}^a_{\ b} = 0. \tag{3.44}$$

3.4.2. Translations

Starting with the structure equation for curvature (3.37) and taking an exterior derivative of both sides gives

$$0 = \mathbf{d}\mathbf{e}^b \wedge \omega^a_{\ b} - \mathbf{e}^b \wedge \mathbf{d}\omega^a_{\ b} + \mathbf{d}\mathbf{T}^a. \tag{3.45}$$

Substituting (3.37) for $\mathbf{d}\mathbf{e}^b$ and (3.36) for $\mathbf{d}\omega^a_{\ b}$ and simplifying gives

$$0 = \mathbf{T}^b \wedge \omega^a_{\ b} - \mathbf{e}^b \wedge \mathbf{R}^a_{\ \ b} + \mathbf{d}\mathbf{T}^a \tag{3.46}$$

or

$$\mathbf{D}\mathbf{T}^a = \mathbf{e}^b \wedge \mathbf{R}^a_{\ \ b}. \tag{3.47}$$

3.5. The covariant derivative and transformation properties

In this section, we find the transformation properties of the connection forms, as well as that of the curvatures, by requiring covariance of the gauge covariant derivative.

3.5.1. Transformation of the connection

The covariant exterior derivative in (3.43) acts on the Riemann curvature tensor. Using a generic connection form given by $\omega^A_{\ B}$ acting on a generic vector, v^A, the covariant derivative is defined as

$$\mathbf{D}v^A = \mathbf{d}v^A + \omega^A_{\ B} v^B. \tag{3.48}$$

Demanding covariance as in (2.4),

$$\tilde{\mathbf{D}}\tilde{\mathbf{v}}^A = \left(\widetilde{\mathbf{D}\mathbf{v}^A}\right) \tag{3.49}$$

or

$$\tilde{\mathbf{D}}\left(g^A_{\ B}\mathbf{v}^B\right) = g^A_{\ B}\mathbf{D}\mathbf{v}^B, \tag{3.50}$$

where

$$\tilde{\mathbf{D}}v^A = \mathbf{d}v^A + \tilde{\omega}^A_B v^B. \tag{3.51}$$

Using g^A_B to represent local gauge transformations (local group elements) and $\bar{g}^A_B$ as its inverse, solving for the transformed connection gives

$$\tilde{\omega}^A_B = g^A_D \omega^D_C \bar{g}^C_B - \mathbf{d}\left(g^A_C\right)\bar{g}^C_B. \tag{3.52}$$

Notice the similarity of this equation to (2.9).

3.5.2. *Transformation of the curvature*

For a general Lie group, the connection in the Maurer-Cartan equation is generalized, resulting in the addition of the curvature $\mathbf{R}^C$ to it:

$$\mathbf{d}\omega^C = -\frac{1}{2}c_{AB}{}^C \omega^A \wedge \omega^B + \mathbf{R}^C. \tag{3.53}$$

This equation can be written in the adjoint representation by multiplying it by a structure constant $c_{DA}{}^E$. Defining

$$\omega^E{}_D \equiv -c_{DA}{}^E \omega^A \tag{3.54}$$

$$\mathbf{R}^E{}_D \equiv -c_{DA}{}^E \mathbf{R}^A, \tag{3.55}$$

Eq. (3.53) becomes

$$-\mathbf{d}\left(\omega^E{}_D\right) = -\frac{1}{2}c_{BC}{}^A c_{DA}{}^E \omega^B \wedge \omega^C - \mathbf{R}^E{}_D. \tag{3.56}$$

Using the Jacobi identity given by $c_{[BC}{}^A c_{D]A}{}^E = 0$, which can be written as

$$c_{BC}{}^A c_{DA}{}^E = -c_{CD}{}^A c_{BA}{}^E - c_{DB}{}^A c_{CA}{}^E, \tag{3.57}$$

and solving for the curvature in (3.56) gives

$$\mathbf{R}^A{}_B = \mathbf{d}\omega^A{}_B - \omega^C{}_B \wedge \omega^A{}_C, \tag{3.58}$$

which transforms as

$$\tilde{\mathbf{R}}^A{}_B = \mathbf{d}\tilde{\omega}^A{}_B - \tilde{\omega}^C{}_B \wedge \tilde{\omega}^A{}_C. \tag{3.59}$$

Substitution of the transformation of the connection (3.52) shows the curvature transforms as a tensor,

$$\tilde{\mathbf{R}}^A_B = g^A_C \mathbf{R}^C_D \bar{g}^D_B. \tag{3.60}$$

3.6. The quotient method and tensors

In this section, we make the transformation properties for both the connection forms and curvatures infinitesimal. We then apply the quotient method by requiring independence of lifting, eliminating all terms containing transformations that lie on the base manifold. From the remaining terms, we identify the tensors, in preparation to forming an action.

3.6.1. Transformation of the connection forms

The transformation of the forms comes from using 3.52 using the allowed indices for the group in question 3.30. Letting the indices $A = \binom{a}{b}$, 3.52 becomes

$$\tilde{\omega}^a_{\ b} = g^a_{\ d}\omega^d_{\ c}\bar{g}^c_{\ b} - \mathbf{d}\left(g^a_{\ c}\right)\bar{g}^c_{\ b}, \tag{3.61}$$

and right away, we see the spin connection is not a tensor.

To see how different objects transform, it is helpful to make the group elements infinitesimal. Infinitesimal group elements behave the same way as ordinary group elements. Letting $g^A_D \to \delta^A_D + \Lambda^A_D$ and $\bar{g}^C_B \to \delta^C_B - \Lambda^C_B$, (3.52) becomes, to first order

$$\tilde{\omega}^A_B = \omega^A_B - \omega^A_C \Lambda^C_B + \Lambda^A_C \omega^C_B - \mathbf{d}\left(\Lambda^A_B\right). \tag{3.62}$$

Setting the lower B index from these terms to $\cdot$ and letting the rest run from 0 to 3, (3.62) gives the transformation of the solder form,

$$\tilde{\omega}^a = \omega^a - \omega^a_{\ c}\Lambda^c + \Lambda^a_{\ d}\omega^d - \mathbf{d}\left(\Lambda^a\right). \tag{3.63}$$

With the understanding $\omega^a \equiv \mathbf{e}^a$, Λ^a is a generator of the translations, and $\Lambda^a_{\ d}$ is the generator of the Lorentz transformations, (3.63) becomes

$$\tilde{\mathbf{e}}^a = \mathbf{e}^a - \omega^a_{\ c}\Lambda^c + \Lambda^a_{\ d}\mathbf{e}^d - \mathbf{d}\left(\Lambda^a\right). \tag{3.64}$$

Since this gauging has only Lorentz symmetry, terms containing generators of the translations must be eliminated, so (3.64) becomes

$$\tilde{\mathbf{e}}^a = \mathbf{e}^a + \Lambda^a_{\ d}\mathbf{e}^d, \tag{3.65}$$

and this is clearly the linearization of

$$\tilde{\mathbf{e}}^a = g^a_{\ d}\mathbf{e}^d. \tag{3.66}$$

The solder form transforms linearly and homogeneously with a Lorentz transformation, and so can be identified as a tensor.

3.6.2. Transformation of the curvatures

Just as in the case for the transformation of the spin connection, taking the transformation of the curvature 3.60 and letting $A = \binom{a}{b}$ gives

$$\tilde{\mathbf{R}}^d_{\ c} = g^d_{\ a}\mathbf{R}^a_{\ b}\bar{g}^b_{\ c}, \tag{3.67}$$

and the Riemann curvature tensor transforms linearly and homogeneously under a Lorentz transformation and an inverse Lorentz transformation.

To find how the torsion transforms, again we allow the transformations to become infinitesimal and up to first order in the generators 3.60 we have

$$\tilde{\mathbf{R}}^A_B = \mathbf{R}^A_B - \mathbf{R}^A_C \Lambda^C_B + \Lambda^A_C \mathbf{R}^C_B. \tag{3.68}$$

Dropping the lower B index and allowing the other indices to run from 0 to 3, 3.68 becomes

$$\tilde{\mathbf{R}}^a = \mathbf{R}^a - \mathbf{R}^a_b \Lambda^b + \Lambda^a_b \mathbf{R}^b. \tag{3.69}$$

With the understanding that $\mathbf{R}^a \equiv \mathbf{T}^a$ and imposing Lorentz symmetry 3.69 becomes

$$\tilde{\mathbf{T}}^a = \mathbf{T}^a + \Lambda^a_b \mathbf{T}^b, \tag{3.70}$$

which is clearly the linearization of

$$\tilde{\mathbf{T}}^d = g^d_{\ a} \mathbf{T}^a. \tag{3.71}$$

The torsion, then, transforms linearly under a Lorentz transformation and is also a tensor.

3.6.3. Transformation of the Levi-Civita symbol

Because Lorentz transformations have determinant one,

$$\varepsilon_{abcd} = \varepsilon_{abcd} \det\left(\Lambda^a_b\right), \tag{3.72}$$

which is equivalent to

$$\varepsilon_{abcd} = \varepsilon_{efgh} \Lambda^e_a \Lambda^f_b \Lambda^g_c \Lambda^h_d. \tag{3.73}$$

Because inverse Lorentz transformations are also Lorentz transformations with a determinant of one, we may write (3.73) as

$$\varepsilon_{abcd} = \varepsilon_{efgh}\bar{\Lambda}^{e}_{\ a}\bar{\Lambda}^{f}_{\ b}\bar{\Lambda}^{g}_{\ c}\bar{\Lambda}^{h}_{\ d}. \tag{3.74}$$

Hence, the Levi-Civita symbol is also a tensor under Lorentz transformations.

3.6.4. Summary

The tensors found by making the transformation properties of the curvature and connection forms infinitesimal are the Riemann curvature tensor, the torsion, and the solder form. The Levi-Civita symbol was also found to be a tensor. The transformation properties of each are given by

$$\tilde{\mathbf{R}}^{d}_{\ c} = g^{d}_{\ a}\mathbf{R}^{a}_{\ b}\bar{g}^{b}_{\ c}, \tag{3.75}$$

$$\tilde{\mathbf{T}}^{d} = g^{d}_{\ a}\mathbf{T}^{a}, \tag{3.76}$$

$$\tilde{\mathbf{e}}^{a} = g^{a}_{\ d}\mathbf{e}^{d}, \text{ and} \tag{3.77}$$

$$\varepsilon_{abcd} = \varepsilon_{efgh}\bar{\Lambda}^{e}_{\ a}\bar{\Lambda}^{f}_{\ b}\bar{\Lambda}^{g}_{\ c}\bar{\Lambda}^{h}_{\ d}. \tag{3.78}$$

From this, we proceed to form an action.

3.7. Action and field equations

In this section, we form an action from the available tensors, and vary it with respect to its independent fields, the spin connection and the solder form.

3.7.1. Action

One possible action to form from (3.75) - (3.78) is

$$S = \int \mathbf{R}^{ab} \wedge \mathbf{e}^{c} \wedge \mathbf{e}^{d}\varepsilon_{abcd}. \tag{3.79}$$

All of the raised indices transform with Lorentz transformations and all of the lowered indices transform with inverse Lorentz transformations. Because of horizontality (3.40), $\mathbf{R}^{ab}$ may be written as $\frac{1}{2}R^{ab}{}_{ef}\mathbf{e}^e \wedge \mathbf{e}^f$, and the action may be written as

$$S = \int \frac{1}{2}R^{ab}{}_{ef}\mathbf{e}^e \wedge \mathbf{e}^f \wedge \mathbf{e}^c \wedge \mathbf{e}^d \varepsilon_{abcd}. \tag{3.80}$$

At this point, we recognize that $\mathbf{e}^e \wedge \mathbf{e}^f \wedge \mathbf{e}^c \wedge \mathbf{e}^d$ may be replaced by $\varepsilon^{efcd}\sqrt{-g}d^4x$. Contracting on the two Levi-Civita symbols, the action (3.79) can be written as

$$S = -\frac{1}{2}\int R\sqrt{-g}d^4x, \tag{3.81}$$

which is the Einstein-Hilbert action (2.51) up to a constant (the field equations remain unchanged).

3.7.2. Variation of the spin connection

The variation of the spin connection only affects the Riemann curvature tensor since that is the only place where it appears, i.e.,

$$\delta_\omega S = \int \left(\delta_\omega \mathbf{R}^{ab}\right) \wedge \mathbf{e}^c \wedge \mathbf{e}^d \varepsilon_{abcd}, \tag{3.82}$$

where

$$\mathbf{R}^a_{\ b} = \mathbf{d}\omega^a_{\ b} - \omega^c_{\ b} \wedge \omega^a_{\ c}. \tag{3.83}$$

Varying the above expression with respect to the spin connection gives $\delta\mathbf{R}^a_{\ b} = \mathbf{D}\left(\delta\omega^a_{\ b}\right)$ and consequently,

$$\delta\mathbf{R}^{ab} = \mathbf{D}\left(\delta\omega^{ab}\right). \tag{3.84}$$

After covariant integration by parts (3.82) becomes

$$\delta_\omega S = \int 2\delta\omega^{ab} \wedge \mathbf{D}\mathbf{e}^c \wedge \mathbf{e}^d \varepsilon_{abcd}. \tag{3.85}$$

Setting the variation to 0 to extract the field equation gives

$$\mathbf{e}^c \wedge \mathbf{D}\mathbf{e}^d \varepsilon_{abcd} = 0. \tag{3.86}$$

From the structure equation of the solder form (3.37), the torsion may be written as $\mathbf{T}^a = \mathbf{d}\mathbf{e}^a + \omega^a_{\ b} \wedge \mathbf{e}^b$. The right side of this equation is recognized as the covariant derivative of the solder form, so it can be written as $\mathbf{T}^a = \mathbf{D}\mathbf{e}^a$. Substitution of this into (3.86) gives

$$0 = \mathbf{e}^c \wedge \mathbf{T}^d \varepsilon_{abcd} = 0. \tag{3.87}$$

Expanding the torsion, wedging with another solder form, and taking the Hodge dual gives

$$T^d_{ef} \varepsilon^{cgef} \varepsilon_{cabd} = 0. \tag{3.88}$$

After expanding the two Levi-Civita symbols, we have

$$T^d_{bd} \delta^g_a + T^g_{ab} = 0. \tag{3.89}$$

Contracting the g with a indices gives $T^d_{bd} = 0$, and consequently

$$T^c_{ab} = 0, \tag{3.90}$$

so there is no torsion as a consequence of the field equation.

Vanishing torsion has an effect on the Bianchi identities. The Bianchi identity resulting from taking the exterior derivative of structure equation of the solder form (3.46) becomes

$$0 = \mathbf{e}^b \wedge \mathbf{R}^a_{\ b}. \tag{3.91}$$

Expanding $\mathbf{R}^a_{\ b}$ in terms of the solder form gives

$$R^a_{\ [bcd]} = 0, \tag{3.92}$$

which is the first Bianchi identity.

3.7.3. Variation of the solder form

Since the Riemann curvature tensor only depends on the spin connection, a variation of the action given by (3.79) only affects what is visible giving

$$\delta_{\mathbf{e}} S = \int 2\mathbf{R}^{ab} \wedge \mathbf{e}^c \wedge \delta \mathbf{e}^d \varepsilon_{abcd}. \tag{3.93}$$

The field equation then becomes

$$2\mathbf{R}^{ab} \wedge \mathbf{e}^c \varepsilon_{abcd} = 0. \tag{3.94}$$

Wedging with another solder form and taking the Hodge dual adds another Levi-Civita symbol:

$$2R^{ab}_{\ \ \ ef} \varepsilon_{abcd} \varepsilon^{efcg} \sqrt{-g} = 0. \tag{3.95}$$

Expanding the two Levi-Civita symbols gives

$$R_{ab} - \frac{1}{2}\eta_{ab} R = 0, \tag{3.96}$$

which is the vacuum Einstein equation (2.58). Adding $\Lambda\mathbf{e}^a\wedge\mathbf{e}^b\wedge\mathbf{e}^c\wedge\mathbf{e}^d\varepsilon_{abcd}$ to the action and following the same procedure yields

$$R_{ab}-\frac{1}{2}\eta_{ab}R-\Lambda\eta_{ab}=0, \tag{3.97}$$

which is the vacuum Einstein equation with a cosmological constant.

Second-order variation of spin connection

For the second-order variation, we assume the compatible connection, which in this case means vanishing torsion. Varying the spin connection in terms of the solder form gives

$$\int\Big(2\mathbf{R}^{ab}\wedge\mathbf{e}^c\wedge\delta\mathbf{e}^d\varepsilon_{abcd}+2\delta_e\omega^{ab}\wedge\mathbf{D}\mathbf{e}^c\wedge\mathbf{e}^d\varepsilon_{abcd}\Big)=0. \tag{3.98}$$

Since $\mathbf{D}\mathbf{e}^a=\mathbf{T}^a$, and that vanishes from the Palatini variation, we are then left with

$$2\mathbf{R}^{ab}\wedge\mathbf{e}^c\varepsilon_{abcd}=0, \tag{3.99}$$

which again yields the Einstein equation.

3.7.4. Summary

The action (3.79) written in forms was shown to be equivalent to the Einstein-Hilbert action (2.51) written with the Ricci scalar. Varying the spin connection led to the condition of vanishing torsion while varying the solder form led to vacuum Einstein equation, as in the case of varying of the metric in standard general relativity.

CHAPTER 4

WEYL GRAVITY

In this chapter, we present the action generally associated with Weyl gravity. We also present the Gauss-Bonnet term and use it to rewrite the action. We give the field equation resulting from varying the action with respect to the metric and discuss its relation to general relativity.

4.1. Action and field equation

Using the ideas of Weyl [2], in 1920 Rudolph Bach [6] proposed a quadratic action that would be invariant under the conformal group, which he constructed from the Weyl curvature tensor or the conformal curvature, given by

$$S = \int C^{abcd} C_{abcd} \sqrt{|g|} d^4 x, \tag{4.1}$$

where the Weyl curvature tensor, C^{abcd}, is defined as the traceless part of the Riemann curvature

$$\begin{aligned} C^{abcd} &= R^{abcd} - \frac{1}{n-2}\left(R^{ac}\eta^{bd} - R^{ad}\eta^{bc} - R^{bc}\eta^{ad} + R^{bd}\eta^{ac}\right) \\ &\quad + \frac{1}{(n-1)(n-2)} R\left(\eta^{ac}\eta^{bd} - \eta^{ad}\eta^{bc}\right). \end{aligned} \tag{4.2}$$

It is constructed so contractions between any two indices gives 0, e.g.

$$\eta_{ac} C^{abcd} = 0. \tag{4.3}$$

Taking (4.2) and solving for R^{abcd} gives

$$\begin{aligned} R^{abcd} &= C^{abcd} + \frac{1}{n-2}\left(R^{ac}\eta^{bd} - R^{ad}\eta^{bc} - R^{bc}\eta^{ad} + R^{bd}\eta^{ac}\right) \\ &\quad - \frac{1}{(n-1)(n-2)} R\left(\eta^{ac}\eta^{bd} - \eta^{ad}\eta^{bc}\right). \end{aligned} \tag{4.4}$$

Contracting with itself gives

$$R^{abcd}R_{abcd} = C^{abcd}C_{abcd} + \frac{4}{(n-2)}R^{ab}R_{ab} - \frac{2}{(n-2)(n-1)}R^2 \tag{4.5}$$

and in $n = 4$ dimensions, we have

$$R^{abcd}R_{abcd} = C^{abcd}C_{abcd} + 2R^{ab}R_{ab} - \frac{1}{3}R^2. \tag{4.6}$$

The action is then

$$\begin{aligned} S &= \int C^{abcd}C_{abcd}d^4x \\ &= \int \left(R^{abcd}R_{abcd} - 2R^{ab}R_{ab} + \frac{1}{3}R^2\right) d^4x. \end{aligned} \tag{4.7}$$

4.1.1. Gauss-Bonnet invariant

The Gauss-Bonnet invariant, or the Euler character, is given by a curvature-squared term as

$$\chi_E = \frac{1}{4}\int \mathbf{R}^{ab} \wedge \mathbf{R}^{cd} \varepsilon_{abcd}. \tag{4.8}$$

Varying this functional with respect to the spin connection or the gauge field of the

Lorentz transformations, ω^a_b gives

$$\begin{aligned} \delta\chi_E &= \frac{1}{2}\int\left(\mathbf{D}\left(\delta\omega^{ab}\right)\wedge\mathbf{R}^{cd}\varepsilon_{abcd}\right) \\ &= \frac{1}{2}\int\left(\mathbf{D}\left(\delta\omega^{ab}\right)\varepsilon_{abcd}\wedge\mathbf{R}^{cd}\right), \end{aligned} \tag{4.9}$$

where $\mathbf{D}$ is a covariant exterior derivative. Since the covariant derivative is Leibniz, we have

$$\begin{aligned} \mathbf{D}\left(\delta\omega^{ab}\varepsilon_{abcd}\wedge\mathbf{R}^{cd}\right) &= \mathbf{D}\left(\delta\omega^{ab}\right)\varepsilon_{abcd}\wedge\mathbf{R}^{cd} - \delta\omega^{ab}\mathbf{D}\left(\varepsilon_{abcd}\right)\wedge\mathbf{R}^{cd} \\ &\quad -\delta\omega^{ab}\varepsilon_{abcd}\wedge\mathbf{D}\left(\mathbf{R}^{cd}\right), \end{aligned} \tag{4.10}$$

and so solving for $\mathbf{D}\left(\delta\omega^{ab}\right)\varepsilon_{abcd}\wedge\mathbf{R}^{cd}$ and substituting

$$\begin{aligned} \delta\chi_E &= \frac{1}{2}\int\left(\mathbf{D}\left(\delta\omega^{ab}\right)\right)\wedge\mathbf{R}^{cd}\varepsilon_{abcd} \\ &= \frac{1}{2}\int\left(\mathbf{D}\left(\delta\omega^{ab}\varepsilon_{abcd}\wedge\mathbf{R}^{cd}\right) - \delta\omega^{ab}\mathbf{D}\left(\varepsilon_{abcd}\right)\wedge\mathbf{R}^{cd} + \delta\omega^{ab}\varepsilon_{abcd}\wedge\mathbf{D}\left(\mathbf{R}^{cd}\right)\right) \\ &= \frac{1}{2}\int\left(\mathbf{d}\left(\delta\omega^{ab}\varepsilon_{abcd}\wedge\mathbf{R}^{cd}\right) - \delta\omega^{ab}\mathbf{D}\left(\varepsilon_{abcd}\right)\wedge\mathbf{R}^{cd} + \delta\omega^{ab}\varepsilon_{abcd}\wedge\mathbf{D}\left(\mathbf{R}^{cd}\right)\right) \\ &= \frac{1}{2}\int\left(-\delta\omega^{ab}\mathbf{D}\left(\varepsilon_{abcd}\right)\wedge\mathbf{R}^{cd} + \delta\omega^{ab}\varepsilon_{abcd}\wedge\mathbf{D}\left(\mathbf{R}^{cd}\right)\right). \end{aligned} \tag{4.11}$$

The first term is a total divergence that depends only on the topology. To evaluate this further, we want to know what the covariant derivative of the Levi-Civita symbol is with respect to the Lorentz transformations. We have

$$\begin{aligned} \mathbf{D}\left(\varepsilon_{abcd}\right) &= \mathbf{d}\varepsilon_{abcd} + \omega^e{}_a\varepsilon_{ebcd} + \omega^e{}_b\varepsilon_{aecd} + \omega^e{}_c\varepsilon_{abed} + \omega^e{}_d\varepsilon_{abce} \\ &= \omega^e{}_a\varepsilon_{ebcd} + \omega^e{}_b\varepsilon_{aecd} + \omega^e{}_c\varepsilon_{abed} + \omega^e{}_d\varepsilon_{abce}. \end{aligned} \tag{4.12}$$

Using the transformation of the Levi-Civita (3.73), expanding the Lorentz transformations in as infinitesimal transformations gives

$$\begin{aligned}\varepsilon_{abcd} &\approx \varepsilon_{efgh}\left(\delta^e_a + \varepsilon^e_a\right)\left(\delta^f_b + \varepsilon^f_b\right)\left(\delta^g_c + \varepsilon^g_c\right)\left(\delta^h_d + \varepsilon^h_d\right) \\ &= \varepsilon_{abcd} + \varepsilon_{abch}\varepsilon^h_d + \varepsilon_{abgd}\varepsilon^g_c + \varepsilon_{ebcd}\varepsilon^e_a + \varepsilon_{afcd}\varepsilon^f_b \end{aligned} \tag{4.13}$$

so

$$0 = \varepsilon_{ebcd}\varepsilon^e_a + \varepsilon_{aecd}\varepsilon^e_b + \varepsilon_{abed}\varepsilon^e_c + \varepsilon_{abce}\varepsilon^e_d, \tag{4.14}$$

up to first order, where ε^a_b is any antisymmetric object. The same relationship holds with forms, as well, replacing $\varepsilon^e_a \to \omega^e_a$. Doing so gives

$$0 = \varepsilon_{ebcd}\omega^e_a + \varepsilon_{aecd}\omega^e_b + \varepsilon_{abed}\omega^e_c + \varepsilon_{abce}\omega^e_d \tag{4.15}$$

which is exactly, $\mathbf{D}(\varepsilon_{abcd})$, so we conclude

$$\mathbf{D}(\varepsilon_{abcd}) = 0 \tag{4.16}$$

and the variation becomes

$$\delta\chi_E = \frac{1}{2}\int\left(\delta\omega^{ab}\varepsilon_{abcd} \wedge \mathbf{D}\left(\mathbf{R}^{cd}\right)\right). \tag{4.17}$$

This vanishes identically by the second Bianchi identity (3.44).

Expanding in components,

$$\mathbf{R}^{ab} \wedge \mathbf{R}^{cd}\varepsilon_{abcd} = R^{ab}{}_{ef} \wedge R^{cd}{}_{gh}\mathbf{e}^g \wedge \mathbf{e}^h \wedge \mathbf{e}^e \wedge \mathbf{e}^f \varepsilon_{abcd}, \tag{4.18}$$

and using

$$\mathbf{e}^g \wedge \mathbf{e}^h \wedge \mathbf{e}^e \wedge \mathbf{e}^f = \varepsilon^{efgh}\Phi, \tag{4.19}$$

where Φ is a generic volume form, we have

$$\mathbf{R}^{ab}\wedge\mathbf{R}^{cd}\varepsilon_{abcd} = R^{ab}{}_{ef}R^{cd}{}_{gh}\varepsilon^{efgh}\varepsilon_{abcd}\Phi. \tag{4.20}$$

Expanding out and simplifying (and taking the Hodge dual to eliminate the generic volume form) gives

$$R^{ab}{}_{ef}R^{cd}{}_{gh}\varepsilon^{efgh}\varepsilon_{abcd} = 4R^2 - 16R^b_cR^c_b + 4R^{ab}{}_{cd}R^{cd}{}_{ab} \tag{4.21}$$

so

$$\chi_E = \int\left(R^{ab}{}_{cd}R^{cd}{}_{ab} - 4R^b_cR^c_b + R^2\right)\Phi. \tag{4.22}$$

4.1.2. Alternate form of the action using the Gauss-Bonnet term

Given the action formed by the square of the conformal curvature (4.7), we may make use of the Gauss-Bonnet form of the Euler character, χ_E, (4.22) whose variation is identically zero. Since this is an invariant, we may equally use

$$\begin{aligned} S_\chi &= S - \chi_E \\ &= 2\int\left(R^{bc}R_{bc} - \frac{1}{3}R^2\right)d^4x \end{aligned} \tag{4.23}$$

for Weyl gravity.

4.1.3. Variation of the metric

Using the techniques in the subsection Second-order variation of the metric, along with the proper scaling, the second-order variation of the action as given by (4.23) with respect to the metric becomes

$$\begin{aligned} 0 &= \frac{1}{3}D_cD_dR - D_aD^a\left(R_{cd} - \frac{1}{6}\eta_{cd}R\right) \\ &\quad + R^{abe}{}_cR_{abed} + 2R^b_cR_{db} - \frac{1}{3}RR_{dc} \\ &\quad + \frac{1}{4}\left(R^{abef}R_{abef} - 2R^{ab}R_{ab} + \frac{1}{3}R^2\right)g_{cd} \end{aligned} \tag{4.24}$$

as also determined by Mannheim [26]. Alternatively, this may be written as

$$0 = 2D_dD_bC^{abcd} - C^{abcd}R_{bd}, \tag{4.25}$$

which is otherwise known as the Bach equation [6]. The Bach equation may also be found by directly varying (4.1).

4.2. Relationship to general relativity

In general relativity, the absence of matter fields in any particular region gives rise to the vacuum Einstein equation (2.60). Taking its trace and contracting it with the metric, g_{ab}, gives $R - 2R = 0$ in four dimensions, and hence, $R = 0$. With this condition,

$$R_{ab} = 0, \tag{4.26}$$

which must be true of all vacuum solutions in general relativity. In Weyl gravity, 4.24 may be written solely in terms of the Ricci tensor and the Ricci scalar, so $R_{ab} = 0$ is a solution to the field equation of this theory. Hence, all vacuum solutions of general relativity are also vacuum solutions of Weyl gravity. Sultana et al. have shown Weyl gravity contains other solutions, so the converse is not true [27]. Indeed, equations (4.24) and (4.25) are fourth-order metric differential equations. For example, they give the following example of a metric that is not conformal to an Einstein metric [21],

$$ds^2 = -B(r)\,dt^2 + \frac{dr^2}{B(r)} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right),$$

where

$$B(r) = 1 - \frac{\beta(2-3\beta\gamma)}{r} - 3\beta\gamma + \gamma r - kr^2.$$

CHAPTER 5

WEYL GRAVITY AS A GAUGE THEORY

In considering general relativity as a gauge theory, we began with the Poincaré group and gauged by the Lorentz transformations, leaving the translations to span the base manifold. The conformal group is a larger group which includes the Poincaré group as a subgroup. In forming a gauge theory that might also lead to general relativity, or at least have a theory that shares many of the same solutions, the quotient formed from this group minimally should have Lorentz symmetry and also leave the translations in the base manifold. Since the conformal group includes scale transformations, which may be attributed to a choice of local units, dilatational symmetry is desirable (some examples of local units include redshift or CMB temperature). The only remaining subgroup is the special conformal transformations and a choice must be made as to whether to include them in the fibers, or not. If we are to not have special conformal symmetry, we must have an eight-dimensional theory – four dimensions spanned by the solder form and four dimensions spanned by the forms dual to the gauge fields of the special conformal transformations. If we have special conformal symmetry, then we have a four-dimensional theory, as we might expect when not specifically considering Weyl gravity as a gauge theory of the conformal group, as in Chapter 3. Taking the conformal group and gauging by the Lorentz transformations, dilatations, and the special conformal transformations is called the auxiliary gauging since the gauge field of the special conformal transformations acts as an auxiliary field rather than a new, physical field [28].

5.1. Generators of the conformal group

Whereas the Poincaré group, the group of translations and Lorentz transformations (rotations), is the group that preserves the metric or the line element, the conformal group is the group that preserves the line element up to a factor [29], and therefore, admits a wider range of transformations. Aside from the two that define the Poincaré group, we also have dilatations and special conformal transformations. We define them and show their infinitesimal transformations.

5.1.1. Special conformal transformations

Special conformal transformations are defined for compactified Minkowski space in Cartesian coordinates by taking a coordinate, inverting it, translating it, then inverting again in the following manner:

$$x^a \to \frac{\frac{x^a}{x^2} + b^a}{\left(\frac{x^a}{x^2} + b^a\right)^2} = \frac{\frac{x^a}{x^2} + b^a}{\frac{1}{x^2} + \frac{2b^c x_c}{x^2} + b^2} = \frac{x^a + b^a x^2}{1 + 2b^c x_c + b^2 x^2}. \tag{5.1}$$

Up to first order in b^a, (5.1) may be written as $\left(x^a + b^a x^2\right)\left(1 + 2b^b x_b + \ldots\right)^{-1}$. Using the Taylor approximation formula of $(1-x)^{-1} = 1 - x + x^2 - x^3 + \ldots$ for $\left(1 + 2b^b x_b + \ldots\right)^{-1}$ along with (3.8), and pulling out the arbitrary coordinate, the generator of this subgroup is given by

$$K_a = -2x_a x^b \partial_b + x^2 \partial_a. \tag{5.2}$$

The free index may be raised using the metric.

5.1.2. Dilatations

Dilatations are performed by taking a coordinate and resizing it. That is,

$$x^a \to e^\lambda x^a. \tag{5.3}$$

Expanding the conformal factor in Taylor series $e^\lambda = 1 + \lambda + \lambda^2 + \ldots$ up to first order in λ, and using (3.8), the generator of the dilatations is given by

$$D = x^a \partial_a. \tag{5.4}$$

5.1.3. Summary of generators for the conformal group

Along with the generators for Lorentz transformations (3.12) and the generators for translations (3.15), the generators for the entire conformal group are given by

$$M^a_b = -\Delta^{ac}_{db} x^d \partial_c, \tag{5.5}$$

$$P_a = \partial_a, \tag{5.6}$$

$$K^a = \left(\frac{1}{2}\eta^{ab}x^2 - x^a x^b\right)\partial_b, \tag{5.7}$$

$$D = x^a \partial_a. \tag{5.8}$$

These are sufficient for finding the Lie algebra.

5.2. Lie Algebra of the conformal group

Taking all possible commutators of the generators as given by Eqs. (5.5) through (5.8) generates the Lie algebra of the conformal group:

$$[M^a_b, M^c_d] = -\frac{1}{2}\left(\delta^c_b \delta^a_f \delta^h_d - \delta^a_d \delta^h_b \delta^c_f - \eta_{bd}\eta^{ch}\delta^a_f - \eta^{ac}\eta_{bf}\delta^h_d\right) M^f_h, \tag{5.9}$$

$$[M^a_b, P_c] = \Delta^{af}_{cb} P_f, \tag{5.10}$$

$$[M^a_b, K^c] = -\Delta^{ca}_{bd} K^d, \tag{5.11}$$

$$\left[P_a, K^b\right] = -\delta^b_a D + 2\Delta^{bd}_{ca} M^c_d, \tag{5.12}$$

$$\left[D, K^b\right] = \delta^b_c K^c, \tag{5.13}$$

$$[D, P_a] = -\delta^b_a P_b. \tag{5.14}$$

Identifying the structure constants from the Lie algebra using (3.16), we have

$$c_{\binom{a}{b}\binom{c}{d}}^{\binom{f}{h}} = -\frac{1}{2}\left(\delta^c_b \delta^a_f \delta^h_d - \delta^a_d \delta^h_b \delta^c_f - \eta_{bd}\eta^{ch}\delta^a_f - \eta^{ac}\eta_{bf}\delta^h_d\right), \tag{5.15}$$

$$c_{\binom{a}{b}\binom{\cdot}{c}}{}^{\binom{\cdot}{f}} = \Delta^{af}_{cb}, \tag{5.16}$$

$$c_{\binom{a}{b}\binom{c}{\cdot}}{}^{\binom{d}{\cdot}} = -\Delta^{ca}_{bd}, \tag{5.17}$$

$$c_{\binom{\cdot}{a}\binom{b}{\cdot}}{}^{\binom{\cdot}{\cdot}} = -\delta^{b}_{a}, \tag{5.18}$$

$$c_{\binom{\cdot}{a}\binom{b}{\cdot}}{}^{\binom{c}{d}} = 2\Delta^{bd}_{ca}, \tag{5.19}$$

$$c_{\binom{\cdot}{\cdot}\binom{b}{\cdot}}{}^{\binom{c}{\cdot}} = \delta^{b}_{c}, \tag{5.20}$$

$$c_{\binom{\cdot}{\cdot}\binom{\cdot}{a}}{}^{\binom{\cdot}{b}} = -\delta^{b}_{a}, \tag{5.21}$$

where $\binom{a}{b}$ denotes the Lorentz transformations, $\binom{a}{\cdot}$ denotes the special conformal transformations, $\binom{\cdot}{a}$ denotes the translations, and $\binom{\cdot}{\cdot}$ denotes the dilatations. All other structure constants are zero.

5.3. Maurer-Cartan structure equations of the conformal group

Using the general Maurer-Cartan structure equation (3.29) and the structure constants (5.15)-(5.21), we arrive at the following structure equations for the conformal group

$$\mathbf{d}\omega^{a}_{\ b} = \omega^{c}_{\ b} \wedge \omega^{a}_{\ c} + 2\Delta^{ad}_{cb}\mathbf{f}_{d} \wedge \mathbf{e}^{c}, \tag{5.22}$$

$$\mathbf{d}\mathbf{e}^{a} = \mathbf{e}^{c} \wedge \omega^{a}_{\ c} + \omega \wedge \mathbf{e}^{a}, \tag{5.23}$$

$$\mathbf{d}\mathbf{f}_{b} = \omega^{c}_{\ b} \wedge \mathbf{f}_{c} + \mathbf{f}_{b} \wedge \omega, \tag{5.24}$$

$$\mathbf{d}\omega = \mathbf{e}^{c} \wedge \mathbf{f}_{c}, \tag{5.25}$$

where $\mathbf{f}_d$ is the gauge field of the special conformal transformations and ω is the gauge field for the dilatations or the Weyl vector. Changing the connection, the same equations acquire a Lorentz curvature, $\Omega^{a}_{\ b}$, special conformal curvature, $\mathbf{S}_b$, translational curvature or torsion, $\mathbf{T}^a$, and dilatational curvature, Ω, to become

$$\mathbf{d}\omega^a_b = \omega^c_b \wedge \omega^a_c + 2\Delta^{ad}_{cb}\mathbf{f}_d \wedge \mathbf{e}^c + \Omega^a_b, \tag{5.26}$$

$$\mathbf{d}\mathbf{e}^a = \mathbf{e}^c \wedge \omega^a_c + \omega \wedge \mathbf{e}^a + \mathbf{T}^a, \tag{5.27}$$

$$\mathbf{d}\mathbf{f}_b = \omega^c_b \wedge \mathbf{f}_c + \mathbf{f}_b \wedge \omega + \mathbf{S}_b, \tag{5.28}$$

$$\mathbf{d}\omega = \mathbf{e}^c \wedge \mathbf{f}_c + \Omega. \tag{5.29}$$

These equations define the curvatures for our conformal gauge theory.

5.4. Bianchi identities of the conformal group

As in the case of the Poincaré group, we take an exterior derivative for each of the structure equations to arrive at its Bianchi identities.

5.4.1. Spin connection

Starting with the structure equation for the solder form (5.27), we take another exterior derivative to get that $\mathbf{d}^2\omega^a_b = 0$, which becomes

$$\begin{aligned} 0 &= \mathbf{d}\left(\omega^c_b\right) \wedge \omega^a_c - \omega^c_b \wedge \left(\mathbf{d}\omega^a_c\right) + \mathbf{d}\left(\mathbf{f}_b\right) \wedge \mathbf{e}^a - \mathbf{f}_b \wedge \mathbf{d}\mathbf{e}^a \\ &\quad -\eta^{ad}\eta_{cb}\mathbf{d}\left(\mathbf{f}_d\right) \wedge \mathbf{e}^c + \eta^{ad}\eta_{cb}\mathbf{f}_d \wedge \left(\mathbf{d}\mathbf{e}^c\right) + \mathbf{d}\Omega^a_b. \end{aligned} \tag{5.30}$$

Substituting from the structure equations (5.26) through (5.29) gives

$$0 = \mathbf{D}\Omega^a_b + 2\Delta^{ad}_{cb}\left(\mathbf{S}_d \wedge \mathbf{e}^c - \mathbf{f}_d \wedge \mathbf{T}^c\right). \tag{5.31}$$

5.4.2. *Solder form*

Starting with the structure equation for the solder form (5.27), we take another exterior derivative to get that $\mathbf{d}^2\mathbf{e}^a = 0$, which becomes

$$0 = (\mathbf{de}^c) \wedge \omega^a_{\ c} - \mathbf{e}^c \wedge (\mathbf{d}\omega^a_{\ c}) + (\mathbf{d}\omega) \wedge \mathbf{e}^a - \omega \wedge (\mathbf{de}^a)\,. \tag{5.32}$$

Substituting from the structure equations (5.26) through (5.29) gives

$$0 = \mathbf{DT}^a + \Omega \wedge \mathbf{e}^a - \mathbf{e}^c \wedge \Omega^a_{\ c}. \tag{5.33}$$

5.4.3. *Weyl vector*

Starting with the structure equation for the Weyl vector (5.29), we take another exterior derivative to get that $\mathbf{d}^2\omega = 0$, which becomes

$$0 = (\mathbf{de}^c) \wedge \mathbf{f}_c - \mathbf{e}^c \wedge (\mathbf{df}_c) + \mathbf{d}\Omega. \tag{5.34}$$

Substituting from the structure equations for the solder form (5.27) and for the special conformal transformations (5.28) gives

$$0 = \mathbf{d}\Omega + \mathbf{T}^c \wedge \mathbf{f}_c - \mathbf{e}^c \wedge \mathbf{S}_c. \tag{5.35}$$

5.4.4. *Special conformal transformations*

Starting with the structure equation for the special conformal transformations (5.28), we take another exterior derivative to get that $\mathbf{d}^2\mathbf{f}_b = 0$, which becomes

$$0 = (\mathbf{d}\omega^c_{\ b}) \wedge \mathbf{f}_c - \omega^c_{\ b} \wedge (\mathbf{df}_c) + (\mathbf{df}_b) \wedge \omega - \mathbf{f}_b \wedge (\mathbf{d}\omega) + \mathbf{dS}_b. \tag{5.36}$$

Substituting from the structure equations (5.26) through (5.29) gives

$$0 = \mathbf{DS}_b + \Omega^c_{\ b} \wedge \mathbf{f}_c - \mathbf{f}_b \wedge \Omega. \tag{5.37}$$

5.4.5. Summary

Equations (5.37), (5.35), (5.31), and (5.33) put together give

$$0 = \mathbf{DS}_b + \Omega^c_{\ b} \wedge \mathbf{f}_c - \mathbf{f}_b \wedge \Omega, \tag{5.38}$$

$$0 = \mathbf{d}\Omega + \mathbf{T}^c \wedge \mathbf{f}_c - \mathbf{e}^c \wedge \mathbf{S}_c, \tag{5.39}$$

$$0 = \mathbf{D}\Omega^a_{\ b} + 2\Delta^{ad}_{cb} \left(\mathbf{S}_d \wedge \mathbf{e}^c - \mathbf{f}_d \wedge \mathbf{T}^c \right), \tag{5.40}$$

$$0 = \mathbf{DT}^a + \Omega \wedge \mathbf{e}^a - \mathbf{e}^c \wedge \Omega^a_{\ c}. \tag{5.41}$$

In the absence of torsion and special conformal curvature, we have

$$0 = \Omega^c_{\ b} \wedge \mathbf{f}_c - \mathbf{f}_b \wedge \Omega, \tag{5.42}$$

$$0 = \mathbf{d}\Omega, \tag{5.43}$$

$$0 = \mathbf{D}\Omega^a_{\ b}, \tag{5.44}$$

$$0 = \Omega \wedge \mathbf{e}^a - \mathbf{e}^c \wedge \Omega^a_{\ c}. \tag{5.45}$$

5.5. The quotient method and tensors

In this section, we use the linear transformation properties of the curvatures and connection forms applied to the conformal group. We also apply the quotient method by eliminating any terms that contain transformations found on the base manifold (translations). From this, we identify the tensors to form an action.

5.5.1. Transformations of the connection forms

Using the infinitesimal transformation relation for the connection forms (3.62), we follow the same procedure we did with the connection forms in the Poincaré group with the difference that the group indices now run from 0 to 5. Also, using the relations between connection forms in the conformal group given by (A.56) - (A.63), we get the following transformations for the forms under infinitesimal conformal transformations,

$$\begin{aligned}
\tilde{\omega}^a_b &= \omega^a_b - \omega^a_c \Lambda^c_b + \Lambda^a_c \omega^c_b - \mathbf{e}^a \Lambda_b, \\
&\quad + \eta^{ac} \Lambda_c \eta_{bd} \mathbf{e}^d - \eta^{ac} \mathbf{f}_c \eta_{bd} \Lambda^d + \Lambda^a \mathbf{f}_b - \mathbf{d}\left(\Lambda^a_b\right), && (5.46) \\
\tilde{\mathbf{e}}^a &= \mathbf{e}^a - \omega^a_c \Lambda^c - \mathbf{e}^a \Lambda + \Lambda^a_c \mathbf{e}^c + \Lambda^a \omega - \mathbf{d}\left(\Lambda^a\right), && (5.47) \\
\tilde{\mathbf{f}}_b &= \mathbf{f}_b + \Lambda_c \omega^c_b - \mathbf{f}_c \Lambda^c_b + \Lambda \mathbf{f}_b - \omega \Lambda_b - \mathbf{d}\left(\Lambda_b\right), && (5.48) \\
\tilde{\omega} &= \omega + \Lambda_c \mathbf{e}^c - \mathbf{f}_c \Lambda^c - \mathbf{d}\left(\Lambda\right), && (5.49)
\end{aligned}$$

where $\omega^a \equiv \mathbf{e}^a$, $\omega_a = \mathbf{f}_a$, Λ^a is an infinitesimal local translation, Λ^a_d is a local Lorentz transformation, Λ_a is a local special conformal transformation, and Λ is a local dilatation. Since in the auxiliary gauging we have only Lorentz, dilatational, and special conformal symmetry, we eliminate all terms with translational generators. Equations (5.46) - (5.49) become

$$\begin{aligned}
\tilde{\omega}^a_b &= \omega^a_b - \omega^a_c \Lambda^c_b + \Lambda^a_c \omega^c_b - \mathbf{e}^a \Lambda_b + \eta^{ac} \Lambda_c \eta_{bd} \mathbf{e}^d - \mathbf{d}\left(\Lambda^a_b\right), && (5.50) \\
\tilde{\mathbf{e}}^a &= \mathbf{e}^a - \mathbf{e}^a \Lambda + \Lambda^a_c \mathbf{e}^c, && (5.51) \\
\tilde{\mathbf{f}}_b &= \mathbf{f}_b + \Lambda_c \omega^c_b - \mathbf{f}_c \Lambda^c_b + \Lambda \mathbf{f}_b - \omega \Lambda_b - \mathbf{d}\left(\Lambda_b\right), && (5.52) \\
\tilde{\omega} &= \omega + \Lambda_c \mathbf{e}^c - \mathbf{d}\left(\Lambda\right). && (5.53)
\end{aligned}$$

From the terms containing the inhomogeneous exterior derivatives, it is clear the spin connection, the gauge field of the special conformal transformations, and the Weyl vector are not tensors. However, up to first order, the three terms from the transformation of the spin connection can be collected to give

$$\tilde{\mathbf{e}}^a = (\delta^a_c + \Lambda^a_c)\,\mathbf{e}^c\,(1-\Lambda)\,, \tag{5.54}$$

which is the linearization of

$$\tilde{\mathbf{e}}^a = g^a_c \mathbf{e}^c \bar{g}. \tag{5.55}$$

(5.55) may also be derived by expanding (3.52) in terms of the indices a, 4, and 5, and requiring that $g^a_4 = g^a_5 = 0$, since these are group elements of the translations. The solder form, therefore, transforms as a tensor under a Lorentz transformation and a dilatation.

5.5.2. Curvature transformations

Using the infinitesimal transformation relation for the curvatures (3.68), we follow the same procedure we used with the curvatures in the Poincaré group with the difference that the group indices now run from 0 to 5. Also, using the relations between curvatures in the conformal group given by (A.66) - (A.73), we get the following transformations for the curvatures under the full conformal group,

$$\begin{aligned}
\tilde{\Omega}^a_b &= \Omega^a_b - \Omega^a_c \Lambda^c_b + \Lambda^a_c \Omega^c_b - \mathbf{T}^a \Lambda_b, \\
&\quad + \Lambda^a \mathbf{S}_b + \eta^{ac} \Lambda_c \eta_{bd} \mathbf{T}^d - \eta^{ac} \mathbf{S}_c \eta_{bd} \Lambda^d, && (5.56)\\
\tilde{\mathbf{T}}^a &= \mathbf{T}^a + \Lambda^a_c \mathbf{T}^c - \Omega^a_c \Lambda^c - \mathbf{T}^a \Lambda + \Lambda^a \Omega, && (5.57)\\
\tilde{\mathbf{S}}_b &= \mathbf{S}_b + \Lambda_c \mathbf{R}^c_b - \mathbf{S}_c \Lambda^c_b + \Lambda \mathbf{S}_b - \Omega \Lambda_b, && (5.58)\\
\tilde{\Omega} &= \Omega + \Lambda_c \mathbf{T}^c - \mathbf{S}_c \Lambda^c, && (5.59)
\end{aligned}$$

where $\mathbf{R}^a \equiv \mathbf{T}^a$ is the torsion, $\mathbf{R}_a = \mathbf{S}_a$ is the special conformal curvature, and $\mathbf{R} = \Omega$ is the dilatational curvature. We will call $\Omega^a{}_b$ the Lorentz curvature to distinguish it from the Riemann curvature. As with the forms, because of the imposed symmetry, we eliminate all terms that contain local translations. Equations (5.56) - (5.59) then become

$$\begin{aligned}
\tilde{\Omega}^a_{\ b} &= \Omega^a_{\ b} - \Omega^a_{\ c}\Lambda^c_{\ b} + \Lambda^a_{\ c}\Omega^c_{\ b} - \mathbf{T}^a\Lambda_b + \eta^{ac}\Lambda_c\eta_{bd}\mathbf{T}^d, && (5.60)\\
\tilde{\mathbf{T}}^a &= \mathbf{T}^a + \Lambda^a_{\ c}\mathbf{T}^c - \mathbf{T}^a\Lambda, && (5.61)\\
\tilde{\mathbf{S}}_b &= \mathbf{S}_b + \Lambda_c\Omega^c_{\ b} - \mathbf{S}_c\Lambda^c_{\ b} + \Lambda\mathbf{S}_b - \Omega\Lambda_b, && (5.62)\\
\tilde{\Omega} &= \Omega + \Lambda_c\mathbf{T}^c. && (5.63)
\end{aligned}$$

A priori, torsion is the only piece of the full curvature that transforms separately as a tensor. All the other curvatures mix with one another. However, since it is a tensor it is consistent to set the torsion to zero $\mathbf{T}^a = 0$ in (5.60) and (5.63). This results in

$$\begin{aligned}
\tilde{\Omega}^a_{\ b} &= \Omega^a_{\ b} - \Omega^a_{\ c}\Lambda^c_{\ b} + \Lambda^a_{\ c}\Omega^c_{\ b}, && (5.64)\\
\tilde{\Omega} &= \Omega, && (5.65)
\end{aligned}$$

while the transformation of $\mathbf{S}_a$ is unchanged. With vanishing torsion, the dilatational curvature becomes an invariant. Using the the full transformation property of the curvature (3.60) and expanding in terms of the indices a, 4, and 5, and requiring $g^a_{\ 4} = g^a_{\ 5} = 0$ results in

$$\tilde{\Omega}^a_{\ b} = g^a_{\ c}\Omega^c_{\ d}\bar{g}^d_{\ b}, \tag{5.66}$$

so the two curvature tensors in the gauging are the Lorentz curvature and the dilatational curvature.

5.6. Action and field equations

There is no action linear in the conformal curvatures that is also scale invariant and we must go to quadratic order. With vanishing torsion, the most general even-parity quadratic action that can be formed from curvatures is given by

$$S=\int\left[\alpha\Omega^a_b\wedge{}^*\Omega^b_a+\beta\Omega\wedge{}^*\Omega\right], \tag{5.67}$$

where α and β are arbitrary constants and * is the Hodge dual [28]. We now find the field equations for this action, once again, using a Palatini-type variation in which all connection forms are varied.

5.6.1. Field equations

Varying the Weyl vector, ω, the gauge field of the special conformal transformations, $\mathbf{f}_a$, the spin connection, ω^a_b, and the solder form, $\mathbf{e}^a$, gives

$$\begin{aligned}
{}^*\mathbf{d}{}^*\Omega &= 0, && (5.68)\\
2\alpha\Delta^{ac}_{db}\mathbf{e}^d\wedge{}^*\Omega^b_a &= \beta\mathbf{e}^c\wedge{}^*\Omega, && (5.69)\\
\mathbf{D}{}^*\Omega^b_a &= 0, && (5.70)\\
4\alpha f^{cd}\Omega_{cadb}+2\beta f_{ac}\Omega^c_b &= 2\beta Q_{ab}-4\alpha\Theta_{ab}, && (5.71)
\end{aligned}$$

respectively, where Θ_{ab} is the energy-momentum tensor built from the Lorentz curvature and Q_{ab} is the energy-momentum tensor built from the Weyl curvature given by

$$\begin{aligned}
\Theta_{ab} &= \Omega^c{}_{dae}\Omega^d{}_{cb}{}^e-\frac{1}{4}\Omega^c{}_{def}\Omega^{d\,ef}_c\eta_{ab}, && (5.72)\\
Q_{ab} &= \Omega_{ac}\Omega^c_b-\frac{1}{4}\Omega_{cd}\Omega^{cd}\eta_{ab}. && (5.73)
\end{aligned}$$

In components, (5.68) - (5.71) become

$$\frac{1}{\sqrt{|g|}}\left(\partial_\alpha\left(\Omega^{\alpha\beta}\sqrt{|g|}\right)\right) = 0, \tag{5.74}$$

$$\alpha 2\Delta^{ac}_{db}\Omega^{b\,ed}_{a}\sqrt{|g|} = \beta\Omega^{ec}\sqrt{|g|}, \tag{5.75}$$

$$D^c\Omega^b_{\ acd} = 0, \tag{5.76}$$

$$4\alpha f^{cd}\Omega_{cadb} + 2\beta f_{ac}\Omega^c_b = 2\beta Q_{ab} - 4\alpha\Theta_{ab}. \tag{5.77}$$

With all the field equations identified, the next step is to find solutions for each of the gauge fields.

CHAPTER 6

SOLVING THE FIELD EQUATIONS

Once an action has been constructed with the desired symmetry and the field equations have been extracted by varying that action with respect to its independent fields, the next step is to solve those field equations using the structure equations and, if needed, their corresponding Bianchi identities. In this section, we find the solutions to each of the gauge fields in closed form, whenever possible, and show how we use those solutions to find the solutions for the remaining fields.

6.1. Special conformal transformations

Taking the field equation of the special conformal transformations, (5.69) adding a third solder form via a wedge product and using (2.45) gives

$$0 = 2\alpha\Omega^{c}_{\ acb} + \beta\Omega_{ab}. \tag{6.1}$$

Taking the trace of Bianchi identity of the solder form (5.45) gives

$$\Omega^{a}_{\ bad} = \Omega_{db}. \tag{6.2}$$

Using this condition in (6.1) gives

$$(2\alpha - \beta)\,\Omega_{ab} = 0, \tag{6.3}$$

which for $(2\alpha - \beta) \neq 0$ gives

$$\Omega_{ab} = 0, \tag{6.4}$$

so the dilatational curvature vanishes. Returning to (6.1), we see

$$\Omega^{a}_{\ bad} = 0. \tag{6.5}$$

Using the definition of the conformal curvature (4.2) and defining the curvature of $\omega^a_{\ b}$ to be

$$\mathbf{R}^a_{\ b} = \mathbf{d}\omega^a_{\ b} - \omega^c_{\ b} \wedge \omega^a_{\ c}, \tag{6.6}$$

the conformal curvature may be written as

$$\mathbf{C}^a_{\ b} = \mathbf{R}^a_{\ b} - \mathbf{e}^a \wedge \boldsymbol{\mathscr{R}}_b + \mathbf{e}_b \wedge \boldsymbol{\mathscr{R}}^a \tag{6.7}$$

or

$$\mathbf{C}^a_{\ b} = \mathbf{R}^a_{\ b} - 2\Delta^{ad}_{cb}\mathbf{e}^c \wedge \boldsymbol{\mathscr{R}}_d, \tag{6.8}$$

where

$$\boldsymbol{\mathscr{R}}_b \equiv \frac{1}{(n-2)}\left(\mathbf{R}_b - \frac{1}{2(n-1)}R\mathbf{e}_b\right) \tag{6.9}$$

is the Schouten tensor [30] and where $\mathbf{R}_a = R_{ab}\mathbf{e}^b$. Solving for the Riemann curvature and substituting into (5.26) gives

$$\Omega^a_{\ bcd} = C^a_{\ bcd} - \Delta^{ae}_{cb}\left(\mathscr{R}_{ed} + f_{ed}\right) + \Delta^{ae}_{db}\left(\mathscr{R}_{ec} + f_{ec}\right). \tag{6.10}$$

Contracting on the a and c indices and imposing the field equation (6.5) gives

$$0 = (n-2)\left(\mathscr{R}_{bd} + f_{bd}\right) + \eta^{ce}\eta_{db}\left(\mathscr{R}_{ec} + f_{ec}\right). \tag{6.11}$$

Contracting with the metric, this becomes

$$0 = 2(n-1)\left(\eta^{db}\mathscr{R}_{bd} + \eta^{db}f_{bd}\right), \tag{6.12}$$

which for dimensions $n > 1$ is

$$\eta^{db}\mathscr{R}_{bd} + \eta^{db}f_{bd} = 0; \tag{6.13}$$

and therefore, substituting (6.13) back into (6.11)

$$f_{bd} = -\mathscr{R}_{bd} \tag{6.14}$$

or equivalently $\mathbf{f}_b = -\boldsymbol{\mathscr{R}}_b$ so the gauge field of the special conformal transformations is identically the negative of the Schouten tensor. This result was shown by Crispim-Romão [31].

6.2. Weyl vector

Vanishing dilatational curvature as a consequence of the field equation for the special conformal transformations (6.5) implies the structure equation for the Weyl vector remains in its original form (5.25). Writing $\mathbf{f}_a = f_{ab}\mathbf{e}^b = -\mathscr{R}_{ab}\mathbf{e}^b$ and substituting into (5.25) gives

$$\mathbf{d}\omega \;=\; -\mathscr{R}_{ab}\mathbf{e}^a \wedge \mathbf{e}^b. \tag{6.15}$$

Because $\mathscr{R}_{ab}$ is symmetric, the right side vanishes, giving that the Weyl vector is closed,

$$\mathbf{d}\omega = 0. \tag{6.16}$$

This implies the Weyl vector takes the pure gauge form

$$\omega = \mathbf{d}\phi, \tag{6.17}$$

giving the Weyl vector as the gradient of an arbitrary function.

6.3. Equivalence to conformal gravity

Substituting the Schouten tensor for the special conformal transformations in (6.10), we have

$$\Omega^a_{\ bcd} = C^a_{\ bcd}. \tag{6.18}$$

The gauge field of the special conformal transformations has acted as an auxiliary field to turn the Lorentz curvature into the conformally invariant Weyl curvature [28, 31]. Using this condition and also the condition that the dilatational curvature be zero (6.5), the action, Eq. (5.67), becomes

$$S = \int \left[\alpha \mathbf{C}^a_{\ b} \wedge {}^*\mathbf{C}^b_{\ a}\right]. \tag{6.19}$$

Expanding out the Hodge dual, we get (4.1) up to a sign (the field equations remain unchanged by this overall sign). The gauge theory, therefore, results in the same action as Weyl gravity.

6.4. Spin connection

Expanding the Hodge dual in the field equation for the spin connection (5.70) and writing it in components gives

$$D_a \Omega^a_{\ bcd} = 0. \tag{6.20}$$

6.5. Solder form

In this section we show how the field equation for the solder form (5.77) simplifies. The energy-momentum tensor constructed from the dilatational curvature vanishes by virtue of a field equation, while the energy-momentum tensor constructed from the Lorentz curvature vanishes from a Gauss-Bonnet identity.

6.5.1. Local identity from the Gauss-Bonnet

We have seen that varying the connection in the Gauss-Bonnet expression for the Euler character, $\chi_E = \int \left(R^{ab}{}_{cd} R^{cd}{}_{ab} - 4R^b_{\ c} R^c_{\ b} + R^2\right)$ vanishes identically by the second Bianchi identity. It must, therefore, also vanish if we restrict the variation of the connection to the metric variation, performing a second-order variation of χ. Explicitly writing the metric, we may write the Gauss-Bonnet integrand as

$$R^{abcd}R_{abcd}-4R^{bc}R_{bc}+R^2=\left(-R^a{}_{dfe}R^d{}_{acb}-4R_{ef}R_{bc}+R_{be}R_{fc}\right)g^{fc}g^{be}, \tag{6.21}$$

so the variation leads to

$$\begin{aligned}0 &= \delta\chi_E,\\ &= \int\left(-2\delta R^a_{bfe}R^b_{acb}g^{fc}g^{be}-8\delta R_{ef}g^{fc}g^{be}R_{bc}+2\delta R_{be}g^{be}R_{fc}g^{fc}\right)g^{fc}g^{be}\sqrt{-g}d^4x\\ &\quad+\int\left(-R^a{}_{dfe}R^d{}_{acb}-4R_{ef}R_{bc}+R_{be}R_{fc}\right)\delta\left(g^{fc}g^{be}\sqrt{-g}\right)d^4x.\end{aligned} \tag{6.22}$$

From the first-order variation of the metric, we had

$$\delta\left(g^{fc}g^{be}\sqrt{-g}\right)=\delta\left(g^{fc}\right)g^{be}\sqrt{-g}+g^{fc}\delta\left(g^{be}\right)\sqrt{-g}+g^{fc}g^{be}\delta\left(\sqrt{-g}\right). \tag{6.23}$$

Utilizing

$$\delta\left(\sqrt{-g}\right)=\frac{1}{2}\sqrt{-g}g^{ab}\delta g_{ab} \tag{6.24}$$

and

$$\delta\left(g^{ae}\right)=-g^{he}g^{ai}\delta\left(g_{hi}\right), \tag{6.25}$$

we find the explicit metric variation gives

$$\delta_1 S_{\rm GB}=\int\left(-2R^{abdf}R_{abef}g^{ce}-4R^{ad}R_{ab}g^{cb}-2RR^{dc}\right)\delta\left(g_{cd}\right)\sqrt{-g}d^4x, \tag{6.26}$$

and the internal variation gives

$$\begin{aligned}\delta_2 S_{\rm GB} &= \int [-\left(2D_bD_aR^{bdca}+2D_bD_aR^{dcba}+2D_bD_aR^{cbad}\right)\delta g_{cd}\sqrt{-g} \\ &\quad -\left(4g^{cd}D_bD_aR^{ab}-8D_aD^dR^{ac}+4D_aD^aR^{cd}\right)\delta g_{cd}\sqrt{-g} \\ &\quad +\left(2g^{cd}D_aD^aR-2D^dD^cR^{cd}\right)\delta g_{cd}\sqrt{-g}]d^4x. \end{aligned} \tag{6.27}$$

Substituting the variation of the Ricci tensor with respect to the metric (2.70) into (6.26), along with integrating the necessary terms by parts twice gives, combining $\delta_1 S_{\rm GB}$ and $\delta_2 S_{\rm GB}$,

$$\begin{aligned} 0 &= -2D_bD_aR^{bdca}-2D_bD_aR^{dcba}-2D_bD_aR^{cbad} \\ &\quad -4g^{cd}D_bD_aR^{ab}+8D_aD^dR^{ac}-4D_aD^aR^{cd} \\ &\quad +2g^{cd}D_aD^aR-2D^dD^cR^{cd} \\ &\quad +\left(-\frac{1}{2}R^{abef}R_{abef}-2R^{ab}R_{ab}+\frac{1}{2}R^2\right)g^{cd} \\ &\quad -2R^{abdf}R_{abef}g^{ce}-4R^{ad}R_{ab}g^{cb}-2RR^{dc}. \end{aligned} \tag{6.28}$$

Using the identities

$$D_eR^{efgh}=D^gR^{hf}-D^hR^{fg} \tag{6.29}$$

along with

$$D_cR^{ca}=\frac{1}{2}D^aR \tag{6.30}$$

and

$$D_bD_aR^{ab}=\frac{1}{2}D_aD^aR \tag{6.31}$$

gives a completely algebraic identity

$$\begin{aligned} 0 &= R^{abfd}R_{abf}{}^{c} - 2R^{ad}R^{c}_{a} - 2R_{ea}R^{edac} \\ &\quad + RR^{dc} - \frac{1}{4}\left(R^{abef}R_{abef} - 4R^{ab}R_{ab} + R^2\right)g^{cd}. \end{aligned} \tag{6.32}$$

6.5.2. Vanishing energy-momentum tensors

The solder form field equation (5.71) contains an energy-momentum tensor, Q_{ab} (5.73), formed from the dilatational curvature and an energy-momentum tensor, Θ_{ab} (5.72), formed from the Lorentz curvature. The dilatational energy-momentum tensor vanishes since the dilatational curvature vanishes (6.4),

$$Q_{ab} = 0. \tag{6.33}$$

Since in this theory the Lorentz curvature, $\Omega^{a}{}_{bcd}$, is equivalent to conformal tensor (6.18), then the energy-momentum tensor of the Lorentz curvature can be written as

$$\Theta_{ab} = C^{c}_{dae}C^{d}_{cb}{}^{e} - \frac{1}{4}C^{c}_{def}C^{d\,ef}_{c}\eta_{ab}. \tag{6.34}$$

Substitution of the conformal curvature in terms of the Riemann tensor, Ricci tensor, and Ricci scalar yields

$$\begin{aligned} \Theta_{ab} &= -R^{cd\mu}{}_{b}R_{cd\mu a} + 2R^{c\mu}R_{cb\mu a} - R_{ab}R \\ &\quad + 2R^{c}_{b}R_{ca} + \frac{1}{4}\left(R_{cdef}R^{cdef} - 4R^{cd}R_{cd} + R^2\right)\eta_{ab}, \end{aligned} \tag{6.35}$$

which, up to a factor of -1, is equivalent to the algebraic expression (6.32), which comes from varying the Gauss-Bonnet term with respect to the metric. This was an unexpected feature of this theory. We, therefore, have

$$\Theta_{ab} = 0. \tag{6.36}$$

6.6. Equivalence to Bach equation

In this section, we recover the Bach equation by performing a second-order variation of the action with respect to the spin connection, and show the Palatini variations result in a more restricted form of the same equation. Right away, we can write the variation using (5.77) and (5.70),

$$\begin{aligned} \delta_{\mathbf{e}} S &= \int \left(4\alpha f^{cd}\Omega_{cadb} + 2\beta f_{ac}\Omega^{c}_{\ b} + 4\alpha\Theta_{ab} - 2\beta Q_{ab}\right) A^{ab} d^4x \\ &\quad + \int 2\alpha \mathbf{D}\left({}^{*}\Omega^{b}_{\ a}\right) \wedge \delta_{\mathbf{e}}\omega^{a}_{\ b}, \end{aligned} \tag{6.37}$$

where $A^{a}_{\ b}\mathbf{e} = \delta \mathbf{e}^{a}$. In the following, we find $\delta_{\mathbf{e}}\omega^{a}_{\ b}$ in order to substitute into (6.37).

In a torsion-free Riemannian geometry, the field equation for the solder form (3.35) is

$$\mathbf{d}\mathbf{e}^{a} = \mathbf{e}^{c} \wedge \omega^{a}_{\ c}. \tag{6.38}$$

Immediately, this can be written as

$$\mathbf{D}\mathbf{e}^{a} \equiv \mathbf{d}\mathbf{e}^{a} - \mathbf{e}^{c} \wedge \omega^{a}_{\ c} = 0, \tag{6.39}$$

or that the covariant derivative of the solder form vanishes. A variation of (6.39) with respect to both the solder form (putting it in coordinates) and the spin connection gives

$$D_{\mu}\delta e^{\ a}_{\nu} - D_{\nu}\delta e^{\ a}_{\mu} = e^{b}_{\mu}\delta\omega^{a}_{\ b\nu} - e^{b}_{\nu}\delta\omega^{a}_{\ b\mu}. \tag{6.40}$$

In order to solve for the variation of the spin connection, we isolate it and cycle the indices giving

$$\begin{aligned}
\delta\omega_{\alpha\mu\nu} - \delta\omega_{\alpha\nu\mu} &= \eta_{ab} e^{b}_{\alpha} \left(D_\mu \delta e^{a}_{\nu} - D_\nu \delta e^{a}_{\mu} \right), \qquad (6.41)\\
\delta\omega_{\mu\nu\alpha} - \delta\omega_{\mu\alpha\nu} &= \eta_{ab} e^{b}_{\mu} \left(D_\nu \delta e^{a}_{\alpha} - D_\alpha \delta e^{a}_{\nu} \right), \text{ and} \qquad (6.42)\\
\delta\omega_{\mu\nu\alpha} - \delta\omega_{\alpha\nu\mu} &= \eta_{ab} e^{b}_{\nu} \left(D_\alpha \delta e^{a}_{\mu} - D_\mu \delta e^{a}_{\alpha} \right). \qquad (6.43)
\end{aligned}$$

Adding the first two and subtracting the third,

$$\begin{aligned}
2\delta\omega_{\alpha\mu\nu} &= -\left(D_\nu \left[\eta_{ab} \delta e^{b}_{\mu} e^{a}_{\alpha} \right] - D_\mu \left[\eta_{ab} \delta e^{b}_{\nu} e^{a}_{\alpha} \right] \right) \\
&\quad - \left(D_\alpha \left[\eta_{ab} e^{a}_{\mu} \delta e^{b}_{\nu} \right] - D_\nu \left[\eta_{ab} e^{a}_{\mu} \delta e^{b}_{\alpha} \right] \right) \\
&\quad + \left(D_\mu \left[\eta_{ab} e^{a}_{\nu} \delta e^{b}_{\alpha} \right] - D_\alpha \left[\eta_{ab} e^{a}_{\nu} \delta e^{b}_{\mu} \right] \right). \qquad (6.44)
\end{aligned}$$

Using the variation of the metric with respect to the solder form

$$\delta \left(g_{\mu\nu} \right) = 2\eta_{ab} \delta \left(e^{a}_{\mu} \right) e^{b}_{\nu} \qquad (6.45)$$

we write (6.44)

$$\begin{aligned}
2\delta\omega_{\alpha\mu\nu} &= -\frac{1}{2} \left(D_\nu \delta g_{\mu\alpha} - D_\mu \delta g_{\nu\alpha} \right) \\
&\quad -\frac{1}{2} \left(D_\alpha \delta g_{\nu\mu} - D_\nu \delta g_{\alpha\mu} \right) \\
&\quad +\frac{1}{2} \left(D_\mu \delta g_{\alpha\nu} - D_\alpha \delta g_{\mu\nu} \right), \qquad (6.46)
\end{aligned}$$

and after combining terms, contracting with basis forms and raising an index, we get

$$\delta\omega^{a}_{b} = -\frac{1}{2} \eta^{ca} e_{c}^{\alpha} e_{b}^{\mu} \left[D_\alpha \delta g_{\nu\mu} - D_\mu \delta g_{\nu\alpha} \right] \mathbf{d}x^\nu, \qquad (6.47)$$

which is the variation of the spin connection with respect to the metric. Substituting into

(6.37) gives

$$\begin{aligned} \delta_{\mathbf{e}}S &= \int \left(4\alpha f^{cd}\Omega_{cadb} + 2\beta f_{ac}\Omega^{c}_{b} + 4\alpha\Theta_{ab} - 2\beta Q_{ab}\right) A^{ab} d^4x \\ &+ \int 4\alpha \left[D_d \mathbf{D}\left({}^{*}\Omega^{d}_{a}\right)\right] \wedge \delta\mathbf{e}^{a}. \end{aligned}$$

Eliminating the basis forms and simplifying gives

$$\begin{aligned} \delta_{\mathbf{e}}S &= \int \left(4\alpha f_{cd}\Omega^{cadb} + 2\beta f^{a}{}_{c}\Omega^{cb} + 4\alpha\Theta^{ab} - 2\beta Q^{ab}\right) A_{ab} d^4x \\ &+ \int \left(4\alpha D_c D_d \Omega^{cadb}\right) A_{ab} d^4x. \end{aligned}$$

The full second-order field equation from the variation of the solder form is

$$4\alpha D_c D_d \Omega^{cadb} + 4\alpha f_{cd}\Omega^{cadb} + 2\beta f^{a}{}_{c}\Omega^{cb} + 4\alpha\Theta^{ab} - 2\beta Q^{ab} = 0. \tag{6.48}$$

Once again, since the dilatational energy-momentum tensor vanishes (6.33) due to zero dilatational curvature (6.4), (6.48) becomes

$$4\alpha D_c D_d \Omega^{cadb} + 4\alpha f_{cd}\Omega^{cadb} + 4\alpha\Theta^{ab} = 0. \tag{6.49}$$

Imposing a vanishing Lorentz or conformal energy-momentum tensor (6.36) results in

$$4\alpha D_c D_d \Omega^{cadb} + 4\alpha f_{cd}\Omega^{cadb} = 0. \tag{6.50}$$

Substitution of the Weyl curvature tensor for the Lorentz curvature (6.18) and the Schouten tensor for the gauge field of the special conformal transformations (6.14) gives

$$4\alpha D_c D_d C^{cadb} - 4\alpha \mathscr{R}_{cd} C^{cadb} = 0. \tag{6.51}$$

Substitution of the Schouten tensor (6.9) in coordinates exactly recovers the Bach equation (4.25). The full field equation would then be written as

$$0 = 2\alpha D_d D_b C^{abcd} - \alpha C^{abcd} R_{bd}. \tag{6.52}$$

However, considering Weyl gravity as a gauge theory, we vary each gauge field independently. In particular, from the field equation of the variation of the spin connection, we have the divergence of conformal curvature is zero (6.20), which implies

$$2\alpha D_d D_b C^{abcd} = 0. \tag{6.53}$$

Imposing that condition on the Bach equation (4.25) gives the more restrictive condition,

$$\alpha C^{abcd} R_{bd} = 0. \tag{6.54}$$

This implies all solutions in this theory solve the Bach equation. However, the converse is not necessarily true; the solutions that satisfy the Bach equation do not imply each of the terms vanishes separately.

6.7. Summary

For the gauge field of the special conformal transformations and the Weyl vector, we found that

$$\mathbf{f}_b = -\mathscr{R}_b, \text{ and} \tag{6.55}$$

$$\omega = \mathbf{d}\phi. \tag{6.56}$$

Precisely because the gauge field of the special conformal transformations is the Schouten tensor, this implies $\Omega^a_{\ bcd} = C^a_{\ bcd}$ and the quadratic Weyl curvature action is equivalent to the quadratic Lorentz curvature action (with an added quadratic dilatational curvature). The field equation for the spin connection was found to be equivalent to the vanishing covariant divergence of the conformal tensor. Since the dilatational curvature was found to be zero, the energy-momentum tensor constructed from it also vanishes. The energy-momentum tensor constructed from the Weyl curvature tensor is also zero because it was found to be equivalent to the second-order variation of the Gauss-Bonnet term. We conclude, while all solutions of the gauge theory satisfy the Bach equation, they must also satisfy the vanishing divergence of the conformal tensor (6.20).

CHAPTER 7

CONSEQUENCES

One of the hallmarks of any conformal theory is it admits classes of solutions related by a conformal transformation. That is, if $g_{\mu\nu}$ is a solution, then so is $e^{2\phi}g_{\mu\nu}$. In this chapter, we explicitly show the necessary condition for this to happen, and show how this condition arises naturally from the field equations (5.68) - (5.71).

7.1. Riemannian gauge

In this section, we show how we recover a Riemannian space by choosing a gauge in which the Weyl vector vanishes.

7.1.1. Structure equations

Under conformal transformations, the structure equations for the conformal group (5.26) - (5.29) remain unmodified; that is, we have invariance under

$$\tilde{\omega}^a_{\ c} = \omega^a_{\ c}, \tag{7.1}$$

$$\tilde{\mathbf{e}}^a = e^{\chi'}\mathbf{e}^a, \tag{7.2}$$

$$\tilde{\mathbf{f}}_c = e^{-\chi'}\mathbf{f}_c, \tag{7.3}$$

$$\tilde{\omega} = \omega + \mathbf{d}\chi', \tag{7.4}$$

where χ' is an arbitrary function of space and time. Since dilatational curvature vanishes as a consequence of the field equations (6.4), and since the Schouten tensor is the gauge field for the special conformal transformations, which is symmetric, the structure equation for the Weyl vector (6.16) becomes $\mathbf{d}\omega = 0$, so the Weyl vector has the form $\omega = \mathbf{d}\phi$ (6.17). This implies there is a gauge $\chi' = -\phi$, where the Weyl vector vanishes, which we call the Riemannian gauge. With a zero Weyl vector, the structure equation for the solder form

(5.23) reduces to

$$\mathbf{de}^a = \mathbf{e}^b \wedge \omega^a_{\ b}, \tag{7.5}$$

which is the torsion-free structure equation for the solder form in the Poincaré group (3.35).

Then $\omega^a_{\ b}$ becomes the usual Poincaré spin connection and the curvature of $\omega^a_{\ b}$ defined in Eq. (6.6) is the Riemann curvature tensor,

$$\mathbf{d}\omega^a_{\ b} = \omega^c_{\ b} \wedge \omega^a_{\ c} + \mathbf{R}^a_{\ b}. \tag{7.6}$$

This equation, along with (7.5), comprise the two structure equations from the Poincaré group in a Riemannian geometry (3.35) and (3.36). The metric in such a theory will be Riemannian.

7.2. Integrability

In this section, we show how the spin connection must transform in a Riemannian geometry, or in the Riemannian gauge, when conformal transformations, χ, of the solder form (or of the metric) are considered. Then, we show how the Riemann curvature tensor changes under conformal transformations. We then find the condition that must be satisfied for the transformed Ricci tensor to vanish, which is a condition on the derivatives of χ, which we write as $\chi_a \equiv e_a^{\ \mu} \partial_\mu \chi$. Since this condition includes the exterior derivative of χ_a, we take another exterior derivative to determine the integrability condition.

7.2.1. Conformal change of basis

A conformal change of basis of the solder form in the Riemannian gauge gives

$$\tilde{\mathbf{e}}^a = e^\chi \mathbf{e}^a, \tag{7.7}$$

where χ is an arbitrary function of space and time. We treat this, for the present, as a change of basis, but not a change of gauge. The structure equation for the solder form in the Riemannian gauge or in a Riemannian geometry (7.5) becomes

$$\mathbf{d}\tilde{\mathbf{e}}^a = \tilde{\mathbf{e}}^c \wedge \tilde{\omega}^a_c. \tag{7.8}$$

To preserve (7.8), we require the spin connection to transform as

$$\tilde{\omega}^a_b = \omega^a_b + 2\Delta^{ac}_{db}\chi_c \mathbf{e}^d. \tag{7.9}$$

The Riemann curvature $\mathbf{R}^a_b = \mathbf{d}\omega^a_b - \omega^c_b \wedge \omega^a_c$ becomes

$$\tilde{\mathbf{R}}^a_b = \mathbf{d}\tilde{\omega}^a_b - \tilde{\omega}^c_b \wedge \tilde{\omega}^a_c, \tag{7.10}$$

and substituting (7.9) we find (7.10) may be written as

$$\tilde{\mathbf{R}}^a_b = \mathbf{R}^a_b + 2\Delta^{ac}_{db}\left[\mathbf{D}\chi_c - \left(\chi_e\chi_c - \frac{1}{2}\eta_{ec}\chi^2\right)\mathbf{e}^e\right] \wedge \mathbf{e}^d. \tag{7.11}$$

Using (6.7) and (6.9), we may write $\mathbf{R}^a{}_b$ and $\tilde{\mathbf{R}}^a{}_b$ in terms of their Weyl and Ricci parts,

$$\mathbf{R}^a_b = 2\Delta^{ac}_{db}\mathbf{e}^d \wedge \mathscr{R}_c + \mathbf{C}^a_b, \text{ and} \tag{7.12}$$

$$\tilde{\mathbf{R}}^a_b = 2\Delta^{ac}_{db}\tilde{\mathbf{e}}^d \wedge \tilde{\mathscr{R}}_c + \mathbf{C}^a_b. \tag{7.13}$$

Substitution of (7.11) into (7.13) and comparing to (7.12) gives the transformation of the Schouten,

$$\tilde{\mathscr{R}}_c = e^{-\chi}\left[\mathscr{R}_c - \mathbf{D}\chi_c + \left(\chi_e\chi_c - \frac{1}{2}\eta_{ec}\chi^2\right)\mathbf{e}^e\right]. \tag{7.14}$$

Since the Schouten tensor can be written in terms of the Ricci tensor, and since this definition may be inverted to write the Ricci tensor solely in terms of $\mathscr{R}_c$, the vanishing of $\tilde{\mathscr{R}}_c$ implies the vanishing of the Ricci tensor and $\tilde{\mathscr{R}}_c$ vanishes if, and only if, there exists χ_e

such that

$$\mathscr{R}_c - \mathbf{D}\chi_c + \left(\chi_e\chi_c - \frac{1}{2}\eta_{ec}\chi^2\right)\mathbf{e}^e = 0. \tag{7.15}$$

7.2.2. Integrability condition

Writing out the covariant exterior derivative in (7.15) gives

$$\mathbf{d}\chi_a = \chi_b\omega^b_{\ a} + \mathscr{R}_a + \left(\chi_b\chi_a - \frac{1}{2}\eta_{ba}\chi^2\right)\mathbf{e}^b. \tag{7.16}$$

Taking an exterior derivative of both sides gives

$$0 = \mathbf{d}\mathscr{R}_a + \mathscr{R}_b \wedge \omega^b_{\ a} + \chi_c\left(\mathbf{R}^c_{\ a} + 2\Delta^{cd}_{ba}\mathscr{R}_d \wedge \mathbf{e}^b\right). \tag{7.17}$$

The first two terms become the covariant exterior derivative of the Schouten tensor. The last term in parentheses is the Weyl curvature as given by (7.12). The integrability condition for χ_a then becomes

$$0 = \mathbf{D}\mathscr{R}_a + \chi_c\mathbf{C}^c_{\ a}. \tag{7.18}$$

Writing this out in components and stripping the basis, we may also write (7.18) as

$$0 = \mathscr{R}_{a[b;c]} + \chi_d C^d_{\ abc}. \tag{7.19}$$

The integrability condition restricts what $\mathscr{R}_a$ and $C^d_{\ abc}$ must be in order for there to exist a solution for χ_a such that $\tilde{\mathscr{R}}_c$ vanishes. In this case, there exists a gauge where (7.15) is satisfied, i.e.,

$$\mathscr{R}_c = \mathbf{D}\chi_c - \left(\chi_e\chi_c - \frac{1}{2}\eta_{ec}\chi^2\right)\mathbf{e}^e, \tag{7.20}$$

which implies

$$\tilde{\mathscr{R}}_c = 0, \tag{7.21}$$

which also implies $R_{ab}\left(\tilde{g}\right)=0$ where

$$\tilde{g}_{\alpha\beta}=e^{2\chi}g_{\alpha\beta}. \tag{7.22}$$

7.3. Field equation implies integrability

In this section, we show how, in a Riemannian geometry, the field equation for the spin connection, or the vanishing of the covariant divergence of the Weyl curvature tensor, leads to the same integrability condition (7.17).

7.3.1. Equivalence to integrability condition

Expanding $\tilde{D}_a\tilde{C}^a_{bcd}$ gives

$$\tilde{D}_a\tilde{C}^a{}_{bdf}=\partial_a\tilde{C}^a{}_{bdf}-\tilde{C}^a{}_{bcf}\tilde{\omega}^c{}_{da}-\tilde{C}^a{}_{bde}\tilde{\omega}^e{}_{fa}+\tilde{C}^c{}_{bdf}\tilde{\omega}^a{}_{ca}-\tilde{C}^a{}_{cdf}\tilde{\omega}^c{}_{ba}, \tag{7.23}$$

where $\tilde{\omega}^a_b$ is the transformed connection given by (7.9) and $\tilde{C}^a{}_{bcd}=e^{-2\chi}C^a_{bcd}$ (B.6). In $n=4$ dimensions, (7.23) simplifies to

$$\tilde{D}_a\tilde{C}^a{}_{bdf}=D_a\tilde{C}^a{}_{bdf}+\chi_c\tilde{C}^c{}_{bdf}. \tag{7.24}$$

In a Riemannian geometry, Bianchi's second identity tells us the covariant exterior derivative of the Riemann curvature two-form is zero (3.44). This identity is valid in the Riemannian gauge. Solving for the Riemann curvature from (6.7) gives

$$\mathbf{R}^a_b=\mathbf{C}^a_b+2\Delta^{ad}_{cb}\mathbf{e}^c\wedge\mathscr{R}_d. \tag{7.25}$$

Substituting this into (3.44) and taking a contraction gives

$$
\begin{aligned}
0 \;=\; & C^{a}_{\;bcd;a}+(n-3)\left(\mathscr{R}_{bd;c}-\mathscr{R}_{bc;d}\right) \\
& +\eta_{db}\left(\mathscr{R}_{;c}-\mathscr{R}^{a}_{\;c;a}\right)+\eta_{cb}\left(\mathscr{R}^{a}_{\;d;a}-\mathscr{R}_{;d}\right). \qquad (7.26)
\end{aligned}
$$

Contracting with η^{cb} gives

$$\mathscr{R}^{c}_{\;d;c}=\mathscr{R}_{;d}, \qquad (7.27)$$

which can then be substituted back into (7.26) to give

$$0=D_{a}C^{a}_{\;bcd}+\mathscr{R}_{b[d;c]} \qquad (7.28)$$

in $n=4$ dimensions. Notice the field equation for the spin connection (6.20) implies (7.28) may be written as

$$0=\mathscr{R}_{b[d;c]}. \qquad (7.29)$$

Under a change of basis, (7.28) becomes

$$0=\tilde{D}_{a}\tilde{C}^{a}_{\;bcd}+\tilde{\mathscr{R}}_{b[d;c]}. \qquad (7.30)$$

Substitution of (7.24) into (7.30) immediately yields

$$0=\tilde{\mathscr{R}}_{b[c;d]}+\chi_{e}\tilde{C}^{e}_{\;\;bcd}, \qquad (7.31)$$

which is precisely the integrability condition for χ_a (7.19) for conformal Ricci flatness of the new metric, $\tilde{\mathbf{e}}^{a}$. This implies $\tilde{\mathbf{e}}^{a}$ is conformal to a Ricci-flat metric or there exists an $\hat{\mathbf{e}}^{a}=e^{\xi}\tilde{\mathbf{e}}^{a}$ such that $R_{ab}\left(\hat{\mathbf{e}}^{a}\right)=0$. However, it follows $\hat{\mathbf{e}}^{a}=e^{(\chi+\xi)}\mathbf{e}^{a}$, so $\mathbf{e}^{a}$ is conformal to

$\hat{\mathbf{e}}^a$. This implies that the integrability condition also holds in the original basis,

$$0 = \mathscr{R}_{b[c;d]} + (\chi + \xi)_e C^e{}_{bcd}. \tag{7.32}$$

However, from (7.29), this reduces to

$$(\chi + \xi)_e C^e{}_{bcd} = 0. \tag{7.33}$$

For a generic $C^e{}_{bcd}$, this implies $(\chi + \xi)_b = 0$ so $\psi = \chi + \xi$ is constant and (7.14) may be written as

$$\boldsymbol{\mathscr{R}}_c(\hat{\mathbf{e}}^a) = e^{-\psi}\left[\boldsymbol{\mathscr{R}}_c(\mathbf{e}^a) - \mathbf{D}\psi_c + \left(\psi_e\psi_c - \frac{1}{2}\eta_{ec}\psi^2\right)\mathbf{e}^e\right]. \tag{7.34}$$

Since ψ_e is zero, $-\mathbf{D}\psi_c + \left(\psi_e\psi_c - \frac{1}{2}\eta_{ec}\psi^2\right)\mathbf{e}^e = 0$, and $\boldsymbol{\mathscr{R}}_c(\hat{\mathbf{e}}^a) = 0$, then (7.34) reduces to

$$0 = \boldsymbol{\mathscr{R}}_c(\hat{\mathbf{e}}^a) = e^{-\psi}\boldsymbol{\mathscr{R}}_c(\mathbf{e}^a), \tag{7.35}$$

which implies $\boldsymbol{\mathscr{R}}_c(\mathbf{e}^a) = 0$. So, in the Riemannian gauge, which is the gauge in which the Weyl vector vanishes, the Schouten tensor must be zero.

Returning to an arbitrary gauge, we recall $\tilde{\mathbf{f}}_a = e^{-\chi'}\mathbf{f}_a$ (7.3) and $\mathbf{f}_a = -\boldsymbol{\mathscr{R}}_a$ (6.14), which means

$$\tilde{\boldsymbol{\mathscr{R}}}_a = e^{-\chi'}\boldsymbol{\mathscr{R}}_a, \tag{7.36}$$

which may also be written as

$$\tilde{\boldsymbol{\mathscr{R}}}_a = e^{-\chi'}\left[\boldsymbol{\mathscr{R}}_c^{(\alpha)}(\mathbf{e}^a) + \mathbf{D}W_c - \left(W_eW_c - \frac{1}{2}\eta_{ec}W^2\right)\mathbf{e}^e\right] \tag{7.37}$$

when decomposing the spin connection into $\omega^a_{\ b} = \alpha^a_{\ b} - 2\Delta^{ac}_{db}W_c\mathbf{e}^d$, where $\alpha^a_{\ b}$ is the metric-compatible connection and W_c is the Weyl vector. Since $\tilde{\boldsymbol{\mathscr{R}}}_a = 0$, then $\tilde{\boldsymbol{\mathscr{R}}}_a = 0$ for any χ', so $\boldsymbol{\mathscr{R}}_a$ is invariant under any conformal transformation. If the Schouten tensor is zero in any particular gauge, then it must be zero in all gauges, and we have demonstrated it vanishes in the Riemannian gauge. If the Schouten tensor vanishes, then so does the Ricci tensor, so as a result, this theory admits conformal classes of Ricci-flat solutions.

7.4. Summary

The solution for the Weyl vector allows us to choose a gauge. From the full conformal space, we were able to choose a gauge in which $\omega = 0$, or the Riemannian gauge, recovering the structure equation for solder form and spin connection for the Poincaré group (5.23) and (3.36). Once in this gauge, we wanted to be able to compute what happened to these same equations in a conformal change of basis. The reason for going from the Riemannian gauge to an arbitrary gauge was to see how objects from this gauge transformed, i.e. the Riemann curvature tensor (7.11). Of particular interest was Schouten tensor, whose vanishing implies the Ricci tensor vanishes. We were able to show how $\mathscr{R}_a$ transformed while leaving the Riemannian gauge and found the condition for it to vanish (7.15). From this, we were able to show the condition on the curvatures for χ_a to exist in order to lead to Ricci-flat solutions, otherwise known as the integrability condition (7.18).

We then approached this problem a little differently. We computed the divergence of the Weyl curvature tensor under a change of basis. Using the Bianchi identity (3.44), as well as the field equation for the spin connection (6.20), we showed how this implied the same integrability condition, which showed the Schouten tensor, as well as the Ricci tensor, must be zero in the Riemannian gauge. From the transformation property of the gauge field of the special conformal transformations, which makes it a tensor, the Ricci tensor must be zero in all gauges, so all conformal transformations of Ricci flat solutions are permissible.

CHAPTER 8

DISCUSSION

From field equation of the spin connection (6.20) and from the integrability condition (7.18), we have shown all metrics that are conformal to Ricci-flat metrics are solutions to this theory. That is, any metric that causes the Ricci tensor to vanish can be multiplied by a conformal factor, and this also satisfies the field equations. That is, all vacuum solutions to General Relativity are solutions to vacuum Weyl gravity as a gauge theory, together with vacuum solutions up to an arbitrary conformal factor. This is a consequence of the inherent local scale freedom, or the freedom to choose units that depend on location in space-time. We now show the field equation for the solder form can be written as the energy-momentum of the Schouten tensor. We also show how the solutions to Weyl vector are related to size change in the following sections.

8.1. Energy-momentum tensor of the Schouten tensor

We recall from the variation of the solder form, we had the energy-momentum tensor of the dilatational curvature (5.73) given by

$$Q_{ab} = \Omega_{ac}\Omega_b^{\ c} - \frac{1}{4}\Omega_{cd}\Omega^{cd}\eta_{ab}, \tag{8.1}$$

and the energy-momentum tensor of the Lorentz curvature or the Weyl curvature tensor (6.34) given by

$$\Theta_{ab} = C^c_{\ dae}C^d_{\ cb}{}^{\ e} - \frac{1}{4}C^c_{\ def}C^{d\ ef}_{\ c}\eta_{ab}.$$

Q_{ab} was zero because the dilatational curvature is zero (6.4). Θ_{ab} is zero because it is equivalent to the second-order variation of the Gauss-Bonnet term (6.32). In each case, the energy-momentum tensor has the same form. Likewise, energy-momentum tensor of the

Schouten tensor can be written compactly as

$$E_{ab} = \mathscr{R}_{bc}\mathscr{R}_a{}^c - \frac{1}{4}\mathscr{R}_{cd}\mathscr{R}^{cd}\eta_{ab}. \tag{8.2}$$

Substituting the definition of the Schouten as given by

$$\mathscr{R}_{ab} = \frac{1}{2}\left(R_{ab} - \frac{1}{6}\eta_{ab}R\right) \tag{8.3}$$

and multiplying by 8 gives

$$8E_{ab} = 2R_{bc}R_a{}^c - \frac{2}{3}RR_{ab} - \frac{1}{2}\eta_{ab}R_{cd}R^{cd} + \frac{1}{6}\eta_{ab}R^2. \tag{8.4}$$

The first-order, or Palatini, variation of the Gauss-Bonnet gives

$$0 = -\frac{1}{4}\left(R^{abef}R_{abef} - 4R^{ab}R_{ab} + R^2\right)g^{cd} + R^{abfd}R_{abf}{}^{c} - 4R^{ad}R^{c}{}_{a} + RR^{dc}. \tag{8.5}$$

The difference between this variation and the second-order variation (6.32) gives the identity

$$R_{ab}R^{adbc} = R^{ad}R^{c}{}_{a}. \tag{8.6}$$

Written out, the field equation for the solder form (6.54) can be expressed as

$$C^{abcd}R_{bd} = R^{abcd}R_{bd} - \frac{2}{3}RR^{ac} + R^a_bR^{bc} - \frac{1}{2}R_{bd}R^{bd}\eta^{ac} + \frac{1}{6}R^2\eta^{ac}. \tag{8.7}$$

Using our identity (8.6), the field equation is identically (8.4) and so (6.54) becomes

$$E_{ab} = 0. \tag{8.8}$$

We believe this is a statement that there is no gravitational self-energy. That is, gravitational fields themselves do not produce gravitational fields through an explicit source term. In terms of just the energy-momentum tensors, the field equation (5.77) may also be written as

$$8\alpha E_{ab} = 2\alpha\Theta_{ab} - \beta\left(-\mathscr{R}_{bc}\Omega^{c}{}_{a} - Q_{ab}\right) \tag{8.9}$$

where each of the components vanishes identically.

One obvious solution to (8.8) is $\mathscr{R}_{ab} = 0$. Using (8.3), this may be written as

$$R_{ab} - \frac{1}{6}\eta_{ab}R = 0, \tag{8.10}$$

which is equivalent to the vacuum Einstein equation (2.58). Once again, the solutions are identical.

8.2. Physical size change

Size change in a conformal theory is brought about by the exponential of the integral of the Weyl vector, $e^{\int\omega}$, where, in general, the integral depends on the path taken in spacetime. Since lengths are determined by comparisons to standards at the same location, we may write

$$L = \frac{l e^{\int_{\mathbf{x}_1 C_1}^{\mathbf{x}_2}\omega}}{l_0 e^{\int_{\mathbf{x}_1 C_2}^{\mathbf{x}_2}\omega}}; \tag{8.11}$$

l can be thought of the length of the object in question, l_0 is the length of the standard, where $\mathbf{x}_1$ is the starting location in spacetime of both and $\mathbf{x}_2$ is the final location of both. $\int_{\mathbf{x}_1 C_1}^{\mathbf{x}_2}\omega$ is then the integral of the object through its own path (path 1), and $\int_{\mathbf{x}_1 C_2}^{\mathbf{x}_2}\omega$ the integral of the standard through its path (path 2), where the two paths need not be the same. If the two integrals return the same values, there will be no size change measured when at point $\mathbf{x}_2$ as compared to point $\mathbf{x}_1$. However, if $\int_{\mathbf{x}_1 C_1}^{\mathbf{x}_2}\omega > \int_{\mathbf{x}_1 C_2}^{\mathbf{x}_2}\omega$, there will be expansion and if $\int_{\mathbf{x}_1 C_1}^{\mathbf{x}_2}\omega < \int_{\mathbf{x}_1 C_2}^{\mathbf{x}_2}\omega$, there will be an observable shrinkage. The solution to the Weyl vector indicates it must be the gradient of a function, ϕ (6.17). This implies the value of each integral depends on the arbitrary function, ϕ, only at positions $\mathbf{x}_1$ and $\mathbf{x}_2$, regardless of path taken. Since the starting point and ending point in each integral is the same, both integrals will take the same value; it will always be the case that $L = l/l_0$, so there will be no size change in this theory, even if units depend on space and time.

8.3. Summary

In this work we presented the standard formulation of general relativity. Using the transformation of the covariant derivative and the Christoffel connection, we were able to construct a scalar action composed the Ricci scalar that was invariant under general coordinate transformations. From this action, the independent fields, the metric and the connection, were varied to find the field equations, and the result was the Einstein equation and the condition for metric compatibility. From here, we presented general relativity as a gauge theory of the Poincaré group. From the group elements, the infinitesimal transformations were found, and from this, the Lie algebra. From the structure constants of the Lie algebra, the structure equations were formed for the corresponding gauge fields and then the Bianchi identities, or the integrability conditions, by administering another exterior derivative to the structure equations. From the transformation properties of the curvature and the connection for the entire group, the transformation of individual curvatures and connections were found, and a scalar linear action, the Einstein-Hilbert action was constructed. The two independent fields, the solder form and the spin connection, were varied to yield the Einstein equation and vanishing torsion, respectively.

As with general relativity, with Weyl gravity as a gauge theory, we followed the same basic procedure. A quadratic scale-invariant action formed from the Weyl curvature tensor was constructed and the metric was varied to yield the Bach equation (4.25). We then considered Weyl gravity as a gauge theory of the full conformal group and formed the most general even-parity, quadratic action from the curvatures that was invariant under local Lorentz transformations, as in the case of general relativity as a gauge theory, but also under local scale transformations, or dilatations. We then proceeded to vary this action with respect to its independent fields. Since the conformal group is a larger group than the Poincaré group, there were more fields to vary, and hence, more field equations. Solving the field equations, together with the structure equations, yielded zero dilatational curvature (6.4), so the Weyl vector could be written as the gradient of an arbitrary function (6.17). It was also found the Lorentz curvature was equivalent to the Weyl curvature tensor (6.18). With these two conditions, the action could equivalently be written as the quadratic conformal tensor action in the standard presentation of the theory. It was also found the gauge field of the special conformal transformations was the negative of the Schouten tensor (6.14). Varying the solder form gave rise to two energy-momentum tensors, one constructed from the dilatational curvature (5.73) and another constructed from the Lorentz curvature (5.72). The dilatational energy-momentum tensor vanishes by virtue of zero dilatational curvature and the Lorentz energy-momentum tensor vanished for being equivalent to the second-order variation of the Euler character (6.32). In the case that the fields are not varied independently, but including a variation of the spin connection in terms of the solder form, we recover the Bach equation. However, a Palatini variation of the spin connection results in the covariant divergence of the Weyl curvature tensor being zero giving the restricted Bach equation (6.54).

Given the gauge freedom inherent in this theory, we were free to choose a gauge in which the Weyl vector was zero. In this gauge, which we called the Riemannian gauge, we recovered the torsion-free structure equation for the solder form (7.5). We were also able to write the structure equation for the spin connection as if from the Poincaré group (7.6). To see how the equations transformed in this gauge, we allowed a change of basis and we were able to see how the Schouten tensor transformed and from this, find a necessary and sufficient condition for it to vanish (7.15). From this condition, we took another exterior derivative to find the integrability condition (7.18) for the existence of χ that caused this tensor to vanish, which is equivalent to the Ricci tensor being zero. We found the same integrability condition from the field equation for the spin connection under conformal transformations (7.31). This showed Weyl gravity as a gauge theory not only admits solutions that are Ricci flat, but equivalence classes of solutions that are conformal to metrics that are Ricci-flat.

Lastly, we showed the restricted Bach equation can be written as the energy-momentum tensor of the Schouten tensor (8.4). Since the field equation sets this to zero, this seems to imply there exists no gravitational self energy. The final result was to show although the conformal factor is arbitrary, the Weyl vector is its gradient, so integration, along a closed loop, equals zero, and so there will be no size change, a phenomenon consistent with observations.

8.4. Future work

In this work, we have considered Weyl gravity as a purely classical theory. Although there are claims this theory is renormalizable and free of ghosts [26, 32], it would nonetheless be noteworthy to check, especially considering that Weyl gravity as a gauge theory of the conformal group is no longer a fourth-order theory. Another aspect of this work is we were considering Weyl gravity in the auxiliary gauging, that is, with dilatational, Lorentz, and special conformal symmetry. Since the Lorentz curvature and the dilatational curvature are also tensors when we gauge by just the Lorentz transformations and dilatations, we can consider the same theory in the biconformal gauging.[1] This would provide a base space that can be identified as a phase space, rather than the conventional four-dimensional configuration space. Lastly, in this work, we only considered the vacuum solutions. It would be interesting to also consider the same theory with matter sources.

[1] It would not be the most general theory quadratic in the curvatures since there are other tensors not present in the auxiliary gauging, notably, the torsion and special conformal curvature.

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APPENDICES

APPENDIX A

SO(4,2) CONFORMAL GROUP REPRESENTATION

The conformal group is the group that preserves light cones. We know that a light cone is described by a four-vector with zero proper length (neither space-like nor time-like). Hence, it is given by

$$x^{\alpha}x_{\alpha} = \eta_{\alpha\beta}x^{\alpha}x^{\beta} = 0, \tag{A.1}$$

where α takes values from 0 to 3. We can multiply this by a constant and the equation is still satisfied,

$$\lambda x^{\alpha}x_{\alpha} = 0. \tag{A.2}$$

Translating the light cone off the origin gives

$$\lambda\left(x^{\alpha} - a^{\alpha}\right)\left(x_{\alpha} - a_{\alpha}\right) = 0. \tag{A.3}$$

Expanding (A.3) gives

$$\lambda x^{2} - 2\lambda a_{\alpha}x^{\alpha} + \lambda a^{2} = 0. \tag{A.4}$$

From this, we make the identification

$$B_{\alpha} \equiv \lambda a_{\alpha}, \tag{A.5}$$

$$B^{4} \equiv \lambda, \tag{A.6}$$

$$B^{5} \equiv \frac{1}{2}\lambda a^{2}, \tag{A.7}$$

and (A.4) becomes

$$B^{4}x^{2} - 2B_{\alpha}x^{\alpha} + 2B^{5} = 0. \tag{A.8}$$

Squaring B_{α} gives

$$B^{\alpha}B_{\alpha} = 2B^{4}B^{5} \tag{A.9}$$

so

$$B^{\alpha}B_{\alpha} - 2B^{4}B^{5} = 0. \tag{A.10}$$

Any map from light cones to light cones is equivalent to mapping

$$(B_{\alpha}, B_{4}, B_{5}) \to \left(\tilde{B}_{\alpha}, \tilde{B}_{4}, \tilde{B}_{5}\right) \tag{A.11}$$

or

$$\lambda\,(x^{\alpha} - a^{\alpha})\,(x_{\alpha} - a_{\alpha}) \to \tilde{\lambda}\,(x^{\alpha} - \tilde{a}^{\alpha})\,(x_{\alpha} - \tilde{a}_{\alpha})\,, \tag{A.12}$$

provided

$$\tilde{B}^{\alpha}\tilde{B}_{\alpha} - 2\tilde{B}^{4}\tilde{B}^{5} = 0. \tag{A.13}$$

We write this constraint using

$$\eta_{AB} = \begin{bmatrix} \eta_{ab} & & \\ & 0 & -1 \\ & -1 & 0 \end{bmatrix}. \tag{A.14}$$

Since $\eta_{45} = \eta_{45} = -1$,

$$\eta_{AB}B^{A}B^{B} = B^{a}B_{a} - 2B^{4}B^{5}. \tag{A.15}$$

Because of (A.10), (A.15) becomes

$$\eta_{AB}B^{A}B^{B} = 0. \tag{A.16}$$

The conformal group is, therefore, the group of transformations preserving the six-dimensional light cone at the origin. When η_{AB} is diagonalized, then

$$\eta_{AB} = \begin{bmatrix} -1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & -1 \end{bmatrix} \tag{A.17}$$

so its signature is $(4,2)$ and its determinant is unity. The group is, therefore, SO(4,2).

We recall with the Lorentz transformations, we were trying to keep the invariant magnitude the same (3.2). In the case of these transformations from light cone to light cone, we're trying to keep the same invariant magnitude of 0 the same so

$$\eta_{AB}\tilde{x}^A\tilde{x}^B = \eta_{AB}x^A x^B, \tag{A.18}$$

or

$$\eta_{AB}\left(\Lambda^A_C x^C\right)\left(\Lambda^B_D x^D\right) = \eta_{CD}x^C x^D, \tag{A.19}$$

where, in this case, Λ^A_B must be a general element of the conformal group. Just as in the Lorentz transformations alone (3.5), we find that letting

$$\Lambda^A_B = \delta^A_B + \varepsilon^A_B \tag{A.20}$$

implies

$$\varepsilon_{CD} = -\varepsilon_{DC}, \tag{A.21}$$

so in this representation, the generators of the conformal group are antisymmetric. The generators are then defined in the following manner:

$$[M_{ab}]_{AB} = \begin{bmatrix} \varepsilon_{ab} & 0 \\ 0 & 0 \end{bmatrix}, \tag{A.22}$$

$$[P_a]_{AB} = \begin{bmatrix} 0 & & & & -1 & \\ & 0 & & & 1 & \\ & & 0 & & 1 & \\ & & & 0 & 1 & \\ 1 & -1 & -1 & -1 & 0 & 0 \\ & & & & 0 & 0 \end{bmatrix}, \tag{A.23}$$

$$[K_a]_{AB} = \begin{bmatrix} 0 & & & & & -1 \\ & 0 & & & & 1 \\ & & 0 & & & 1 \\ & & & 0 & & 1 \\ & & & & 0 & 0 \\ 1 & -1 & -1 & -1 & 0 & 0 \end{bmatrix}, \tag{A.24}$$

$$[D]_{AB} = \begin{bmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 0 & -1 \\ & & & & 1 & 0 \end{bmatrix}, \tag{A.25}$$

where, as mentioned before, M_{ab} is the generator of the Lorentz transformations, P_a is the generator of the translations, K_a is the generator of the special conformal transformations, and D is the generator of the dilatations. ε_{ab} are the same generators of the Lorentz group as defined in (3.4). Since the group indices on each generator are naturally mixed indices, we may use η^{AC} to raise the first index of each generator:

$$[P_a]^A_{\ B} = \begin{bmatrix} 0 & & & & 1 & \\ & 0 & & & 1 & \\ & & 0 & & 1 & \\ & & & 0 & 1 & \\ & & & & 0 & 0 \\ -1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}, \tag{A.26}$$

$$[K_a]^A_{\ B} = \begin{bmatrix} 0 & & & & & 1 \\ & 0 & & & & 1 \\ & & 0 & & & 1 \\ & & & 0 & & 1 \\ -1 & 1 & 1 & 1 & 0 & 0 \\ & & & & 0 & 0 \end{bmatrix}, \tag{A.27}$$

$$[D]^A_{\ B} = \begin{bmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & -1 & 0 \\ & & & & 0 & 1 \end{bmatrix}. \tag{A.28}$$

Exponentiating the generators to get to the group elements gives

$$e^{(b^\alpha P_\alpha)} = \begin{bmatrix} 1 & & & & b^0 & \\ & 1 & & & b^1 & \\ & & 1 & & b^2 & \\ & & & 1 & b^3 & \\ & & & & 1 & 0 \\ -b^0 & b^1 & b^2 & b^3 & \frac{1}{2}b_\alpha b^\alpha & 1 \end{bmatrix}, \tag{A.29}$$

$$e^{(c^\alpha K_\alpha)} = \begin{bmatrix} 1 & & & & & c^0 \\ & 1 & & & & c^1 \\ & & 1 & & & c^2 \\ & & & 1 & & c^3 \\ -c^0 & c^1 & c^2 & c^3 & 1 & \frac{1}{2}c_\alpha c^\alpha \\ & & & & 0 & 1 \end{bmatrix}, \tag{A.30}$$

and

$$e^{[\lambda D]} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & e^{-\lambda} & 0 \\ & & & & 0 & e^{\lambda} \end{bmatrix}. \tag{A.31}$$

To show what these group elements do, we have them each act on a vector given by $V = \begin{bmatrix} B^0 & B^1 & B^2 & B^3 & B^4 & B^5 \end{bmatrix}^T$ or otherwise, $\begin{bmatrix} B^\alpha & B^4 & B^5 \end{bmatrix}^T$ and return a vector given by $\begin{bmatrix} \tilde{B}^\alpha & \tilde{B}^4 & \tilde{B}^5 \end{bmatrix}^T$.

Translations

A translation acting on V returns

$$\begin{bmatrix} \tilde{B}^{\alpha} \\ \tilde{B}^{4} \\ \tilde{B}^{5} \end{bmatrix} = \begin{bmatrix} B^{\alpha} + b^{\alpha} B^{4} \\ B^{4} \\ b_{\alpha} B^{\alpha} + \frac{1}{2} b_{\alpha} b^{\alpha} B^{4} + B^{5} \end{bmatrix}. \tag{A.32}$$

Solving for a^{α} in (A.5) (with raised indices) gives $a^{\alpha} = \frac{B^{\alpha}}{\lambda}$, which is equivalent to

$$a^{\alpha} = \frac{B^{\alpha}}{B^{4}} \tag{A.33}$$

when using (A.6). Under a translation, (A.33) becomes

$$\tilde{a}^{\alpha} = \frac{\tilde{B}^{\alpha}}{\tilde{B}^{4}}. \tag{A.34}$$

Substituting $\tilde{B}^{\alpha}$ and $\tilde{B}^{4}$ from (A.32) into (A.33) returns

$$\tilde{a}^{\alpha} = a^{\alpha} + b^{\alpha}. \tag{A.35}$$

Letting $a^{\alpha} \to x^{\alpha}$ gives

$$\tilde{x}^{\alpha} = x^{\alpha} + b^{\alpha}, \tag{A.36}$$

which is clearly a translation. An inverted coordinate is given by dividing a coordinate by its squared magnitude. Taking (A.33), dividing both sides by a^{2} to get an inverted coordinate and multiplying the numerator and denominator in the fraction to the right by $\frac{1}{2}\lambda$ gives

$$\frac{a^{\alpha}}{a^{2}} = \frac{B^{\alpha}}{2B^{5}}. \tag{A.37}$$

After a translation, this becomes

$$\frac{\tilde{a}^\alpha}{\tilde{a}^2} = \frac{\tilde{B}^\alpha}{2\tilde{B}^5}. \tag{A.38}$$

Substitution of $\tilde{B}^\alpha$ and $\tilde{B}^5$ from (A.32) yields

$$\frac{\tilde{a}^\alpha}{\tilde{a}^2} = \frac{\frac{B^\alpha}{2B^5} + \frac{b^\alpha B^4}{2B^5}}{\frac{2b_\alpha B^\alpha}{2B^5} + \frac{b_\alpha b^\alpha B^4}{2B^5} + 1}. \tag{A.39}$$

Designating $y^\alpha \equiv \frac{B^\alpha}{2B^5}$ and recognizing $\frac{B^4}{2B^5} = \frac{1}{a^2}$, which is the same as y^2 (if $a^\alpha = x^\alpha$), then (A.39) becomes

$$\tilde{y}^\alpha = \frac{y^\alpha + b^\alpha y^2}{2b_\alpha y^\alpha + b^2 y^2 + 1}. \tag{A.40}$$

Comparing this transformation to (5.1), we recognize (A.40) as a special conformal transformation. That is, translations translate ordinary coordinates but transform inverse coordinates by special conformal transformations.

Special conformal transformations

The special conformal group element (A.30) on the same vector, V, returns

$$\begin{bmatrix} \tilde{B}^\alpha \\ \tilde{B}^4 \\ \tilde{B}^5 \end{bmatrix} = \begin{bmatrix} B^\alpha + c^\alpha B^5 \\ c_\alpha B^\alpha + B^4 + \frac{1}{2} c_\alpha c^\alpha B^5 \\ B^5 \end{bmatrix}. \tag{A.41}$$

As in the case with the translations, substituting $\tilde{B}^\alpha$ and $\tilde{B}^4$ from (A.41) gives

$$\tilde{x}^\alpha = \frac{\frac{B^\alpha}{B^4} + c^\alpha \frac{B^5}{B^4}}{c_\alpha \frac{B^\alpha}{B^4} + \frac{B^4}{B^4} + c_\alpha c^\alpha \frac{1}{2} \frac{B^5}{B^4}}. \tag{A.42}$$

From (A.33), we recognize

$$x^a = \frac{B^a}{B^4}. \tag{A.43}$$

Squaring x^a in (A.43) and utilizing (A.9), we get

$$x^\alpha x_\alpha = \frac{2B^5}{B^4}. \tag{A.44}$$

Substituting (A.43) and (A.44) into (A.42) and letting $c^\alpha \to 2c^\alpha$ yields

$$\tilde{x}^\alpha = \frac{x^\alpha + c^\alpha x^2}{2c_\alpha x^\alpha + c^2 x^2 + 1}. \tag{A.45}$$

To see what the inverse coordinates are doing under special conformal transformations, we write (A.38) as

$$\tilde{y}^\alpha = \frac{\tilde{B}^\alpha}{2\tilde{B}^5}. \tag{A.46}$$

Substituting $\tilde{B}^\alpha$ and $\tilde{B}^5$ from (A.41) gives

$$\tilde{y}^\alpha = y^\alpha + c^\alpha \tag{A.47}$$

after the same relabeling of the arbitrary constant as in (A.45). Special conformal transformations translate inverse coordinates.

Dilatations

The special conformal group element (A.30) on the same vector, V, returns

$$\begin{bmatrix} \tilde{B}^\alpha \\ \tilde{B}^4 \\ \tilde{B}^5 \end{bmatrix} = \begin{bmatrix} B^\alpha \\ e^{-\lambda} B^4 \\ e^{\lambda} B^5 \end{bmatrix}. \tag{A.48}$$

Following the same procedure as with the translations and special conformal transformations gives

$$\tilde{x}^\alpha = e^\lambda x^\alpha \tag{A.49}$$

and

$$\tilde{y}^{\alpha} = e^{-\lambda} y^{\alpha}, \tag{A.50}$$

so coordinates and inverse coordinates transform oppositely.

Forms and curvatures

From the antisymmetry of the generators ε_{AB} (A.21), we have

$$\varepsilon^A_B = -\eta^{AC}\eta_{BD}\varepsilon^D_C, \tag{A.51}$$

where η^{AB} is the metric for the full representation space (A.14). The same relation also holds for forms, that is,

$$\omega^A_B = -\eta^{AC}\eta_{BD}\omega^D_C. \tag{A.52}$$

Restricting the group indices to be Lorentz (0 to 3) or 4 and 5, we have

$$\omega^A_B \in \left\{\omega^a_b, \omega^4_b, \omega^a_4, \omega^4_4, \omega^4_5, \omega^5_b, \omega^b_5, \omega^5_5, \omega^5_4\right\}. \tag{A.53}$$

Likewise, we have

$$\eta_{AB} \in \{\eta_{ab}, \eta_{45}, \eta_{54}\} \tag{A.54}$$

with $\eta_{45} = \eta_{54} = 1$ and all other elements zero. Since η_{AB} is its own inverse, we also have

$$\eta^{AB} \in \left\{\eta^{ab}, \eta^{45}, \eta^{54}\right\} \tag{A.55}$$

with $\eta^{45} = \eta^{54} = 1$. Expanding the relation for forms (A.52) using the allowed forms (A.53) and the nonzero metric elements (A.54)-(A.55), we find

$$\omega^a_4 = \eta^{ac}\omega^5_c, \tag{A.56}$$

$$\omega^4_b = \eta_{bd}\omega^d_5, \tag{A.57}$$

$$\omega^4_4 = -\omega^5_5, \tag{A.58}$$

$$\omega^4_5 = 0, \tag{A.59}$$

$$\omega^a_5 = \eta^{ac}\omega^4_c, \tag{A.60}$$

$$\omega^5_b = \eta_{bd}\omega^d_4, \tag{A.61}$$

$$\omega^5_5 = -\omega^4_4, \tag{A.62}$$

$$\omega^5_4 = 0. \tag{A.63}$$

The only independent forms are given by

$$\omega^A_B \in \left\{\omega^a_b, \omega^4_b, \omega^a_4, \omega^4_4\right\}, \tag{A.64}$$

and these same forms may also be expressed as

$$\omega^A_B = \{\omega^a_b, \mathbf{f}_b, \mathbf{e}^a, \omega\}, \tag{A.65}$$

which is the spin connection, gauge field of special conformal transformations, solder form, and Weyl vector.

Since the curvatures in the conformal group have the same sets of relations as the forms, we also have

$$\mathbf{R}^a_4 = \eta^{ac}\mathbf{R}^5_c, \tag{A.66}$$

$$\mathbf{R}^4_b = \eta_{bd}\mathbf{R}^d_5, \tag{A.67}$$

$$\mathbf{R}^4_4 = -\mathbf{R}^5_5, \tag{A.68}$$

$$\mathbf{R}^4_5 = 0, \tag{A.69}$$

$$\mathbf{R}^a_5 = \eta^{ac}\mathbf{R}^4_c, \tag{A.70}$$

$$\mathbf{R}^5_b = \eta_{bc}\mathbf{R}^c_4, \tag{A.71}$$

$$\mathbf{R}^5_5 = -\mathbf{R}^4_4, \tag{A.72}$$

$$\mathbf{R}^5_4 = 0. \tag{A.73}$$

The independent vectors are then given by

$$\mathbf{R}^A_B \in \left\{\mathbf{R}^a_b, \mathbf{R}^4_b, \mathbf{R}^a_4, \mathbf{R}^4_4\right\}, \tag{A.74}$$

and these same curvatures may be expressed as

$$\mathbf{R}^A_B \in \left\{\mathbf{R}^a_b, \mathbf{S}_b, \mathbf{T}^a, \Omega\right\}, \tag{A.75}$$

which is the Riemann curvature, special conformal curvature, torsion, and the dilatational curvature.

APPENDIX B

INVARIANCE OF WEYL CURVATURE TENSOR

Under conformal transformations, the decomposition of the Riemann curvature tensor into the Weyl curvature tensor and the Schouten tensor (7.12) becomes

$$\tilde{\mathbf{R}}^a_b = \tilde{\mathbf{C}}^a_b - 2\Delta^{ac}_{db}\tilde{\mathscr{R}}_c \wedge \tilde{\mathbf{e}}^d. \tag{B.1}$$

Substituting the transformation of the Riemann curvature (7.11) and explicitly antisymmetrizing to write in coordinates gives

$$\begin{aligned} 2\tilde{C}^a_{bcd}e^{2\phi} - 2\Delta^{ae}_{db}\tilde{\mathscr{R}}_{ec}e^{2\phi} + 2\Delta^{ae}_{cb}\tilde{\mathscr{R}}_{ed}e^{2\phi} &= 2C^a_{bcd} - 2\Delta^{ae}_{db}\mathscr{R}_{ec} + 2\Delta^{ae}_{cb}\mathscr{R}_{ed} \\ &\quad +2\Delta^{ae}_{db}\left[D_c\phi_e - \left(\phi_c\phi_e - \frac{1}{2}\eta_{ce}\phi^2\right)\right] \\ &\quad -2\Delta^{ae}_{cb}\left[D_d\phi_e - \left(\phi_d\phi_e - \frac{1}{2}\eta_{de}\phi^2\right)\right] \end{aligned} \tag{B.2}$$

Taking a trace on the a and c indices gives

$$\begin{aligned} \eta_{db}\tilde{\mathscr{R}}e^{2\phi} + (n-2)\tilde{\mathscr{R}}_{bd}e^{2\phi} &= \eta_{db}\mathscr{R} + (n-2)\mathscr{R}_{bd} \\ &\quad -(n-2)\left[D_d\phi_b - \left(\phi_d\phi_b - \frac{1}{2}\eta_{db}\phi^2\right)\right] \\ &\quad -\eta_{db}\left[\eta^{ae}D_a\phi_e - \left(\eta^{ae}\phi_a\phi_e - \frac{1}{2}n\phi^2\right)\right]. \end{aligned} \tag{B.3}$$

Contracting on the b and d indices gives the condition for the transformation of the Schouten scalar:

$$\tilde{\mathscr{R}}e^{2\phi} = \mathscr{R} - D^b\phi_b + \left(\frac{2-n}{2}\right)\phi^2. \tag{B.4}$$

Substitution of (B.4) into (B.3) gives the transformation of the Schouten tensor in coordinates:

$$\tilde{\mathscr{R}}_{bd}e^{2\phi} = \mathscr{R}_{bd} - \left[D_d\phi_b - \left(\phi_d\phi_b - \frac{1}{2}\eta_{db}\phi^2\right)\right]. \tag{B.5}$$

Notice this is essentially the same equation as (7.14). Substitution of (B.5) into (B.2) gives

$$\tilde{C}^{a}_{bcd} = e^{-2\phi} C^{a}_{bcd}, \tag{B.6}$$

so, up to a conformal factor, the Weyl curvature tensor is indeed invariant under conformal transformations.

CURRICULUM VITAE

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