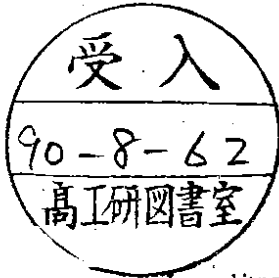


**ANISOTROPIC, TIME-DEPENDENT SOLUTIONS
IN MAXIMALLY GAUSS-BONNET EXTENDED GRAVITY**

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In an arbitrary number of dimensions, we find the full exact anisotropic, time-dependent, diagonal-metric solutions to maximally Gauss-Bonnet extended gravity theory. This class of theories, for which the lagrangian is an arbitrary linear combination of dimensionally extended Euler forms, is the most general gravitational theory in which the field equations contain no more than second derivatives of the metric.

We show that the spacetime exponentially approaches an asymptotic state of constant, anisotropic curvature and prove three theorems concerning two generic types of singularities. The first theorem gives conditions for the existence of Kasner-like curvature singularities. For these the metric diverges as t^{p_i} where $\sum p_i = 2k_{\max} - 1$ and k_{\max} is the highest power of the curvature in the lagrangian. Other critical point singularities can arise from the polynomial nature of the theory. The remaining theorems demonstrate that the generic solution is extendible at all of these other critical points and that the generic critical points occur at moments of extremal volume density of spacetime. We give an explicit coordinate transformation which produces a smooth extension through the critical point. The spacetime may therefore alternately expand and contract for many cycles before expanding forever or contracting to a singularity. Many particular cases are treated in detail including several power series solutions, the generalized Kasner solution to general relativity with or without cosmological constant, the perturbative solution for quadratic string gravity, and 5-d extended gravity.

I. Introduction

Consider the class of all densities of the form¹

$$\mathcal{L}_k = \mathcal{R}^{ab} \mathcal{R}^{cd} \dots \mathcal{R}^{ef} e^g \dots e^h \varepsilon_{abcd \dots efgh \dots} \quad (1.1)$$

in a d-dimensional spacetime. If $k = d/2$, then there are no factors of the vielbein and these densities become the total divergences whose integrals are proportional to the Euler characteristic. When $0 \leq k < d/2$ they are known as dimensionally extended Euler characteristic densities [1] or Lipschitz-Killing curvatures [2]. In this case, these densities are no longer total derivatives and may therefore serve as lagrangian densities for gravitational field theories. The most general such gravitational theory is composed of an arbitrary linear combination of all of the \mathcal{L}_k :

$$\mathcal{L} = \sum_{k=0}^{k_{\max}} \bar{c}_k \mathcal{L}_k \quad (1.2)$$

where k_{\max} is the integer part of $(d-1)/2$ and the \bar{c}_k are arbitrary constants. Several properties strongly suggest the use of \mathcal{L} as the gravitational lagrangian:

1. \mathcal{L} gives the most general theory of gravity with field equations which contain no more than second derivatives of the metric [3].
2. The first order density, \mathcal{L}_1 , is the usual Einstein lagrangian, while the lowest order density, \mathcal{L}_0 , gives rise to a cosmological constant. \mathcal{L}_2 arises as the order α' correction in the low-energy expansion of string models [4, 5, 6]. Theories involving \mathcal{L}_k therefore make close contact with standard theories of gravity.

¹ \mathcal{R}^{ab} is the curvature 2-form, e^a is the vielbein 1-form and $\varepsilon_{a \dots b}$ is the Levi-Civita tensor. \mathcal{L}_k contains k factors of \mathcal{R}^{ab} and enough factors of the vielbein to saturate the remaining indices on the Levi-Civita tensor. The wedge product is assumed between forms.

3. \mathcal{L} is the most general parity preserving lagrangian constructible from the curvature two-form, the vielbein one-form, and invariant tensors in the tangent space [1].
4. All of the densities \mathcal{L}_k are free of ghosts [1,4], ie., there are no negative norm eigenstates in the quantized theory. All are divergences to lowest order in perturbation theory.

Points 1 and 2 are particularly important. This is the most general geometric theory of gravity with 2nd-order field equations. Exact solutions for this class of theories contain all of the following as special cases:

1. Einstein gravity with or without cosmological constant.
2. Einstein gravity, with or without cosmological constant, generalized to arbitrary dimensions.
3. Second order string gravity, with or without cosmological constant, in arbitrary dimensions.

These special cases will be discussed in section IV.

Several solutions to this class of theories have been studied previously. Static, spherically symmetric solutions have been found by Wheeler [7] and, with $k_{\max} = 2$, by Boulware and Deser [8,9] and Wheeler [10]. Boulware and Deser also studied the stability of the two classes of solution which arise due to the quadratic nature of the $k_{\max} = 2$ case.

Cosmological solutions were first studied by Wheeler [10] and Müller-Hoissen [11]. The recent review of cosmological studies in extended gravity by Deruelle and Farifia-Busto [12] provides extensive references to further work. The Kasner case of a time-dependent diagonal metric forms a gap in this research. Deruelle [13] has studied the Kasner-like singularities occuring in the $k = 2$ case and the equations of motion for the same problem up to $k = 4$ were generated on a computer by Caprasse, Demaret and Popadopoulos [4]. However, no exact solution has been calculated.

In this work, we find the complete exact solution to the extended, generalized Kasner problem: time-dependent anisotropic solutions are found for the maximally Gauss-Bonnet extended gravity theory, generalized to an arbitrary finite number of dimensions, described by eqs.(1.1) and (1.2).

In section II below \mathcal{L} (eq.(1.2)) is varied and the resulting equations of motion are evaluated for the case of a diagonal, time-dependent vacuum. The full anisotropic time-dependent solution is derived in section III. In section IV, we examine the particular cases mentioned above: Einstein gravity and quadratic string gravity, with or without cosmological constant. Next, we turn to an analytic study of generic properties of these models, beginning in section V with a study of those solutions which are asymptotically of constant curvature. The singularity structure of this class of gravity theories is considered in sections VI and VII. We first look in the region of the Kasner-like singularities (sec. VI) to second order in perturbation theory and show that there are important differences from the usual Kasner model: the initial power law expansion may have a power larger than those allowed by the standard Kasner exponents, and during the subsequent motion this rate may increase still further. Additional differences from standard models are treated in section VII, where we show that the polynomial nature of the field equations leads to critical points. At these critical points it is possible to find additional singularities, but we prove two theorems showing that the typical behavior is oscillation: the universe may experience a modest finite number of aperiodic oscillations before ultimately expanding or contracting maximally. The final section includes a summary of these results.

II. Anisotropic Equations of Motion

We vary the vielbein, e^a , and the connection, ω_a^b , independently. Since ω_a^b occurs only in the curvature 2-form and the variation of the curvature,

$$\mathcal{R}^{ab} = d\omega^{ab} + \omega^a{}_c \omega_c{}^b, \quad (2.1)$$

is given by

$$\delta_\omega \mathcal{R}^{ab} = D(\delta\omega^{ab}) \quad (2.2)$$

the variation of \mathcal{L}_k gives

$$\delta_\omega \mathcal{L}_k = k D(\delta\omega^{ab}) \mathcal{R}^{cd} \dots \mathcal{R}^{ef} e^g \dots e^h \varepsilon_{abcd \dots efg \dots h} \quad (2.3)$$

Integration by parts together with the Bianchi identity $D\mathcal{R}^{ab} = 0$ and the vanishing of the torsion,

$$De^a = de^a + e^b \omega_b{}^a = T^a = 0, \quad (2.4)$$

then leaves the connection variation identically zero (see [7] for further detail). The vielbein variation, $\delta_e \mathcal{L}_k$ gives

$$\delta_e \mathcal{L}_k = (d-2k) (\delta e^c) \mathcal{R}^{ab} \dots \mathcal{R}^{cd} e^f \dots e^g \varepsilon_{ab \dots cdef \dots g} \quad (2.5)$$

The full field equations are

$$0 = \sum_k \tilde{c}_k (d-2k) e^a \mathcal{R}^{cd} \dots \mathcal{R}^{ef} e^g \dots e^h \varepsilon_{bcd \dots efg \dots h} \quad (2.6)$$

Next, equation (2.6) is evaluated for an anisotropic, time-dependent spacetime. In particular, we assume the generalized Kasner form for the metric tensor. Latin indices (a,b,c,...) refer to the orthonormal basis, while Greek indices (μ, ν, α, \dots) refer to the metric coordinates, with the exception that $dx^0 = dt$ refers to the manifold.

The vielbein and resulting metric are assumed to be of the form

$$e^a = \begin{pmatrix} A_0 dt \\ A_1 dx^1 \\ \vdots \\ A_{d-1} dx^{d-1} \end{pmatrix} \quad g_{\mu\nu} = \begin{pmatrix} -A_0^2 & & & \\ & A_1^2 & & \\ & & \ddots & \\ & & & A_{d-1}^2 \end{pmatrix} \quad (2.7)$$

where all of the A_a are functions of time only. Notice that the function A_0 may be redefined arbitrarily by redefining the time coordinate. Defining

$$\alpha_a = \frac{\dot{A}_a}{A_a} = \frac{d}{dt} \ln(A_a) \quad (2.8)$$

we use eq.(2.4) to calculate the connection 1-forms, ω^{ab} . Letting Latin indices from the center of the alphabet (i,j,k,...) take values from 1 to (d-1) we find:

$$\omega_0^i = A_0^{-1} \alpha_i \quad \omega_j^i = 0 \quad (2.9)$$

Finally, the nonvanishing curvature two-forms, calculated from eq.(2.1), are given by

$$\mathcal{R}^{0i} = A_0^{-2} (\dot{\alpha}_i + \alpha_i^2 - \alpha_i \alpha_0) e^0 e^i \quad (2.10)$$

$$\mathcal{R}^{ij} = A_0^{-2} (\alpha_i \alpha_j) e^i e^j \quad (2.11)$$

Eq.(2.6) may now be evaluated for the given symmetry. There are two distinct cases depending on where the 0-index falls. When $a=0$ we find:

$$0 = \sum_k c_k (-)^k A_0^{-2k} \sum_{ijl;lm;n;p} \alpha_i \alpha_j \dots \alpha_l \alpha_m e^0 e^i e^j \dots e^l e^m e^n \dots e^p \varepsilon_{0ij \dots lmn \dots p} \quad (2.12)$$

Notice that each α in eq.(2.12) is paired with a corresponding diagonal 1-form. This means that each product of α 's is such that no index is repeated. Moreover, the summation gives all possible combinations of $2k$ indices. We therefore define the symbols C^k and C_m^k as the coefficients in the polynomial expressions:

$$(x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_{d-2})(x-\alpha_{d-1}) = \sum_k (-1)^k C^k x^{d-k-1} \quad (2.13)$$

$$(x-\alpha_1)\dots(x-\alpha_{m-1})(x-\alpha_{m+1})\dots(x-\alpha_{d-1}) = \sum_k (-1)^k C_m^k x^{d-k-2} \quad (2.14)$$

For example, C^k may be written as the sum of all possible products of k different α 's:

$$C^k \equiv \sum_{\kappa} \alpha_{i_1} \alpha_{i_2} \dots \alpha_{i_k} \quad \kappa = \{ \{i_1, i_2, \dots, i_k\} \mid 0 < i_1 < i_2 < \dots < i_k \leq k_{\max} \} \quad (2.15)$$

while subscripts, $C_{m, \dots, n}^k$ indicate the absence of indices m, \dots, n from the set κ . Some identities involving the $C_{m, \dots, n}^k$ are given in appendix A. In terms of the C^k eq.(2.5) with $a = 0$ takes the form

$$0 = \sum_k c_k A_0^{-2k} C^{2k} \quad (2.16)$$

where we have defined new constants by

$$c_k \equiv \tilde{c}_k (-1)^k (2k-1)(d-2k)!(2k)! \quad (2.17)$$

We next turn to the case when $a = m \neq 0$ in eq.(2.6). Now there are two terms which must be considered separately depending on whether the 0-index falls on the curvature or the vielbein:

$$0 = \sum_k \tilde{c}_k (d-2k)e^m \left[(2k) \sum_{c,d,e,g,h} \mathcal{R}^{0c} \dots \mathcal{R}^{de} e^g \dots e^h \epsilon_{m0c\dots deg\dots h} \right. \\ \left. + (d-2k-1) \sum_{bc,de,h} \mathcal{R}^{bc} \dots \mathcal{R}^{de} e^0 \dots e^h \epsilon_{mbc\dots de0\dots h} \right] \quad (2.18)$$

Substituting for the curvatures and replacing \tilde{c}_k with c_k ultimately results in

$$0 = \sum_k c_k \frac{A_0^{-2k}}{2k-1} \left[\sum_{i \neq m} (\alpha_i + \alpha_i^2 - \alpha_i \alpha_0) C_{i,m}^{2k-2} + (2k-1) C_m^{2k} \right] \quad m = 1, 2, \dots, d-1 \quad (2.19)$$

Eqs.(2.16) and (2.19) are the complete field equations for the problem.

III. Solution of the Equations of Motion

We wish to solve the set of d coupled equations

$$0 = E_0 \equiv \sum_k c_k A_0^{-2k} C^{2k} \quad (3.1)$$

$$0 = E_m \equiv \sum_k c_k \frac{A_0^{-2k}}{2k-1} \left[\sum_{i \neq m} \left(\frac{d\alpha_i}{dt} + \alpha_i^2 - \alpha_i \alpha_0 \right) C_{i,m}^{2k-2} + (2k-1) C_m^{2k} \right] \quad m = 1, 2, \dots, d-1 \quad (3.2)$$

However, since we know the function A_0 to be arbitrary, there must be a relationship between these equations. The required relation is provided by exploring the consequences of coordinate invariance for the original lagrangian. The change in the metric at a fixed point, \mathcal{P} , is given by

$$\delta g_{\mu\nu}(\mathcal{P}) = g'_{\mu\nu}(\mathcal{P}) - g_{\mu\nu}(\mathcal{P}) = h_{\mu,\nu} + h_{\nu,\mu} \quad (3.3)$$

for an infinitesimal coordinate change h_{μ} . The change in the action is therefore

$$0 = \delta S = \int \frac{\delta \mathcal{L}}{\delta g_{\mu\nu}} (h_{\mu,\nu} + h_{\nu,\mu}) \\ = - \int 2 \left(\frac{\delta \mathcal{L}}{\delta g_{\mu\nu}} \right)_{\nu} h_{\mu} \quad (3.4)$$

Thus, the divergence of the field equation following from the metric variation must vanish. In the present case this condition reduces to

$$\frac{d}{dt} E_0 + C^1 E_0 = 0 \quad (3.5)$$

That this vanishes independently of the equation $E_0 = 0$ follows immediately by noting that

$$0 = \sum \alpha_m E_m = \frac{d}{dt} E_0 + C^1 E_0 \quad (3.6)$$

Therefore, of the d equations (2.16) and (2.19) only $d-1$ are independent. The simplest complete set to work with is provided by the vanishing of the differences

$$E_m - E_1 = 0 \quad m = 2, 3, \dots, d-1 \quad (3.7)$$

together with

$$E_0 = 0 \quad (3.8)$$

The difference equations (3.7) are easily found using the identities presented in appendix A. They are:

$$0 = E_m - E_1 = \sum_k c_k \frac{A_0^{-2k}}{2k-1} \left[\frac{d}{dt} [(\alpha_m - \alpha_1) C_{1;m}^{2k-2}] + (\alpha_m - \alpha_1)(C^1 - \alpha_0) C_{1;m}^{2k-2} \right] \quad (3.9)$$

These take the simplest form if the time coordinate is chosen so that $A_0 = 1$ and $\alpha_0 = 0$. Then t gives the proper time for an observer at rest with respect to the spatial coordinates. This t will (oxymoronically) be called the proper time coordinate. Then we find

$$\sum_k c_k C^{2k} = 0 \quad (3.10)$$

$$(\alpha_m - \alpha_1) \sum_k \frac{c_k}{2k-1} C_{1;m}^{2k-2} = b_m e^{-\int C^1 dt} \equiv b_m \tau \quad m = 2, \dots, d-1 \quad (3.11)$$

where the b_m are constants of integration and we have introduced a new time coordinate τ . Note that τ equals the inverse volume density $(-g)^{-1/2}$. Equations (3.10) and (3.11) provide a complete solution to the problem.

We finish this section with some general observations concerning the solution. First, it is easy to show that all of the α_m satisfy similar polynomial equations since repeated use of eq.(A.6) allows the replacement

$$C_{1;m}^{2k-2} = \sum_{n=0}^{2k-2} (-\alpha_m)^n C_1^{2k-2-n} \quad (3.12)$$

Eq.(3.11) then becomes

$$(\alpha_m - \alpha_1) \sum_k \frac{c_k}{2k-1} \sum_{n=0}^{2k-2} (-\alpha_m)^n C_1^{2k-2-n} = b_m \tau \quad (3.13)$$

If we suppose that we have already solved the system of equations (3.10) and (3.11) for $\alpha_1(\tau)$, then we can find all of the coefficients C_1^{2k-2-n} as functions of τ . According to eq.(3.13), each α_m therefore satisfies *the same* polynomial equation up to the substitution of a different constant of

integration on the right. This form will be suggestive in our later discussion of the singularities of the general theory.

Eq.(3.13) might be thought useful for finding the explicit form of solutions if $c_k = 0$ for many of the allowed k , since the equation may be used to replace the $d-1$ unknown α_1 with however many of the $2k_{max}$ coefficients C_1^{2k-2-n} still remain in eqs.(3.10) and (3.13). However, for the cases which can be solved algebraically there appear to be simpler methods of solution. Moreover, when $c_k \neq 0$ for $k > 3$ the resulting polynomial is greater than fifth order and no explicit algebraic solution is possible even in principle.

From eq.(3.11) it is also easy to see that if two α 's coincide at some time, $\tau_1 \neq 0$, then either they are identical for all time or a certain polynomial must vanish. For, taking the difference of their respective equations, we see that

$$(\alpha_m - \alpha_n) \sum_k \frac{c_k}{2k-1} C_{m;n}^{2k-2} = (b_m - b_n) \tau \quad (3.14)$$

If $\alpha_m(\tau_1) = \alpha_n(\tau_1)$ then $b_m = b_n$ and the right side vanishes. The left-hand expression must now vanish for all τ , requiring either $\alpha_m(\tau) = \alpha_n(\tau)$ or the vanishing of the summation.

Finally, we note that it follows immediately from the equations of motion that if $\alpha_m(\tau)$ is a solution for $\tau \geq 0$ then $-\alpha_m$ is a solution to the equations when τ is replaced by $-\tau$.

IV. Particular Solutions

We now consider the special cases corresponding to Einstein gravity with nonzero cosmological constant, pure Einstein gravity, and second-order string gravity. We begin with the case of Einstein gravity with cosmological constant, so that $c_k = 0$ for all $k > 1$. The field equations reduce to

$$c_0 + C^2 = 0 \quad (4.1)$$

$$(\alpha_i - \alpha_1) = b_i \tau \quad i = 1, \dots, d-1 \quad (4.2)$$

where c_0 and the integration constant, b_i have been normalized by c_1 . Solving eq.(4.2) for α_m , $m > 1$, in terms of α_1 , we write C^1 and C^2 as

$$C^1 = (d-1) \alpha_1 + \Sigma b_i \tau \equiv (d-1) \alpha_1 + b \tau \quad (4.3)$$

$$C^2 = \frac{1}{2} [(C^1)^2 - \Sigma (\alpha_i)^2] = \frac{1}{2} [(d-1)(d-2)\alpha_1^2 + 2b(d-2)\alpha_1 \tau + (b^2 - \Sigma b_i^2)\tau^2] \quad (4.4)$$

Eq.(4.1) is then immediately solved for α_1 , giving

$$\alpha_i = \lambda_i \tau \quad i = 1, 2, \dots, d-1 \quad (4.5)$$

with

$$\lambda_i(\tau) \equiv \left[b_i - \bar{b} \pm \frac{\sigma}{(d-1)^{1/2}} \sqrt{1 - \frac{a}{\tau^2}} \right] \quad (4.6)$$

where \bar{b} and σ are the mean and standard deviation, respectively, of the integration constants, b_m , and where the constant, a , is proportional to the cosmological constant:

$$\bar{b} \equiv \frac{1}{d-1} \Sigma b_m \quad \sigma \equiv \sqrt{\frac{\Sigma b_m^2 - (d-1) \bar{b}^2}{d-2}} \quad a \equiv \frac{2c_0}{(d-2)\sigma^2} \quad (4.7)$$

To find the metric components A_i^2 with the usual time coordinate, t , we must integrate

$$\ln A_i = \int \alpha_i(t) dt \quad (4.8)$$

where

$$\frac{d\tau}{dt} = -C^1 \tau \quad (4.9)$$

These integrations result in the expressions

$$\tau = \begin{cases} \sqrt{-a} \operatorname{csch} \omega t & a < 0 \\ \sqrt{a} \operatorname{csc} \omega t & a > 0 \end{cases} \quad (4.10)$$

for τ where the absolute value signs insure the positivity of τ , the phases have been chosen to place the singularity at $t = 0$ and ω is given by:

$$\omega \equiv \sqrt{(d-1)a} \sigma \quad (4.11)$$

For the metric we have the corresponding expressions

$$A_i = \begin{cases} A_{0i} (\tanh \frac{\omega t}{2})^{\pm \beta_i} (\sinh \omega t)^{1/(d-1)} & a < 0 \\ A_{0i} (\tan \frac{\omega t}{2})^{\pm \beta_i} (\sin \omega t)^{1/(d-1)} & a > 0 \end{cases} \quad (4.12)$$

where the power β_i is defined to be

$$\beta_i = \frac{b_i - \bar{b}}{\sqrt{d-1} \sigma} \quad (4.13)$$

For vanishing cosmological constant we may recover the case of pure Einstein gravity. Taking the limit of eq.(4.12) as $\omega \rightarrow 0$, the metric reduces to the generalized Kasner solution:

$$A_i^2 = t^{2p_i} \quad (4.14)$$

with

$$p_i = \frac{1}{d-1} \pm \beta_i \quad (4.15)$$

The relations $\Sigma p_i = \Sigma p_i^2 = 1$ are automatically satisfied by the form of eq.(4.15) with β_i given by eq.(4.13).

Next, we turn to the case when $c_2 \neq 0$. Unfortunately the solution becomes more difficult as the dimension of spacetime increases. For this reason we will defer the full analytic discussion to later sections where we consider the singularity structure and asymptotic properties of the general solution. In the remainder of this section we look at two cases of particular interest. First,

we give a complete treatment of the $d = 5$ case because it is algebraically tractable; then we provide a perturbative treatment of quadratic string gravity.

When $d = 5$, the field equations reduce to:

$$c_0 + C^2 + 3\gamma C^4 = 0 \quad (4.16)$$

$$(\alpha_2 - \alpha_1)[1 + \gamma\alpha_3\alpha_4] = b_2\tau \quad (4.17a)$$

$$(\alpha_3 - \alpha_1)[1 + \gamma\alpha_2\alpha_4] = b_3\tau \quad (4.17b)$$

$$(\alpha_4 - \alpha_1)[1 + \gamma\alpha_1\alpha_2] = b_4\tau \quad (4.17c)$$

where c_0 and b_m have been normalized by c_1 and $\gamma = \frac{c_2}{3c_1}$. Eqs.(4.17b) and (4.17c) are easily used to eliminate α_3 and α_4 in favor of α_1 and α_2 :

$$\alpha_3^\pm = \alpha^\pm + \frac{(b_3 - b_4)\tau}{2D} \quad (4.18a)$$

$$\alpha_4^\pm = \alpha^\pm - \frac{(b_3 - b_4)\tau}{2D} \quad (4.18b)$$

where

$$\alpha^\pm = \frac{D-2}{2\gamma\alpha_2} \pm \frac{1}{2D\gamma\alpha_2} [D^4 + 2D^2\gamma(b_3+b_4)\alpha_2\tau + \gamma^2(b_3-b_4)^2\alpha_2^2\tau^2]^{1/2} \quad (4.19)$$

and

$$D = 1 + \gamma\alpha_1\alpha_2 \quad (4.20)$$

When these are substituted into eqs(4.16) and (4.17a), the result may be written as two cubic equations for α_1 , with α_2 -dependent coefficients. The difference between these leaves a quadratic equation for α_1 , with the solution:

$$\alpha_1 = \frac{1}{3\gamma\alpha_2} \left\{ 1 \pm \left[1 - \frac{3}{1+\gamma\alpha_2^2} (6\gamma^2\alpha_2^4 - 3\gamma^2b\alpha_2^3\tau + \gamma(1+2\gamma c_0)\alpha_2^2 + \gamma(b_3+b_4)\alpha_2\tau + 1) \right]^{1/2} \right\} \quad (4.21)$$

where $b = 2b_2 - b_3 - b_4$. The final equation for α_2 requires the substitution of this expression for α_1 , its square and its cube, into the remaining cubic equation for α_2 :

$$-\alpha_2^3[\gamma(2+\gamma\alpha_1^2)] + \alpha_2^2[\gamma b\tau + 2\gamma\alpha_1 + \gamma^2\alpha_1^3] + \alpha_2[\gamma(b_3+b_4)\alpha_1\tau + 1] + \alpha_1 = 0 \quad (4.22)$$

The substitution results in a 14th order polynomial to be solved for α_2 . Thus, it is impossible even in this simple case to write a closed *algebraic* expression for each of the α 's. It is impractical and unimportant to list this polynomial, which has more than 100 terms, since the main interest of these solutions is to study the singularity structure in detail. This study is most easily accomplished by working with the coupled α_1, α_2 system, ie, the cubic for α_2 in terms of α_1 and the solution for α_1 in terms of α_2 above, in various limiting cases. In particular, we have verified that the result above agrees with the general results to be presented in the following sections, when the appropriate limits are taken.

We conclude this section with a brief discussion of the quadratic approximation to string gravity, since it involves a limit somewhat different from those to be presented in sections V and VI. For string,

$$-\gamma = (8\pi)^{3/2}\alpha' \ll 1 \quad (4.23)$$

where α' is the Regge slope parameter. In this limit we find those solutions which lie near to the solutions to the Einstein case. Writing α_i in the form

$$\alpha_i = \lambda_i\tau + \epsilon_i \quad (4.24)$$

and letting $\lambda = \sum \lambda_i$ we find that to first order in γ :

$$\epsilon_i = \begin{cases} \frac{-\gamma}{(d-2)\lambda} \left\{ [3C^4(\lambda_i) + \sum_m (\lambda_m^4 + \lambda^2\lambda_m^2 - 2\lambda\lambda_m^3 + (\lambda_1^2\lambda - \lambda_1^3)\lambda_m + \lambda_1^2\lambda(\lambda_1 - \lambda))] \tau^3 \right. \\ \left. + (\sum \lambda_m^2 - \lambda^2) c_0 \tau \right\} & i = 1 \\ \epsilon_1 - \gamma [(\lambda_i^3 - \lambda\lambda_i^2 + \lambda_1^2\lambda - \lambda_1^3)\tau^3 + (\lambda - \lambda_1)c_0\tau] & i > 1 \end{cases} \quad (4.25)$$

Here $C^4(\lambda_j)$ is the usual C^4 defined by eq.(2.13), but with λ_j replacing α_j . When the cosmological constant is nonvanishing λ_i depends on τ and the integration of these expressions becomes tedious. However, when $c_0 = 0$ the integrations become easy because each α_i has the simple form

$$\alpha_i = \lambda_i \tau + \gamma_i \tau^3 \quad (4.26)$$

for constants λ_i as before and $\gamma_i \ll 1$ following from the expressions for ϵ_i . Letting $\eta = \Sigma \gamma_i$ we integrate eqs.(4.8) and (4.9) to find:

$$t = \begin{cases} \sqrt{\eta/\lambda^3} \left(\frac{1}{\bar{\tau}} + \tan^{-1} \bar{\tau} \right) & \lambda \eta > 0 \\ \sqrt{-\eta/\lambda^3} \left(\frac{1}{\bar{\tau}} - \tanh^{-1} \bar{\tau} \right) & \lambda \eta < 0 \end{cases} \quad (4.27)$$

where $\bar{\tau}$ is defined by

$$\bar{\tau} = \sqrt{\frac{|\eta|}{|\lambda|}} \tau \quad (4.28)$$

The series expansion, eqs.(4.22) and (4.23), will hold as long as $\bar{\tau} \ll 1$, corresponding to $t \gg 1$. It is simple to find the metric components in this approximation. Letting $t \approx \frac{1}{\lambda \tau}$ we find for both cases (4.25):

$$A_i = t^{\frac{\lambda_i}{\lambda}} \exp\left(-\frac{\gamma_i}{2\lambda^3 t^2}\right) \quad (4.29)$$

Since the exponential factor will be unmeasurably close to one, the long time behavior is a power law. The limiting exponent, $\frac{\lambda_i}{\lambda}$, is the same as that of the dimensionally extended Kasner solution, eq.(4.15). As we show in the next section, the general asymptotic behavior can be quite different from this. The behavior of this and other solutions near singularities, which require different expansions, is taken up in sections VI and VII.

V. General Properties: Asymptotic Behavior

As we have seen above, the writing of a closed algebraic expression for each of the α 's is impossible even in the simplest case, when $d = 5$. Nonetheless, it is possible to obtain some useful results for general values of the coefficients in all dimensions. In this section we obtain series solutions which hold in the asymptotic, $t \rightarrow \pm\infty$ limit. Then we turn to a study of the singularity structure of the solution, proving three theorems about its generic behavior. In section VI we study those regions which correspond to the usual Kasner singularity, and in section VII we explore further singularities which occur due to the polynomial nature of the original lagrangian density.

The $t \rightarrow \pm\infty$ limit allows α_i to be expanded in negative powers of t , implying a power series in positive powers of τ . Therefore we begin with:

$$\alpha_i = \sum_{n=0}^{\infty} a_{in} \tau^{-n\lambda} \quad (5.1)$$

where λ remains to be determined. For this and a subsequent power series expansion it will be convenient to define the following functions, which are simply the first and second order expansions of the summands in the equations of motion:

$$B_0(a_{j0};k) \equiv c_k C^{2k}(a_{j0}) \quad (5.2a)$$

$$B_{m0}(a_{j0};k) \equiv (a_{m0} - a_{j0}) \frac{c_k}{2^{k-1}} C_{1;m}^{2k-2}(a_{j0}) \quad (5.2b)$$

$$B_{01}(a_{j1};a_{j0};k) \equiv c_k \sum_{i=1}^k a_{i1} C_i^{2k-1}(a_{j0}) \quad (5.2c)$$

$$B_{m1}(a_{j1};a_{j0};k) \equiv (a_{m0} - a_{j0}) \frac{c_k}{2^{k-1}} \sum_{j=1}^k a_{j1} C_{1;m;j}^{2k-3}(a_{j0}) \\ + (a_{m1} - a_{j1}) \frac{c_k}{2^{k-1}} C_{1;m}^{2k-2}(a_{j0}) \quad (5.2d)$$

First consider the $n = 0$ term in eq.(5.1) for α_i . The a_{i0} are fixed by:

$$\sum_k B_0(a_{i0};k) = 0 \quad (5.3)$$

$$\sum_k B_{m0}(a_{i0};k) = 0 \quad (5.4)$$

At this order the a_{i0} are determined up to multiplicity of roots; at least one must be nonzero unless the cosmological constant vanishes. One simple solution is to set $a_{m0} = a_{10}$ for all $m > 1$, then choose a_{10} to solve eq.(5.3). This solution and a second with $\alpha_1 = \alpha_2 \neq \alpha_3 = \alpha_4$ exist for the $d = 5$ case; for a particular choice of γ there is also a geometrically free solution [7]. More generally, eq.(5.4) implies, after some algebra, that the a_{i0} cannot all be different. Generically, at this order, the curvature is constant and anisotropic.

At first order in τ , the coefficients a_{i1} are related to the constants of integration, b_m by:

$$\sum_k B_{01}(a_{i1};a_{i0};k) = 0 \quad (5.5)$$

$$\sum_k B_{m1}(a_{i1};a_{i0};k) = b_m \quad (5.6)$$

where we have now chosen $\lambda = 1$. Because of the integration constants in eq.(5.6) we can choose $d-2$ of the a_{i0} arbitrarily, with the final one determined by eq.(5.5). To this order, the α_i have the form

$$\alpha_i = a_{i0} + a_{i1} \tau \quad (5.7)$$

Setting $a_n = \Sigma a_{in}$, the relation between t and τ is given by:

$$\tau = \frac{a_0}{e^{a_0 t} - a_1} \quad (5.8)$$

confirming that $\tau \rightarrow 0$ as $a_0 t$ tends to $+\infty$. The α_i approach a_{i0} exponentially in terms of proper time t , leading very quickly to the anisotropic, constant curvature state. The components of the metric tensor vary as

$$A_i = e^{a_{i0} t} (1 - a_1 e^{-a_0 t})^{a_{i1}/a_1} \quad (5.9)$$

The exponential character of the asymptotic behavior is dictated by the constants $a_0 = \Sigma a_{i0}$ and the individual a_{i0} . Notice that any direction which has $a_{i0} < 0$ undergoes exponential dynamical dimensional reduction. In a relatively short time such a damping exponential could reduce the extent of the corresponding direction far beyond a detectable scale even if the solution for that direction changed to a power law expansion in a later, matter-dominated era.

Recalculating when a_0 vanishes, the metric components become

$$A_i = t^{a_{i1}/a_1} e^{a_{i0} t + C(t^{-1})} \quad (5.10)$$

and if the cosmological constant vanishes it is possible to choose the solution $a_{i0} = 0$, giving the asymptotically Kasner behavior found in the previous section.

Finally, observe that the power series of eq.(5.1) will break down as τ becomes large. This means that we will require a different approximation in for large τ . This will take the solution into the region where the curvature can diverge.

VI. General Properties: Type I Singularities

The existence of singularities in extended gravity solutions is complicated by the polynomial nature of the solutions. Here, these complications give rise to two important

differences from Einstein gravity. First, $t = 0$ is not necessarily a divergent point as it is for the Kasner solution. We will display series solutions and provide an exact solution which demonstrate this. Second, there exist critical point singularities. These critical points are studied in detail in section VII.

We begin our study of curvature singularities by looking at certain conditions which must hold for curvature divergences to occur. This divides the potential singularities into two types: type I, consisting of those cases in which the α 's diverge, and type II encompassing those cases for which the α 's remain finite but the time derivatives, $\frac{d\alpha}{dt}$, diverge. Type I cases typically occur as $t \rightarrow 0$, while type II cases are due to the polynomial character of the class of theories considered and in particular to the existence of critical points. We first examine the α -divergent singularities in detail by looking for Kasner-like approximations. Such approximate solutions have been studied to lowest order by Deruelle [13]. Here we give a fuller treatment, carrying the second-order term as well. Then, in section VII, we go on to discuss the existence and character of the type II critical point singularities. These extremal points are not found in the Kasner solution and cannot be found from the first order expansion to extended gravity.

Type I singularities happen only if one or more of the α_i diverge. We prove the following theorem concerning the times when this can occur:

Theorem I: The curvature diverges if:

1. $\tau \rightarrow \infty$
2. $\tau \rightarrow 0$ as $t \rightarrow 0$.

Proof: First let $\tau \rightarrow \infty$. Then, it is clear from eq.(3.11) that one or more of the α_i must diverge with it. This case is treated in detail below, where we show that this divergence in α_i produces a divergence of the curvature at time $t = 0$.

Next, suppose that $\tau \rightarrow 0$ as $t \rightarrow 0$. Then in order for

$$t = - \int \frac{d\tau}{C^1 \tau}$$

to hold, C^1 must be dominated by τ^{-s} for some power $s > 0$; the power series for C^1 will be of the form:

$$C^1 = \tau^{-s} \sum_{n=0}^{\infty} a_n \tau^n \quad (6.1)$$

For any positive s , C^1 will diverge as $\tau \rightarrow 0$, and for generic solutions the curvature scalar, R , will also diverge.

Notice that this case can only occur if the divergent part of the α 's cancels between different α 's since at $\tau = 0$ the right hand side of eq.(3.13) vanishes.

There is an alternative to the two possibilities covered by the theorem. As $t \rightarrow 0$, τ may approach some finite, nonzero number. We may always arrange $\tau \rightarrow 1$ in such a case by our choice of the t -origin. Such cases do occur in general. We need only let

$$\alpha_i = \sum_{n=0}^{\infty} a_{in} (\tau - 1)^n \quad (6.2)$$

Then the difference equations may be rewritten as

$$(\alpha_m - \alpha_1) \sum_k \frac{\alpha_k}{2k-1} C_{1;m}^{2k-2}(\alpha_i) = b_m + b_m(\tau-1) \quad (6.3)$$

At lowest order, we have

$$\sum_k B_{m0}(a_{j0}; k) = b_m \quad (6.4)$$

where b_m is arbitrary, so a_{m0} may be chosen arbitrarily for $m > 1$. Then a_{10} is determined by

$$\sum_k B_0(a_{j0}; k) = 0 \quad (6.5)$$

At the next order the constants b_m are regarded as functions of the a_{j0} :

$$\sum_k B_{m1}(a_{j1}; a_{j0}; k) = b_m(a_{j0}) \quad (6.6)$$

The higher order equations are then homogeneous. One exact solution which leaves τ finite at the t -origin occurs when all of the integration constants vanish. Then the difference equations may be solved by taking $\alpha_m = \alpha_1$, with α_1 determined by

$$\sum_k c_k C^{2k}(\alpha_1) = \sum \binom{d-1}{2k} c_k (\alpha_1)^{2k} = 0 \quad (6.7)$$

A necessary condition for the existence of this solution in a particular theory is that $c_k/c_{k'} < 0$ for some k, k' .

We now turn to an examination of the singularities which occur when $\tau \rightarrow \infty$. These generalize the Kasner singularity. In this limit the α_i may be expanded in a power series

$$\alpha_i = \sum_{n=0}^{\infty} a_{in} \tau^{-(2n-1)\kappa} \quad (6.8)$$

where $\kappa \equiv \frac{1}{2k_{\max}-1}$. The divergent part of α_i then goes as

$$\alpha_i = a_{i0} \tau^{\kappa} \quad (6.9)$$

This means that C^k will diverge as $\tau^{k\kappa}$ and the constant coefficients of the series are determined by the algebraic equations. To first order we can neglect all but the maximum value of k :

$$B_0(a_{j0}; k_{\max}) = 0 \quad (6.10)$$

$$B_{m0}(a_{j0}; k_{\max}) = b_m \quad (6.11)$$

Notice that the choice made above for κ insures that the τ dependence has cancelled from eq.(6.11). The appearance of the arbitrary integration constants on the right-hand side of eq.(6.11)

means that we can take $(d-2)$ of the $(d-1)$ constants, a_{m0} , to be completely arbitrary, with the last one determined by the single constraint, eq.(6.10). When t is used as the time coordinate, the metric components take the form:

$$A_i = t^{p_i} \quad (6.12)$$

where p_i is given by

$$p_i = \frac{a_{j0}}{\kappa \sum a_{j0}} = (2k_{\max} - 1) \frac{a_{j0}}{\sum a_{j0}} \quad (6.13)$$

These exponents now satisfy the sum rule

$$\sum p_i = 2k_{\max} - 1 \quad (6.14)$$

This solution requires $p_i > 1$ for some i when $k_{\max} > 1$. In even dimensions eq.(6.10) requires at least one of the exponents to be negative and at least one to be positive, while in odd dimensions one exponent must vanish. For $k_{\max} = 1$ we recover the generalized Kasner solution.

That these singularities are in fact curvature singularities almost always follows, for $k_{\max} > 1$, from the Ricci scalar:

$$R = 2[\sum(\dot{\alpha}_i + \alpha_i^2) + C^2] \approx \tau^{2\kappa} [\sum(-a_{j0} + (a_{j0})^2) + C^2(a_{j0})] \quad (6.15)$$

For the pure Einstein case ($k_{\max}=1$), C^2 and the sum in parentheses each vanish separately as a consequence of the field equations. However, for the full class of $k_{\max} > 1$ Gauss-Bonnet extended gravity theories the vanishing of R is independent of the field equations and therefore provides a constraint on the solutions (ie., the vanishing of the a_{j0} -dependent terms in brackets in eq.(6.15)). Therefore the Ricci scalar diverges for all but a zero-measure subset of solutions.

At second order, the coefficients a_{i1} are completely determined in terms of the a_{j0} by the k_{\max} and $k_{\max}-1$ terms of the equations of motion:

$$B_0(a_{j0}; k_{\max}-1) + B_{01}(a_{j1}; a_{j0}; k_{\max}) = 0 \quad (6.16)$$

$$B_{m0}(a_{j0}; k_{\max}-1) + B_{m1}(a_{j1}; a_{j0}; k_{\max}) = 0 \quad (6.17)$$

To this order, the α_i have the form

$$\alpha_i = a_{j0} \tau^\kappa + a_{j1} \tau^{-\kappa} \quad (6.18)$$

Setting $a_0 = \sum a_{j0}$ and $\omega = \kappa \sqrt{|a_0 a_1|}$, the relation between t and τ is given by:

$$\tau^\kappa = \begin{cases} \frac{\omega}{\kappa |a_0|} \cot \omega t & a_0 a_1 > 0 \\ \frac{\omega}{\kappa |a_0|} \coth \omega t & a_0 a_1 < 0 \end{cases} \quad (6.19)$$

The resulting metric components when $a_0 a_1 > 0$ are

$$A_i = A_{j0} (\sin \omega t)^{\frac{a_{0j}}{\kappa a_0}} (\cos \omega t)^{-\frac{a_{j1}}{\kappa a_1}} \quad (6.20)$$

When $a_0 a_1 < 0$ there are two branches:

$$A_i = \begin{cases} A_{j0} (\sinh \omega t)^{\frac{a_{0j}}{\kappa a_0}} (\cosh \omega t)^{-\frac{a_{j1}}{\kappa a_1}} & \left| \frac{a_0}{a_1} \right| \tau^\kappa > 1 \\ A_{j0} (\cosh \omega t)^{\frac{a_{0j}}{\kappa a_0}} (\sinh \omega t)^{-\frac{a_{j1}}{\kappa a_1}} & \left| \frac{a_0}{a_1} \right| \tau^\kappa < 1 \end{cases} \quad (6.21)$$

In the $a_0 a_1 > 0$ case a second branch can be found by shifting the t -origin by $\frac{\pi}{2}$. These functions can increase faster than the original power law; for small t the first is of the general form

$$A_i = A_{j0} (t^{P_i} + \lambda t^{P_i+2}) \quad (6.22)$$

It is therefore possible that the generalized, extended Kasner universe would have an inflationary stage even though at the earliest times the expansion goes as a power law. Unfortunately, as seen in the discussion of the $d = 5$ case, it is difficult to characterize the solution at intermediate times.

VII. General Properties: Type II Singularities

In addition to the singularities described above at which the curvature becomes infinite due to the divergence of one or more α_i , there are many other singular points in the general solution. These are critical points which follow from the polynomial nature of the lagrangian density. At any of these critical points $\dot{\alpha}$ may either diverge or become discontinuous, resulting in divergence of the curvature or inextendibility of the manifold. In this section we characterize these points, and prove that in almost all cases neither of these pathologies actually occur.

Though our main results depend on direct examination of the original form of the solution, the character of the solution near the critical points is easier to intuit through a study of eq.(3.13):

$$P_{2k_{\max}-1}(\alpha_m; \tau) \equiv (\alpha_m - \alpha_1) \sum_k \frac{C_k}{2k-1} \sum_{n=0}^{2k-2} (-\alpha_m)^n C_1^{2k-2-n} - b_m \tau = 0 \quad (3.13)$$

The polynomial nature of eqs.(3.13), with the coefficients C_1^{2k-2-n} regarded as functions of τ , give the functions $\alpha_m(\tau)$ implicitly. The solution for α_m is the inverse function,

$$\alpha_m(\tau) = P_{2k_{\max}-1}^{-1}(0; \tau) \quad (7.1)$$

but this inverse only exists when the first derivative of $P_{2k_{\max}-1}$ with respect to α_m is nonzero. Conversely, the solution will break down whenever:

$$\frac{d}{d\alpha_m} P_{2k_{\max}-1}(\alpha_m) = 0 \quad (7.2)$$

Thus, singularities occur for any value of the time, τ_0 , which makes $\alpha_m(\tau_0)$ a double root of $P_{2k_{\max}-1}(\alpha_m; \tau_0)$. Notice such an extremal point of a polynomial typically occurs as one lobe of the graph of the polynomial crosses the α -axis. This either causes the simultaneous vanishing or appearance of two roots. Because each of the roots approaches (or departs from) the extremum

where the slope of the curve vanishes, the rate of approach to (departure from) the extremum diverges:

$$\left. \frac{d\alpha}{d\tau} \right|_{\text{Critical pt.}} \rightarrow \infty \quad (7.3)$$

This fact will be of importance in our proof of theorem III below.

While these arguments demonstrate the existence of and rate of approach to extremal points in most solutions, they do not give a complete classification of those critical points since the coefficient functions $C_1^{2k-2-n}(\tau)$ may also be singular. Neither is it possible to explicitly solve eqs.(3.10) and (3.11), even in a given dimension. Instead, we will look at the analytic properties of this system of equations near the branch points.

Before proceeding to our theorems it will be illustrative to have two concrete examples of critical points. We consider the cases of Einstein gravity with cosmological constant and the general $d = 5$ solution, both discussed in section IV .

The simplest example is provided by Einstein gravity with a positive cosmological constant. Combining eqs.(4.5) and (4.6) displays a critical point when $\tau^2 = a$:

$$\alpha_i = (b_1 - \bar{b})\tau \pm \frac{\sigma}{(d-1)^{1/2}} \sqrt{\tau^2 - a} \quad (4.6')$$

The derivative of α_i with respect to τ is divergent when $\tau^2 = a$. Of course, the Ricci scalar vanishes for this case, but $R_{abcd}R^{abcd}$ contains the square root term. Even though the critical point may be reached in a finite proper time, the solution simply stops. However all components of the curvature tensor prove to be finite at the extremum. This suggests that the solution may be extendible and we will show below that it is.

For the $d = 5$ case, recall that the equations of motion lead to three quadratics and a 14th order polynomial and could therefore have as many as $2^3 \times 14 = 112$ different critical points. We

can see that some of these branches do indeed occur by choosing $b_2 = b_3 = b_4 = b$ in eqs.(4.17), and setting $c_0 = 0$ in (4.16). Then with

$$\alpha_2 = \alpha_3 = \alpha_4 \equiv \beta \quad \alpha_1 \equiv \alpha \quad (7.4)$$

we find the solution

$$\alpha = \beta - \frac{\tau}{1+\gamma\beta^2} \quad (7.5)$$

$$P(\beta) \equiv \beta^3 + \frac{2}{\gamma}\beta - \frac{\tau}{\gamma} = 0 \quad (7.6)$$

Since τ only occurs in the $(\beta)^0$ part of eq.(7.6), changing τ simply raises or lowers the cubic curve for β , shifting the roots of P . When $\gamma < 0$ (as in string theory), there is a double root at the value of τ for which $P'(\beta) = 0$:

$$\beta_0 = \pm \sqrt{-\frac{2}{3\gamma}} \quad (7.7)$$

$$\tau_0 = \pm(\gamma\beta_0^3 + 2\beta_0) \quad (7.8)$$

A solution evolving to τ_0 from above (forward in t) will cease at τ_0 ; evolution downward from $\tau < \tau_0$ must end at $-\tau_0$; and the solution with $\tau < -\tau_0$ is bounded above in t . Again, the τ derivatives of α and β diverge at the critical times. The curvature is finite throughout this evolution, but a simple calculation shows that the Ricci scalar, R , contains a term with the singular factor. Perturbation about the singular point shows that the singularity can be reached in a finite proper time.

In each of the two examples above, direct calculation shows that while $\frac{d\alpha}{d\tau}$ diverges as claimed above, $\frac{d\alpha_i}{d\tau}$ remains finite. As a result, the curvature is finite at the critical point and it is easy to show that the solution is extendible. In fact, the simplest way to extend the solution is to take advantage of the fact that two solutions for α_i merge by letting the time run forward along one branch to the critical point, then backward as α_i moves out the other branch. Matching the branches of eq.(4.6) in this way gives:

$$\alpha_i(\tau) = \begin{cases} (b_i - \bar{b})\tau + \frac{\sigma}{(d-1)^{1/2}} \sqrt{\tau^2 - a} & \tau > \sqrt{a} \\ (b_i - \bar{b})(2\sqrt{a} - \tau) - \frac{\sigma}{(d-1)^{1/2}} \sqrt{(2\sqrt{a} - \tau)^2 - a} & 0 < \tau < \sqrt{a} \end{cases} \quad (7.9)$$

In terms of proper time, $\alpha_i(t)$ automatically includes both parts of eq.(7.9). A similar patching is possible in the cubic example.

Remarkably, extendibility is a generic property of the critical points of the full solution.

We prove:

Theorem II: In an arbitrary number of dimensions, to all possible orders of generic Gauss-Bonnet extension, the generic solution given by eqs.(3.10) and (3.11) is extendible at every type II critical point.

Proof: The general form of the implicit function theorem states that given a set of n once-differentiable equations in $n + m$ variables, $\{P_i(a_j; x_k) = 0 \mid i, j = 1, \dots, n; k = 1, \dots, m\}$ satisfied at a point $\mathcal{P} = (a_{j0}, x_{k0})$, there exists a unique set of differentiable functions, $\{a_j(x_k)\}$, defined in a neighborhood of \mathcal{P} , provided

$$\det \left(\frac{\partial P_i}{\partial a_j} \right) \Big|_{\mathcal{P}} \neq 0 \quad (7.10)$$

Applied to the equations of motion we have $d-1$ equations and d variables:

$$P_1(\alpha_i, \tau) \equiv \sum c_k C^{2k} = 0 \quad (7.11)$$

$$P_m(\alpha_i, \tau) \equiv (\alpha_m - \alpha_1) \sum_{k=1}^{2k-1} \frac{c_k}{2k-1} C_i^{2k-2} - b_m \tau = 0 \quad (m = 2, 3, \dots, d-1) \quad (7.12)$$

The theorem requires that differentiable solutions $\alpha_i(\tau)$ exist provided

$$\det \left(\frac{\partial P_i}{\partial \alpha_j} \right) \neq 0. \quad (7.13)$$

Unless two or more of the α 's are identical, this determinant is nonvanishing at all but the critical points. If two or more α 's do coincide in some neighborhood then the no-crossing result from section III implies that they are the same for all τ , and we can simply consider the reduced set of equations gotten by eliminating the redundant α 's. So, away from the critical points the matrix $\frac{\partial P_i}{\partial \alpha_j}$ is already of

maximal rank and solutions exist.

The point of the theorem is that the critical points of P_i , where $\frac{\partial P_i}{\partial \alpha_j}$ is only of rank $(d-2)$, arise because τ no longer provides an adequate parameter for the curve and not because the solution breaks down. The choice of the time coordinate provides an additional degree of freedom allowing the continuation of the solution; we only need the $(d-1) \times d$ dimensional matrix

$$\begin{Bmatrix} \frac{\partial P_i}{\partial \alpha_j} & \frac{\partial P_i}{\partial \tau} \end{Bmatrix} \quad (7.14)$$

to have rank $(d-1)$. Because these time derivatives are simply the integration constants, generic solutions will retain the maximum rank.

Avoidance of the critical point is easily achieved by making a different choice of time coordinate. Concretely, suppose $\frac{\partial P_i}{\partial \alpha_j}$ is of rank $(d-2)$. Then there exists a single linear dependency among the partials and we can always choose a basis so that the dependency is restricted to the gradients of just two functions. Let this basis be such that the remaining $(d-3)$ gradients

$$\begin{Bmatrix} \frac{\partial P_m}{\partial \alpha_j} \end{Bmatrix} \quad m = 1, 4, 5, \dots, d-1 \quad (7.15)$$

are orthogonal to $\frac{\partial P_2}{\partial \alpha_1}, \frac{\partial P_3}{\partial \alpha_1}$. For some nonzero constant λ and all i we have

$$\lambda \frac{\partial P_2}{\partial \alpha_i} + \frac{\partial P_3}{\partial \alpha_i} = 0 \quad (7.16)$$

Now perform a rotation between α_1 and τ . Let

$$\bar{\alpha}_1 = \frac{1}{2}(\alpha_1 + \tau) \quad (7.17)$$

$$\bar{\tau} = \frac{1}{2}(\alpha_1 - \tau) \quad (7.18)$$

and consider whether there still exists a linear dependency. Since eq.(7.16) still holds for all $i \geq 2$, the constant λ must remain the same. However,

$$\begin{aligned} \lambda \frac{\partial P_2}{\partial \bar{\alpha}_1} + \frac{\partial P_3}{\partial \bar{\alpha}_1} &= \lambda \left(\frac{\partial P_2}{\partial \alpha_1} - b_2 \right) + \left(\frac{\partial P_3}{\partial \alpha_1} - b_3 \right) \\ &= -\lambda b_2 - b_3 \end{aligned} \quad (7.19)$$

which only vanishes when

$$\lambda = -\frac{b_3}{b_2} \quad (7.20)$$

This does not hold in general and if it did we could simply choose a different angle of rotation. Therefore, except for double degeneracies of the Jacobian matrix (which require nongeneric Gauss-Bonnet extensions) or degeneracies of $\frac{\partial P_1}{\partial \tau}$ with two of the other partials (which requires special choices of the integration constants b_j), the condition for the existence of solutions will be satisfied and the critical point must be only a coordinate singularity. All components of the curvature and curvature invariants will be finite at the critical point.

We now prove another result based on the rate of approach of the α_i to the critical points:

Theorem III: Generic critical points occur at moments of extremal volume density of spacetime.

Proof: Let $\alpha_{i0}(\tau)$ be a curve in α - τ space such that $P_m(\alpha_{i0}, \tau) = 0$; i.e., $\alpha_{i0}(\tau)$ is a solution to the equations of motion. Then along that curve

$$\left. \frac{dP_m}{dt} \right|_{\alpha_{i0}} = 0 \quad (7.21)$$

Starting at a point near a critical point we expand

$$\begin{aligned} 0 = \frac{dP_m}{dt} &= \frac{\partial P_m}{\partial \alpha_i} \frac{d\alpha_i}{dt} + \frac{\partial P_m}{\partial \tau} \\ &= \frac{\partial P_m}{\partial \alpha_i} \frac{d\alpha_i}{dt} - b_m \end{aligned} \quad (7.22)$$

Now let the matrix $\frac{\partial P_m}{\partial \alpha_i}$ project into a subspace \mathcal{S}_c at the critical point. For generic initial conditions, b_m , there will be some component, b_{cm} , of b_m orthogonal to \mathcal{S}_c . Then as $\frac{\partial P_m}{\partial \alpha_i}$ approaches a projection, $\frac{d\alpha_i}{dt}$ must diverge in order to keep b_{cm} fixed. Therefore, for at least one value of i , $\frac{d\alpha_i}{dt}$ must diverge at the critical point.

Next, consider the curvature. By theorem II, there exists a coordinate choice for which all of the α_i and their first derivatives are bounded. It follows that the curvature and all curvature invariants are also bounded. In particular

$$R^{abcd}R_{abcd} = \Sigma (\dot{\alpha}_i)^2 + F(\alpha_i) \quad (7.23)$$

is bounded where the dot denotes the time derivative with respect to the proper time coordinate t . This means that for every value of i ,

$$\dot{\alpha}_i = -C^i \tau \frac{d\alpha_i}{dt} \quad (7.24)$$

is bounded even though the τ -derivative is diverging for at least one value of i . Since τ is nonzero in general we must have

$$C^i = 0 \quad (7.25)$$

at the critical point. Therefore

$$\frac{d}{dt} \sqrt{-g} = -C^i e^{-\int C_i dt} = 0 \quad (7.26)$$

We conclude that the critical point occurs at a local extremum of the spacetime volume density.

As a consequence of this theorem we have a clear picture of the meaning of the type II critical points as times when the volume is maximal or minimal. It is easy to show from the existence of such extrema that for some particular direction, the spacetime may stop expanding and start contracting, or vice-versa. This sort of reversal does not typically happen at exactly the same time as the extremum, but some fluctuation in the rate of expansion is a necessary consequence of the extremum. In some of these extended gravity theories the universe may therefore go through many bounces without reaching any singularity. The maximum possible number of bounces increases with dimension with the number of solutions to the equation:

$$\det \left(\frac{\partial P_i}{\partial \alpha_j} \right) = 0 \quad (7.27)$$

This number is $(d-1)(2k_{\max}-2) + 1$, which is of order d^2 . This large number of bounces could provide substantial mixing.

It is also interesting to note the times at which bounces may occur. Of course in general, any time is possible. However, in second order string gravity where the times of the bounces are determined by the extremely small slope parameter α' , or in other unified theories where the higher order terms have a small coupling, the bounces are likely to occur at either very early or very late times.

If we consider nongeneric solutions as well, there are three types of solution when the rank of the Jacobian matrix reduces to $(d-2)$ at a point. Choosing a set of $(d-2)$ of the equations $P_m = 0$ such that their gradients span the critical subspace S_c , we see that their intersection will give a smooth curve in α -space. The intersection of this curve with the $(d-1)$ dimensional surface determined by the remaining equation may be:

1. A set of isolated points. The merging of a pair of these points is the generic condition covered by theorem III.
2. Empty. In this case the intersection of the curve with the subspace must have run off to infinity, typically giving a curvature singularity. The singularities studied in section VI are of this type.
3. An entire segment of the curve. These solutions are geometrically free [7]. The metric will contain undetermined functions.

Other possibilities arise if the rank of the Jacobian matrix falls below $(d-2)$.

VIII. Summary

In an arbitrary number of dimensions, we have found the complete anisotropic, time-dependent, diagonal-metric solutions to maximally Gauss-Bonnet extended gravity theory. This class of theories, for which the lagrangian is an arbitrary linear combination of dimensionally extended Euler forms, is the most general gravitational theory in which the field equations contain no more than second derivatives of the metric.

We considered the asymptotic behavior of the theory and showed that the curvature for large t becomes constant and anisotropic. We develop the solution in a power-series near this limit and find that the approach to the asymptotic region is exponential.

We have divided the singularities of the problem into two types. Type I singularities are characterized by divergences of the logarithmic derivatives, α_i , of the metric components and always lead to curvature singularities. Type II singularities are produced by critical points in the solution due to the polynomial nature of the original class of theories.

We proved three theorems concerning the two types of singularities.

The first theorem states that Type I curvature singularities occur whenever $\tau = \exp(-\int C^i dt)$ tends to infinity, or when τ and t tend to zero together. Examples are given for which neither of these conditions occurs. When τ diverges the Type I metric components are shown to diverge as t^{p_i} where $\sum p_i = 2k_{\max} - 1$ and k_{\max} is the highest power of the curvature in the original lagrangian. The expansion away from the singular region typically proceeds faster than in the corresponding Kasner or generalized Kasner solution. There will always be at least one direction which expands and at least one direction which contracts.

The proof of the second theorem demonstrates that almost all Type II singularities are coordinate singularities only, and we give an explicit coordinate transformation which produces a smooth extension through the branch point. True type II critical point singularities require at least two constraints on the constants c_k defining the theory or they require one constraint on the c_k and one on the integration constants, b_m .

Our third theorem shows that type II critical points occur at extreme values of the spacetime volume density. The spacetime may therefore alternately expand and contract for many cycles before expanding forever or contracting to a singularity. While this behavior is impossible in

Einstein gravity, it is quite natural in extended gravity as first one term in the lagrangian, then another, dominates the time evolution.

We have treated many particular cases in detail, verifying or demonstrating the claims above. In addition to several power series solutions we have displayed:

- (i) The generalized (ie., higher dimensional) Kasner solution to general relativity.
- (ii) The generalized Kasner solution to general relativity with nonzero cosmological constant.
- (iii) The perturbative solution for string gravity to quadratic order in the curvature.
- (iv) The complete solution for one branch of five dimensional extended gravity.

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Appendix A: Identities involving the C^k Coefficients

The symbols C^k and C_m^k are defined to be the coefficients in the polynomial expressions:

$$\prod (x - \alpha_i) = \sum_k (-1)^k C^k(\alpha_i) x^{d-k-1} \quad (2.13)$$

$$\prod_{i \neq m} (x - \alpha_i) = \sum_k (-1)^k C_m^k(\alpha_i) x^{d-k-2} \quad (2.14)$$

In general, C^k is the sum of all possible products of k of the $d-1$ different α 's. In particular

$$C^1 = \alpha_1 + \alpha_2 + \dots + \alpha_{d-1} \quad (A.3)$$

$$C^2 = \alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \dots + \alpha_{d-2} \alpha_{d-1} \quad (A.4)$$

It is simple to derive a variety of useful relations between these coefficients. Some are:

$$C^k = \sum_i \alpha_i C_i^{k-1} \quad (A.5)$$

$$C^k = \alpha_1 C_1^{k-1} + C_1^k \quad (\text{no sum}) \quad (A.6)$$

$$\sum_i C_i^k = (d-k-1) C^k \quad (A.7)$$

$$\sum_i C_{ij}^k = (d-k-2) C_j^k \quad (A.8)$$

$$\sum_i \alpha_i C_i^{k-1} = k C^k \quad (A.9)$$

$$\frac{dC^k}{dt} = \sum_i \dot{\alpha}_i C_i^{k-1} \quad (A.10)$$

$$C_i^k - C_j^k = (\alpha_j - \alpha_i) C_{ij}^{k-1} \quad (A.11)$$

The following sequence of identities used to manipulate eq.(3.2) is a little more difficult:

$$\begin{aligned} \sum_{i \neq j} (\alpha_i)^2 C_{ij}^{k-2} &= \sum_{i \neq j} [\alpha_i C_j^{k-1} - C_j^k + C_{ij}^k] \\ &= C_j^1 C_j^k - (d-2) C_j^k + (d-k-2) C_j^k \\ &= C_j^1 C_j^k - k C_j^k \end{aligned} \quad (A.12)$$

Finally, the number of terms in any given expression is a convenient check on equations. Let $[E]$ be the number of terms in the expression E . Then for example:

$$[C^k] = \binom{d-1}{k} = \frac{(d-1)!}{k!(d-k-1)!} \quad (A.13)$$

$$[C_i^k] = \binom{d-2}{k} \quad (A.14)$$

$$[\sum \alpha_i C_i^k] = (d-1) \binom{d-2}{k} \quad (A.15)$$

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