PHYSICAL REVIEW D VOLUME 43, NUMBER 8 15 APRIL 1991

Spacetime dimension from a variational principle

David Hochberg
Bartol Research Institute, University of Delaware, Newark, Delaware 19716

James T. Wheeler
Department of Physics and Astronomy, Swarthmore College, Swarthmore, Pennsylvania 19081
(Received 14 September 1990)

We consider spacetime as having an *a priori* arbitrary, possibly fractional dimension $p>0$ and propose a new variational principle for actions defined on p-dimensional spaces. Demanding that the action be stationary with respect to variations in p leads to a constraint equation whose solution yields an explicit determination of the dimension at the classical level. We illustrate these concepts by analyzing a model which reduces to free scalar field theory when p is any positive integer.

One of the most fundamental concepts in physics is the very notion of the dimension of spacetime. We are accustomed to taking for granted that we inhabit a four-dimensional world—one time and three spatial dimensions—insofar as this is what we perceive from our limited observations. When we do envision the possibility that the world is other than four dimensional, we invariably have in mind some specific *integer* value for its dimensionality. For example, the general Kaluza-Klein scheme for unifying the elementary interactions via the introduction of extra spatial dimensions tacitly assumes the precompactified space to have an integer dimensionality. Determining *what* value this integer should have is outside the scope of these types of unification programs, although within the context of Kaluza-Klein supergravity, the allowed (integer) range for p is necessarily bounded above: $p \le 11$.[1] A notable exception is provided by superstring theory, where the internal consistency of the theory demands that $p=10$.[2] Yet even here one is faced with the nontrivial task of dynamically justifying the compactification of the extra six spatial dimensions which must take place *a posteriori* in order that we be left over with an effective four-dimensional spacetime. Given the present status of string theory, however, we have as yet no proof that the preferred vacuum locally takes the form $\mathcal{M}\otimes\mathcal{K}$, where $\mathcal{M}$ corresponds to our (apparent) four-dimensional Minkowski spacetime and $\mathcal{K}$ is some suitable compact six-dimensional manifold. Clearly, if strings are to provide a theory of everything, we must expect the dimensionality of our low-energy world $\mathcal{M}$ to be a *derivable* consequence of string dynamics, and not merely a rigid number which is put in by hand as another working assumption.

In this paper we shall regard the dimensionality of spacetime as a dynamical property of the fundamental interactions, to be determined from a well-defined variational principle. In other words, we assert that the topology we ascribe to spacetime, the arena within which all acts of observation and measurement take place, is itself determined by the fundamental interactions. Any act of measurement presupposes the existence of such interactions. Some vivid examples of the intimate linkage between interaction and topology are provided by string dynamics and Brownian motion. In the former case, the world-sheet topology is governed by the string-string interactions: in type-I closed-string theory, the one-loop "diagram" is topologically equivalent to a Klein bottle, whereas in type-II theories, it is a torus. In the latter case, the (Hausdorff) dimension of the trajectory of the Brownian particle is $p=2$. What leads to this particular value are the detailed collisions between the particle and the surrounding molecules. One can imagine that fluctuations in these interactions can lead to fluctuations in the dimension. The dimension is itself a fundamental topological property of a space. Here we mean topology in the most general (weakest) sense: a topology of a space Ω is the collection of all open sets in Ω, satisfying the usual axioms.[3] When Ω is endowed with a metric, or distance, function, we can define the corresponding Hausdorff dimension, for which the nature of these open sets plays a crucial role. With the concept of Hausdorff dimension, we can speak meaningfully of two spaces having nearby dimensions, and thereby formulate a variational principle involving the response of an action to continuous changes in dimension. Demanding that this variation vanish yields a constraint equation whose solution in certain cases provides specific values for p. These values are then interpreted as the preferred dimensions of space that the given interactions (as described by the action) determine.

In general, we may expect noninteger, or fractional, values of dimensionality to result from this approach. This poses no conceptual problem; indeed, calculations based on quite a distinct line of reasoning suggest that the dimension of our world may be somewhat *less* than four.[4]

Much of the mathematical machinery needed to deal with fractional dimension spaces was developed long ago by Hausdorff.[5] Let (Ω,d) denote a metric space (Ω is a set of points and d is a metric, or distance, function on Ω). Take O to be the family of all open sets in Ω and put

$$O_\epsilon=\{U\in O\,|\,\delta(U)\le\epsilon\}$$

where

$$\delta(U)=\sup\{d(x,y)|x,y\in U\}$$

is the diameter of U. For $E\subset\Omega$ and U_n a countable open cover of E, define

$$\mu^p(E,\epsilon)=\inf\left\{\sum_{n=1}^{\infty}\delta(U_n)^p|U_n\in O_\epsilon,E\subset\bigcup_{n=1}^{\infty}U_n\right\}. \tag{1}$$

The infimum is over all countable open covers of E. Then the Hausdorff p-dimensional measure on E is

$$\mu^p(E)=\lim_{\epsilon\to 0^+}\mu^p(E,\epsilon), \tag{2}$$

and the Hausdorff dimension is given by

$$\dim(E)=\sup\{p\geq 0|\mu^p(E)=+\infty\}. \tag{3}$$

It is worthwhile to illustrate these definitions with a concrete example. Consider the Cantor set C obtained from the unit interval by successive deletions of middle thirds. So, $\Omega=[0,1]$, $d(x,y)=|x-y|$, and put $F_1=\Omega-(\frac{1}{3},\frac{2}{3})$ so that F_1 is $[0,1]$ with the middle third removed. Similarly, delete the open intervals $(\frac{1}{9},\frac{2}{9})$ and $(\frac{7}{9},\frac{8}{9})$ from F_1 and call this F_2. Continuing in this way, we delete 2^{n-1} open intervals of diameter 3^{-n} at the nth stage, and $\Omega\supset F_1\supset\cdots\supset F_n$. Then $C=\bigcap_{n=1}^{\infty}F_n$ is the Cantor set.[6] Let $\epsilon>3^{-n}$, then $\{F_{nj}|1\leq j\leq 2^n\}$ is an ϵ covering of C, where F_{nj} denotes an open set of diameter ϵ barely containing the jth segment of diameter 3^{-n} remaining at the nth stage of deletion. Since there are 2^n such segments, we have

$$\mu^p(C,\epsilon)=\inf\left\{\sum_{j=1}^{2^n}[\delta(F_{nj})]^p\right\}=2^n3^{-pn}.$$

Thus,

$$\mu^p(C)=\lim_{\epsilon\to 0^+}\mu^p(C,\epsilon)=\lim_{n\to\infty}(2/3^p)^n=\begin{cases}0, & 2<3^p,\\ 1, & 2=3^p,\\ +\infty, & 2>3^p.\end{cases} \tag{4}$$

The measure is nonzero and finite only when

$$p=p_c\equiv\frac{\ln(2)}{\ln(3)}=0.63093\ldots,$$

and this is the dimension of the Cantor set as well: indeed, for $0<p<p_c$ (and $\epsilon<1$),

$$\mu^p(C,\epsilon)=\inf\left\{\sum_{n=1}^{\infty}\delta(U_n)^p|\delta(U_n)\leq\epsilon,C\subset\bigcup_{n=1}^{\infty}U_n\right\}$$
$$\geq\epsilon^{p-p_c}\inf\left\{\sum_{n=1}^{\infty}\delta(U_n)^{p_c}|\delta(U_n)\leq\epsilon,C\subset\bigcup_{n=1}^{\infty}U_n\right\}.$$

So, $\mu^p(C,\epsilon)\geq\epsilon^{p-p_c}\mu^{p_c}(C)\to+\infty$ as $\epsilon\to 0^+$. Thus, the dimension of the Cantor set

$$\dim(E)=\sup\{p>0|\mu^p(C)=+\infty\}\equiv p_c.$$

Integration of functions f over p-dimensional spaces is defined as

$$\int_E f\,d\mu^p=\lim_{\epsilon\to 0^+}\inf\left\{\sum_{n=1}^{\infty}f^*(x_n)\delta(U_n)^p|U_n\in O_\epsilon,E\subset\bigcup_{n=1}^{\infty}U_n\right\},$$

where

$$f^*(x_n)=\sup\{f(x)|x\in U_n\}. \tag{5}$$

Note that $\int_E d\mu^p=\mu^p(E)$. For the applications we are interested in, we shall be integrating Lagrangian densities $\mathcal{L}$ over these spaces thereby defining the corresponding actions. The action S, in addition to being a functional of the fields, is also now a function of p. This latter dependence comes from the explicit dependence in the measure [see (1) and (2)] as well as from the p dependence of the Lagrangian density $\mathcal{L}$:

$$S[\Phi](p)=\int_E\mathcal{L}(\Phi,\partial\Phi;p)d\mu^p. \tag{6}$$

The fields Φ are just functions over E ($\Phi:E\to\mathbb{R}$), and ∂ stands for the p-dimensional generalization of derivation, which is given by the usual limiting procedure.

To proceed with the formulation of our variational principal for S, let us first consider the response of the measure to changes in p: from (1) and (2) we see that

$$\frac{\partial\mu^p(E)}{\partial p}=\lim_{\epsilon\to 0^+}\inf\left\{\sum_{n=1}^{\infty}\ln[\delta(U_n)]\delta(U_n)^p|U_n\in O_\epsilon,E\subset\bigcup_{n=1}^{\infty}U_n\right\},$$

and this is divergent. This should not be surprising in view of how the Hausdorff dimension is defined, Eq. (3). The point is that, for a given $E\subset\Omega$, there is only one value of p for which $\mu^p(E)$ is finite and nonzero. Thus, when we vary (6) with respect to p, the change in the measure must be accompanied by a corresponding change of the integration domain E, in such a way that the resulting new measure (over the new domain) be finite (and nonzero). But this is what we should expect from a dimensional variation. Thus, to effect this variation, we must first consider the set of topological spaces over which the variation takes place. We may imagine organizing the totality of fractional dimensional subspaces of some space (Ω,d) into equivalence classes of subspaces (Ω_p,d_p) of Ω all of which have the same Hausdorff dimension p. Here d_p is the restriction of d to $\Omega_p\subset\Omega$. Given a domain of integration $E\subset\Omega$, we integrate over the p-dimensional set $E_p=E\cap\Omega_p$.

However, such domains are unsatisfactory since the values of a function $\Phi:E_p \to \mathbb{R}$ are unrelated for different elements of the same equivalence class. For example, with $\Omega=[0,1]$ let Ω_c be the usual Cantor set defined above and let Ω_c' be the set of the same dimension obtained by beginning with the interval $[\frac{1}{3},\frac{2}{3}]$ and deleting middle thirds. Then the range $\Phi(\Omega_c)$ is totally unrelated to the range $\Phi(\Omega_c')$.

As a second attempt we may choose a *single* space for each value of p. Define the uniform Cantor set C_p of dimension p to be the limit as $n \to \infty$ of r_n intervals of length s_n spaced uniformly in $[0,1]$, where

$$\lim_{n\to\infty} \frac{r_n}{(s_n)^p} = 1 \ .$$

Then, for an arbitrary integer N, let $\Omega = \mathbb{R}^N$ and redefine

$$\Omega_p \equiv \mathbb{R}^{[p]} \times C_{p-[p]} \ , \quad 0 \le p \le N \ .$$

Domains of integration are still given by $E_p = E \cap \Omega_p$, so if we consider continuous functions $\Phi:E\to\mathbb{R}$, the values of $\Phi(x\in E_p)$ and $\Phi(y \in E_{p'})$ will be close whenever $d(x,y)$ is small. Now, as we vary p, the domain of integration in (6) shall run over the E_p and the above finiteness condition will automatically be satisfied since $0 < \mu^p(E_p) < \infty$. In other words, we define the dimensional difference of S to be

$$\Delta_{p,p'}S \equiv \int_{E_p} \mathcal{L}(\Phi,\partial\Phi;p)\,d\mu^p - \int_{E_p'} \mathcal{L}(\Phi,\partial\Phi;p')\,d\mu^{p'} \ . \tag{7}$$

The variational principle states that the action shall be stationary with respect to change in the dimension

$$\lim_{p\to p'} \frac{\Delta_{p,p'}S}{p-p'} = 0 \ . \tag{8}$$

This limit, when it exists, is a function of p whose zeros yield specific values for p corresponding to the (classically) stable spacetime dimensions singled out by the theory whose action is S.

While the definitions presented here are straightforward, and the nature of the variation is clear, it turns out that the class of functions given by simple restriction misses crucial elements of dimensional dependence. Thus, for this class and a simple scalar field Lagrangian, the variation given in Eq. (8) vanishes identically. However, it is clear that $\mathcal{L}$ must have nontrivial p dependence, and that this dependence will be most important in the derivative terms. For example, consider the most general solution of Laplace's equation $\Box f = 0$, in $p=1$ and 2 dimensions, respectively. In the former case, $f(x) = a + bx$—a simple linear function in one variable. But for the latter dimensionality,

$$f(x,y) = h(x-y) + g(x+y) \ ,$$

where h and g are twice differentiable but otherwise arbitrary functions of their respective arguments. This suggests the need for a different generalization of dimension since the richness of solutions to higher-dimensional Laplace equations stems from the *vector space* nature of the Laplace equation. Our treatment so far has been restricted to variations of the Hausdorff dimension of space, which, loosely speaking, counts the number of points in a given space. As our example below shows, it may be more fruitful to consider variation of the topological dimension, as characterized by the number of basis elements of a vector space.

To this end, we comment on the possible forms one can write down for $\mathcal{L}$. We recall that, for standard, integer-dimensional spaces, the search is guided by the usual spacetime symmetries, elegantly summarized by the Poincaré and Lorentz groups. We then classify the fields according to the group representation they carry. This allows us to speak of vectors, tensors, spinors, etc. The points of conventional spacetime transform as vectors, and the entire edifice of its linear transformation theory rests on its possessing a *basis*, in the linear algebraic sense of the term. The fractional dimension spaces, in general, will not have bases (in this sense), and it may be difficult to see what symmetry, if any, they possess. Nonetheless, absence of a basis shall present no real obstacle. We will take advantage of the lack of a basis by generalizing the notion of coordinates. Thus, while x_j, $j=1,\ldots,n$, denote coordinate variables for standard integer dimension spaces labeled by some discrete index set (in this case, the integers), we shall introduce a continuous index set and make the replacement $x_j \to x(z)$. The new index set may be chosen to be the real or complex numbers. For each value z, $x(z)$ is an independent coordinate, i.e.,

$$\langle x(z)|x(z')\rangle = \delta(z-z') \ .$$

Given this generalization, we are lead to make the following obvious transcriptions as well: $\Phi(x)\to\Phi[x(z)]$,

$$\partial/(\partial x_j) \to \delta/[\delta x(z)]$$

and

$$\sum_j \to \int dz\ \rho(z) \ .$$

In words, we replace fields by functionals, partial by functional derivatives, and discrete summation by integration with some density or weight function $\rho(z)$. The only conditions that ρ must satisfy are that $p=\int dz\,\rho(z)$ and that the weighted integral reduces to a discrete summation whenever p is integer valued. This constraint still leaves much freedom in the possible choices for ρ.

Note that this particular prescription for making the transition from integer to fractional dimensions means that the action is now a functional integral instead of the p-dimensional integral defined in Eq. (6):

$$S = \int [\mathcal{D}x(z)]\mathcal{L}\left[\Phi[x], \frac{\delta\Phi}{\delta x};p\right] \ . \tag{9}$$

Nevertheless, this will prove useful for exploring different extensions of $\mathcal{L}$ into fractional dimensional spaces. With S as given here, the dimensional variation is now only sensitive to the p dependence in $\mathcal{L}$ since the measure in (9) is a functional, not Hausdorff, measure. We now illustrate these concepts by constructing a p-dimensional model which reduces to a theory of a free scalar field in integer dimensions. Recall the Lagrangian in the case when $p=n$ is given by

$$\sum_{j=1}^{n}\left[\frac{\partial\Phi(x)}{\partial x_j}\right]^2 .$$

The basic task at hand is to generalize this expression employing the above transcriptions. In addition, a choice for the density ρ must be made. For concreteness, we will make the arbitrary, but explicit, choice

$$\mathcal{L}=\sum_{n=1}^{[p]}\frac{1}{2\pi i}\oint_{\Gamma}\frac{dz}{z-n}\left[\frac{\delta\Phi[x]}{\delta x(z)}\right]^2 +\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{\gamma_\epsilon}\frac{dz}{z-p}\left[\frac{\delta\Phi[x]}{\delta x(z)}\right]^2 , \tag{10}$$

where the only condition to be imposed on $x(z)$ is that the functional derivatives be analytic functions. Here $[p]$ is the largest integer $\leq p$, Γ is a closed contour enclosing the first $[p]$ integers, and γ_ϵ is a semicircular contour of radius ϵ centered at p subtending the angle $2\pi(p-[p])$. The contour Γ can be deformed into $[p]$ simple contours, each one encircling the simple pole at each integer $1\leq n\leq[p]$, which allows immediate evaluation of the first term on the right-hand side of (10). This, together with the change of variable $z=p+\epsilon e^{i\theta}$ in the second term, gives

$$\begin{aligned}\mathcal{L}&=\sum_{n=1}^{[p]}\left[\frac{\delta\Phi[x]}{\delta x(n)}\right]^2 +\frac{1}{2\pi}\int_0^{2\pi(p-[p])}d\theta\left[\frac{\delta\Phi[x]}{\delta x(p)}\right]^2\\ &=\sum_{n=1}^{[p]}\left[\frac{\delta\Phi[x]}{\delta x(n)}\right]^2+(p-[p])\left[\frac{\delta\Phi[x]}{\delta x(p)}\right]^2 .\end{aligned} \tag{11}$$

This automatically reduces to Euclidean free scalar field theory whenever p is an integer. The corresponding equation of motion is

$$\nabla^2\Phi[x]+(p-[p])\frac{\delta^2\Phi[x]}{\delta x(p)^2}=0 . \tag{12}$$

The choice of $\rho(z)$ made here gives a graphic interpretation of noninteger dimensions in terms of poles and contours in the complex plane. The amount by which p deviates from being an integer is represented by the failure of the final contour γ_ϵ to completely close around the singular point $z=p$. The limit indicated in (8) is easy to compute for the action given by (9) and (11). As the functional measure is independent of p, then

$$\lim_{p\to p'}\frac{\Delta_{p,p'}S}{p-p'}=\int[\mathcal{D}x]\frac{\partial}{\partial p}\mathcal{L} .$$

So,

$$\begin{aligned}\frac{\partial S}{\partial p}&=\int[\mathcal{D}x]\frac{\partial}{\partial p}\left[\left[\frac{\delta\Phi}{\delta x}\right]^2(p-[p])+\sum_{n=1}^{[p]}\left[\frac{\delta\Phi}{\delta x(n)}\right]^2\right]\\ &=\int[\mathcal{D}x]\left[\left[\frac{\delta\Phi}{\delta x}\right]^2+(p-[p])\frac{\partial}{\partial p}\left[\frac{\delta\Phi}{\delta x}\right]^2\right]\\ &=0 .\end{aligned} \tag{13}$$

As one would hope, the derivatives at the nondistinguished point $[p]$ cancel. With the explicit form for $\mathcal{L}$ given above, we can now obtain the equation of motion and investigate the constraint Eq. (13) for this model. To solve Eq. (12), we appeal to linearity and construct the most general solution by superposing plane waves. Thus, we take the functional Fourier transform

$$\Phi[x]=\int[\mathcal{D}K]A[K]\exp\left[i\int_0^p ds\; K(s)x(s)\right] , \tag{14}$$

where $A[K]$ is the envelope of the wave packet. This is a solution provided that

$$\sum_{n=1}^{[p]}K_n^2+(p-[p])K(p)^2=0 ,$$

which is just the mass-shell condition for the field Φ. To be even more concrete, take a Gaussian envelope for the wave packet:

$$A[K]=\exp\left[\int\int ds\; ds' K(s)\delta(s-s')K(s')\right] .$$

Then we have that (for $0<t<p$)

$$\frac{\delta\Phi[x]}{\delta x(t)}=-\tfrac{1}{2}x(t)\exp\left[-\tfrac{1}{4}\int_0^p ds\; x^2(s)\right] ,$$

up to an irrelevant overall constant. The constraint equation is now easy to evaluate. Substituting this expression into the above functional integral Eq. (13), we obtain

$$\int[\mathcal{D}x]x^2\exp\left[-\frac{1}{2}\int_0^p ds\; x^2(s)\right][1-\tfrac{1}{2}(p-[p])x^2(p)]=0 . \tag{15}$$

The integrations over $x(s)$ for $s\neq p$ just give an overall p-independent constant, the only relevant integration occurring when $s=p$; hence,

$$\int_{-\infty}^{\infty}dx_p x_p^2 e^{-x_p^2/2}[1-\tfrac{1}{2}(p-[p])x_p^2]=0 ,$$

so the allowed dimensions are solutions to

$$p-[p]=\frac{\sqrt{2}}{3}=0.471\,40\ldots . \tag{16}$$

The zeros of this simple equation are at $p_0=\sqrt{2}/3, 1+\sqrt{2}/3,\ldots,n+\sqrt{2}/3,\ldots$.

Some comments regarding the example developed in (9)–(15) are in order. The functional measure, with its "spectral" function ρ, takes into account the varying dimension by interpolating smoothly between integer dimensions. Our explicit realization of ρ as a continuously deforming contour in the complex plane (space of dimensions) is but one way to model this. However, the physical connection between our functional approach and fractional dimension is clear: it is well known that it is the nondifferentiable, *fractal* paths which make the most important contributions to a path integral.[7] The functional integral samples all paths, having all possible fractional dimensions. Since these paths are defined in dimension space (as modeled above), an Euler-Lagrange-type variation yields preferred values for the dimension.

This example is by no means intended to represent a fundamental theory of spacetime. For that, we would like to have a putative, consistent quantum theory of gravity. There, the quantum fluctuations in the metric could be related to fluctuations in the dimension of the underlying spacetime. However, the model-independent feature we can abstract from this is the *origin* of the metric fluctuations: namely, the changing microscopic topology of the spacetime. This can perhaps be made even more vivid if one thinks of a pregeometric spacetime foam, characterized by wildly fluctuating topological features. It is unlikely that the foam could have a single, well-defined value for its dimension. Yet, it evolves into a Riemannian manifold of dimension four, so the challenge, among other things, is to explain this particular dimensionality.

In conclusion, we have proposed that the spacetime dimension be treated at a fundamental level on par with the elementary interactions by promoting its status to that of a dynamical variable. Thus, dimension is a computable number, determined by a new variational principle which involves being able to continuously vary the dimension. In practice, this has necessitated the use of either Hausdorff or functional measure, concepts which have already enjoyed extensive application in other areas of physics.[8] An explicit calculation has been presented illustrating how specific values of dimension p_0 emerge from the variational principle.

The research of D.H. was supported in part by Grant No. DE-AC02-78ER05007.

[1] W. Nahm, Nucl. Phys. **B135**, 149 (1978); E. Cremmer and B. Julia, *ibid.* **B159**, 141 (1979).

[2] M. Green, J. Schwarz, and E. Witten, *Superstring Theory* (Cambridge University, Cambridge, England, 1986).

[3] J. R. Munkres, *Topology* (Prentice-Hall, Englewood Cliffs, NJ, 1974).

[4] A. Zeilinger and K. Svozil, Phys. Rev. Lett. **54**, 2353 (1985); B. Müller and A. Schäfer, *ibid.* **56**, 1215 (1986).

[5] F. Hausdorff, Math. Ann. **79**, 157 (1919); H. Hurewicz and H. Wallman, *Dimension Theory* (Princeton University, Princeton, 1941).

[6] W. Rudin, *Principles of Mathematical Analysis* (McGraw-Hill, New York, 1976).

[7] F. W. Wiegel, *Introduction to Path Integral Methods in Physics and Polymer Science* (World Scientific, Singapore, 1986).

[8] B. B. Mandelbrot, *Fractals: Form, Chance and Dimension* (Freeman, San Francisco, 1977).