December 2014

Gauge transformations of the biconformal connection

Gauge transformations of the biconformal connection

James T. Wheeler

jim.wheeler@usu.edu USU Department of physics, 4415 Old Main Hill, Logan, UT 84322-4415

Abstract

We study the changes of the biconformal gauge fields under the local rotational and dilatational gauge transformations. We find two tensors built from the symmetric parts ot the connection and the Weyl vector.

1 Definitions

In the conformally orthonormal frame our structure equations are

$$\begin{aligned}
\mathbf{d}\boldsymbol{\omega}^a_{\ b} &= \boldsymbol{\omega}^c_{\ b}\boldsymbol{\omega}^a_{\ c} + \Delta^{ah}_{gb}\eta_{hj}\mathbf{e}^j\mathbf{e}^g - \Delta^{ah}_{gb}\eta^{gi}\mathbf{f}_h\mathbf{f}_i + 2\Delta^{ah}_{gb}\Xi^{gi}_{jh}\mathbf{f}_i\mathbf{e}^j + \boldsymbol{\Omega}^a_{\ b} \\
\mathbf{d}\mathbf{e}^a &= \mathbf{e}^b\Theta^{ac}_{db}\boldsymbol{\tau}^d_{\ c} + \frac{1}{2}\eta_{cb}\mathbf{d}\eta^{ac}\mathbf{e}^b + \frac{1}{2}\mathbf{D}\eta^{ae}\mathbf{f}_e + \mathbf{T}^a \\
\mathbf{d}\mathbf{f}_a &= \Theta^{bc}_{da}\boldsymbol{\tau}^d_{\ c}\mathbf{f}_b + \frac{1}{2}\eta^{bc}\mathbf{d}\eta_{ab}\mathbf{f}_c - \frac{1}{2}\mathbf{D}\eta_{ac}\mathbf{e}^c + \mathbf{S}_a \\
\mathbf{d}\boldsymbol{\omega} &= \mathbf{e}^a\mathbf{f}_a + \boldsymbol{\Omega}
\end{aligned}$$

where the $\mathbf{e}^c$ components of $\mathbf{D}\eta^{ab}$ become

$$\begin{aligned}
\mathbf{D}\eta^{ab} &= \mathbf{d}\eta^{ab} + \eta^{cb}\boldsymbol{\alpha}^a_{\ c} + \eta^{ac}\boldsymbol{\alpha}^b_{\ c} - 2W_c\mathbf{e}^c\eta^{ab} \\
&= \mathbf{d}\eta^{ab} + \eta^{cb}\left(\boldsymbol{\sigma}^a_{\ c} + \boldsymbol{\mu}^a_{\ c}\right) + \eta^{ac}\left(\boldsymbol{\sigma}^b_{\ c} + \boldsymbol{\mu}^b_{\ c}\right) - 2W_c\mathbf{e}^c\eta^{ab} \\
&= \mathbf{d}\eta^{ab} + \left(\eta^{cb}\boldsymbol{\sigma}^a_{\ c} + \eta^{ac}\boldsymbol{\sigma}^b_{\ c}\right) + \eta^{cb}\boldsymbol{\mu}^a_{\ c} + \eta^{ac}\boldsymbol{\mu}^b_{\ c} - 2W_c\mathbf{e}^c\eta^{ab} \\
&= \mathbf{d}\eta^{ab} + 2\eta^{ac}\boldsymbol{\mu}^b_{\ c} - 2W_c\mathbf{e}^c\eta^{ab}
\end{aligned}$$

2 Lorentz transformations

We compute the gauge transformation properties of the different pieces of the connection. Notice first that we have the antisymmetry of the inhomogeneous part, $\bar{\Lambda}^b_{\ e}\partial^d\Lambda^e_{\ c}$, of the transformation. Using the relations

$$\begin{aligned}
\eta_{ab} &= \eta_{ef}\Lambda^e_{\ a}\Lambda^f_{\ b} \\
\delta^c_b &= \eta^{ca}\eta_{ef}\Lambda^e_{\ a}\Lambda^f_{\ b} \\
\delta^c_b\bar{\Lambda}^b_{\ d} &= \eta^{ca}\eta_{ef}\Lambda^e_{\ a}\Lambda^f_{\ b}\bar{\Lambda}^b_{\ d} \\
\bar{\Lambda}^c_{\ d} &= \eta^{ca}\eta_{de}\Lambda^e_{\ a} \\
\eta_{ac}\eta^{de}\bar{\Lambda}^c_{\ d} &= \Lambda^e_{\ a} \\
\eta^{de}\bar{\Lambda}^c_{\ d} &= \eta^{cd}\Lambda^e_{\ d}
\end{aligned}$$

we have

$$\begin{aligned}
\bar{\Lambda}^b{}_e \partial^d \Lambda^e{}_c &= \bar{\Lambda}^b{}_e \partial^d \left(\eta^{ef} \eta_{cg} \bar{\Lambda}^g{}_f \right) \\
&= \eta^{ef} \eta_{cg} \bar{\Lambda}^b{}_e \partial^d \bar{\Lambda}^g{}_f \\
&= \eta^{eb} \eta_{cg} \Lambda^f{}_e \partial^d \bar{\Lambda}^g{}_f \\
\bar{\Lambda}^b{}_e \partial^d \Lambda^e{}_c &= -\eta^{eb} \eta_{cg} \bar{\Lambda}^g{}_f \partial^d \Lambda^f{}_e \\
\delta^b_g \delta^e_c \bar{\Lambda}^g{}_f \partial^d \Lambda^f{}_e + \eta^{eb} \eta_{cg} \bar{\Lambda}^g{}_f \partial^d \Lambda^f{}_e &= 0 \\
\Xi^{eb}_{cg} \left(\bar{\Lambda}^g{}_f \partial^d \Lambda^f{}_e \right) &= 0
\end{aligned}$$

or equivalently,

$$\Theta^{eb}_{cg} \left(\bar{\Lambda}^g{}_f \partial^d \Lambda^f{}_e \right) = \bar{\Lambda}^b{}_f \partial^d \Lambda^f{}_c$$

Now consider the basis equation (dropping the involution terms):

$$\begin{aligned}
\mathbf{d}\mathbf{e}^a &= \mathbf{e}^b \Theta^{ac}_{db} \boldsymbol{\tau}^d{}_c + \frac{1}{2} \eta_{cb} \mathbf{d} \eta^{ac} \mathbf{e}^b + \frac{1}{2} \mathbf{D} \eta^{ae} \mathbf{f}_e + \mathbf{T}^a \\
\mathbf{d}_{(x)} \mathbf{e}^a + \mathbf{d}_{(y)} \mathbf{e}^a &= \mathbf{e}^b \left(\boldsymbol{\sigma}^a{}_b + \boldsymbol{\gamma}^a{}_b \right) + \frac{1}{2} \eta_{cb} \left(\mathbf{d}_{(x)} \eta^{ac} + \mathbf{d}_{(y)} \eta^{ac} \right) \mathbf{e}^b + \frac{1}{2} \left(\mathbf{d}_{(x)} \eta^{ab} + 2\eta^{ac} \boldsymbol{\mu}^b{}_c - 2W_c \mathbf{e}^c \eta^{ab} \right) \mathbf{f}_b \\
&\quad + T^{ab}{}_c \mathbf{f}_b \mathbf{e}^c + \frac{1}{2} T^a_{bc} \mathbf{e}^b \mathbf{e}^c
\end{aligned}$$

so we have two pieces,

$$\begin{aligned}
\mathbf{d}_{(x)} \mathbf{e}^a &= \mathbf{e}^b \boldsymbol{\sigma}^a{}_b + \frac{1}{2} \eta_{cb} \left(\mathbf{d}_{(x)} \eta^{ac} \right) \mathbf{e}^b + \frac{1}{2} T^a_{bc} \mathbf{e}^b \mathbf{e}^c \\
\mathbf{d}_{(y)} \mathbf{e}^a &= \left(-\gamma^a_c{}^{\ b} + \frac{1}{2} \eta_{cd} \partial^b \eta^{ad} - \frac{1}{2} \partial_c \eta^{ab} - \eta^{ad} \mu^b{}_{dc} + W_c \eta^{ab} + T^{ab}{}_c \right) \mathbf{f}_b \mathbf{e}^c
\end{aligned}$$

If we choose a different orthonormal basis, $\tilde{\mathbf{e}}^a = \Lambda^a{}_b \mathbf{e}^b$, $\tilde{\eta}_{ab} = \eta_{ab}$, and $\mathbf{f}_b = \bar{\Lambda}^c{}_b \mathbf{f}_c$, the first piece becomes

$$\begin{aligned}
\mathbf{d}_{(x)} \left(\Lambda^a{}_b \mathbf{e}^b \right) &= \left(\Lambda^b{}_c \mathbf{e}^c \right) \tilde{\boldsymbol{\sigma}}^a{}_b + \frac{1}{2} \eta_{cb} \left(\mathbf{d}_{(x)} \eta^{ac} \right) \left(\Lambda^b{}_d \mathbf{e}^d \right) + \frac{1}{2} T^a_{bc} \Lambda^b{}_d \Lambda^c{}_e \mathbf{e}^d \mathbf{e}^e \\
\Lambda^a{}_b \mathbf{d}_{(x)} \mathbf{e}^b &= \Lambda^b{}_c \mathbf{e}^c \left(\tilde{\boldsymbol{\sigma}}^a{}_b + \bar{\Lambda}^d{}_b \mathbf{d}_{(x)} \Lambda^a{}_d \right) + \frac{1}{2} \eta_{cb} \left(\mathbf{d}_{(x)} \eta^{ac} \right) \Lambda^b{}_d \mathbf{e}^d + \frac{1}{2} T^a_{bc} \Lambda^b{}_d \Lambda^c{}_e \mathbf{e}^d \mathbf{e}^e \\
\Lambda^a{}_b \left(\mathbf{e}^c \boldsymbol{\sigma}^b{}_c + \frac{1}{2} \eta_{dc} \left(\mathbf{d}_{(x)} \eta^{bd} \right) \mathbf{e}^c + \frac{1}{2} T^b_{cd} \mathbf{e}^c \mathbf{e}^d \right) &= \Lambda^b{}_c \mathbf{e}^c \left(\tilde{\boldsymbol{\sigma}}^a{}_b + \bar{\Lambda}^d{}_b \mathbf{d}_{(x)} \Lambda^a{}_d \right) + \frac{1}{2} \eta_{cb} \left(\mathbf{d}_{(x)} \eta^{ac} \right) \Lambda^b{}_d \mathbf{e}^d + \frac{1}{2} T^a_{bc} \Lambda^b{}_d \Lambda^c{}_e \mathbf{e}^d \mathbf{e}^e
\end{aligned}$$

The second is

$$\mathbf{d}_{(y)} \mathbf{e}^a = \left(-\gamma^a_c{}^{\ b} + \frac{1}{2} \eta_{cd} \partial^b \eta^{ad} - \frac{1}{2} \partial_c \eta^{ab} - \eta^{ad} \mu^b{}_{dc} + W_c \eta^{ab} + T^{ab}{}_c \right) \mathbf{f}_b \mathbf{e}^c$$

$$\begin{aligned}
\mathbf{d}_{(x)} \left(\Lambda^a{}_b \mathbf{e}^b \right) + \mathbf{d}_{(y)} \left(\Lambda^a{}_b \mathbf{e}^b \right) &= \left(\Lambda^b{}_c \mathbf{e}^c \right) \tilde{\boldsymbol{\sigma}}^a{}_b + \frac{1}{2} \tilde{T}^a_{bc} \left(\Lambda^b{}_d \mathbf{e}^d \right) \left(\Lambda^c{}_e \mathbf{e}^e \right) \\
&\quad + \left(-\tilde{\gamma}^a_c{}^{\ b} - \tilde{\eta}^{af} \tilde{\mu}^b{}_{fc} + \tilde{\eta}^{ab} \tilde{W}_c + \tilde{T}^{ab}{}_c \right) \left(\bar{\Lambda}^d{}_b \mathbf{f}_d \right) \left(\Lambda^c{}_e \mathbf{e}^e \right) \\
\Lambda^a{}_b \mathbf{e}^c \boldsymbol{\sigma}^b{}_c + \mathbf{d}_{(x)} \Lambda^a{}_b \mathbf{e}^b + \mathbf{d}_{(y)} \Lambda^a{}_b \mathbf{e}^b &= \mathbf{e}^c \tilde{\boldsymbol{\sigma}}^a{}_b \Lambda^b{}_c + \frac{1}{2} \tilde{T}^a_{bc} \Lambda^b{}_d \Lambda^c{}_e \mathbf{e}^d \mathbf{e}^e \\
&\quad + \left(-\tilde{\gamma}^a_c{}^{\ b} - \tilde{\eta}^{af} \tilde{\mu}^b{}_{fc} + \tilde{\eta}^{ab} \tilde{W}_c + \tilde{T}^{ab}{}_c \right) \bar{\Lambda}^d{}_b \Lambda^c{}_e \mathbf{f}_d \mathbf{e}^e \\
&\quad - \frac{1}{2} \Lambda^a{}_b T^b{}_{cd} \mathbf{e}^c \mathbf{e}^d + \left(\Lambda^a{}_b \gamma^b_c{}^{\ d} + \Lambda^a{}_b \eta^{be} \mu^d{}_{ec} - \Lambda^a{}_b \eta^{bd} W_c - \Lambda^a{}_b T^{bd}{}_c \right) \mathbf{f}_d \mathbf{e}^c
\end{aligned}$$

so that

$$
\begin{aligned}
0 &= \mathbf{e}^c\left(\tilde{\boldsymbol{\sigma}}^a{}_b\Lambda^b{}_c - \Lambda^a{}_b\left(\boldsymbol{\sigma}^b{}_c - \bar{\Lambda}^b{}_e\mathbf{d}_{(x)}\Lambda^e{}_c\right)\right) + \frac{1}{2}\left(\tilde{T}^a_{bc}\Lambda^b{}_d\Lambda^c{}_e - \Lambda^a{}_bT^b_{de}\right)\mathbf{e}^d\mathbf{e}^e \\
0 &= \left(\left(-\tilde{\gamma}^a_e{}^{\ \ b}\bar{\Lambda}^d{}_b\Lambda^e{}_c - \partial^d\Lambda^a{}_c + \Lambda^a{}_b\gamma^b_c{}^{\ \ d}\right) + \Lambda^a{}_b\eta^{be}\left(\mu^d{}_{ec} - \tilde{\mu}^g{}_{fh}\bar{\Lambda}^d{}_g\Lambda^h{}_c\Lambda^f{}_e\right)\right)\mathbf{f}_d\mathbf{e}^c \\
&\quad + \left(\left(\tilde{\eta}^{ab}\tilde{W}_e\bar{\Lambda}^d{}_b\Lambda^e{}_c - \Lambda^a{}_b\eta^{bd}W_c\right) + \left(\tilde{T}^{ab}{}_e\bar{\Lambda}^d{}_b\Lambda^e{}_c - \Lambda^a{}_bT^{bd}{}_c\right)\right)\mathbf{f}_d\mathbf{e}^c
\end{aligned}
$$

and therefore,

$$
\begin{aligned}
\tilde{\boldsymbol{\sigma}}^a{}_d &= \Lambda^a{}_b\boldsymbol{\sigma}^b{}_c\bar{\Lambda}^c{}_d - \Lambda^a{}_b\bar{\Lambda}^b{}_e\mathbf{d}_{(x)}\Lambda^e{}_c\bar{\Lambda}^c{}_d \\
\tilde{T}^a_{bc} &= \Lambda^a{}_bT^b_{de}\bar{\Lambda}^d{}_b\bar{\Lambda}^e{}_c \\
\tilde{\gamma}^a_e{}^{\ \ b}\bar{\Lambda}^d{}_b\Lambda^e{}_c &= \Lambda^a{}_b\gamma^b_c{}^{\ \ d} - \partial^d\Lambda^a{}_c \\
\tilde{\mu}^g{}_{fh}\bar{\Lambda}^d{}_g\Lambda^h{}_c\Lambda^f{}_e &= \mu^d{}_{ec} \\
\tilde{W}_e\Lambda^e{}_c &= W_c \\
\tilde{T}^{ab}{}_e\bar{\Lambda}^d{}_b\Lambda^e{}_c &= \Lambda^a{}_bT^{bd}{}_c
\end{aligned}
$$

Solving for the new objects,

$$
\begin{aligned}
\tilde{\boldsymbol{\sigma}}^a{}_d &= \Lambda^a{}_b\boldsymbol{\sigma}^b{}_c\bar{\Lambda}^c{}_d - \mathbf{d}_{(x)}\Lambda^a{}_c\bar{\Lambda}^c{}_d \\
\tilde{T}^a_{bc} &= \Lambda^a{}_bT^b_{de}\bar{\Lambda}^d{}_b\bar{\Lambda}^e{}_c \\
\tilde{\boldsymbol{\gamma}}^a{}_g &= \Lambda^a{}_b\boldsymbol{\gamma}^b{}_c\bar{\Lambda}^c{}_g - \bar{\Lambda}^c{}_g\mathbf{d}_{(y)}\Lambda^a{}_c \\
\tilde{\mu}^a_{bc} &= \mu^d{}_{ef}\Lambda^a{}_d\bar{\Lambda}^e{}_b\bar{\Lambda}^f{}_c \\
\tilde{W}_a &= W_b\bar{\Lambda}^b{}_a \\
\tilde{T}^{ab}{}_c &= \Lambda^a{}_d\Lambda^b{}_eT^{de}{}_f\bar{\Lambda}^f{}_c
\end{aligned}
$$

For the co-solder form,

$$
\begin{aligned}
\mathbf{d}_{(x)}\mathbf{f}_b + \mathbf{d}_{(y)}\mathbf{f}_b &= \boldsymbol{\gamma}^a{}_b\mathbf{f}_a + \frac{1}{2}S_b{}^{cd}\mathbf{f}_c\mathbf{f}_d \\
&\quad + \boldsymbol{\sigma}^a{}_b\mathbf{f}_a + \eta_{ae}\boldsymbol{\rho}^e_b\mathbf{e}^a - \eta_{ab}W^c\mathbf{f}_c\mathbf{e}^a + S_b{}^c{}_d\mathbf{f}_c\mathbf{e}^d
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
0 &= -\left(\mathbf{d}_{(x)}\left(\bar{\Lambda}^b{}_a\mathbf{f}_b\right) + \mathbf{d}_{(y)}\left(\bar{\Lambda}^b{}_a\mathbf{f}_b\right)\right) \\
&\quad \tilde{\boldsymbol{\gamma}}^b{}_a\left(\bar{\Lambda}^c{}_b\mathbf{f}_c\right) + \frac{1}{2}\tilde{S}_a{}^{cd}\left(\bar{\Lambda}^e{}_c\mathbf{f}_e\right)\left(\bar{\Lambda}^f{}_d\mathbf{f}_f\right) \\
&\quad + \tilde{\boldsymbol{\sigma}}^b{}_a\left(\bar{\Lambda}^c{}_b\mathbf{f}_c\right) + \tilde{\eta}_{de}\tilde{\boldsymbol{\rho}}^e{}_a\left(\Lambda^d{}_c\mathbf{e}^c\right) - \eta_{ab}\tilde{W}^c\left(\bar{\Lambda}^e{}_c\mathbf{f}_e\right)\left(\Lambda^b{}_d\mathbf{e}^d\right) \\
&\quad + \tilde{S}_a{}^c{}_d\left(\bar{\Lambda}^e{}_c\mathbf{f}_e\right)\left(\Lambda^d{}_f\mathbf{e}^f\right) \\
0 &= \left(\Lambda^b{}_d\boldsymbol{\gamma}^d{}_c\bar{\Lambda}^c{}_a - \bar{\Lambda}^c{}_a\mathbf{d}_{(y)}\Lambda^b{}_c\right)\bar{\Lambda}^e{}_b\mathbf{f}_e + \frac{1}{2}\tilde{S}_a{}^{cd}\bar{\Lambda}^e{}_c\bar{\Lambda}^f{}_d\mathbf{f}_e\mathbf{f}_f \\
&\quad + \tilde{S}_a{}^c{}_d\bar{\Lambda}^e{}_c\Lambda^d{}_f\mathbf{f}_e\mathbf{e}^f - \mathbf{d}_{(x)}\bar{\Lambda}^b{}_a\mathbf{f}_b - \mathbf{d}_{(y)}\bar{\Lambda}^b{}_a\mathbf{f}_b \\
&\quad - \bar{\Lambda}^b{}_a\left(\boldsymbol{\gamma}^c{}_b\mathbf{f}_c + \frac{1}{2}S_b{}^{cd}\mathbf{f}_c\mathbf{f}_d + \boldsymbol{\sigma}^c{}_b\mathbf{f}_c + \eta_{ce}\boldsymbol{\rho}^e_b\mathbf{e}^c - \eta_{bd}W^c\mathbf{f}_c\mathbf{e}^d + S_b{}^c{}_d\mathbf{f}_c\mathbf{e}^d\right) \\
&\quad + \left(\Lambda^b{}_d\boldsymbol{\sigma}^d{}_c\bar{\Lambda}^c{}_a - \mathbf{d}_{(x)}\Lambda^b{}_c\bar{\Lambda}^c{}_a\right)\bar{\Lambda}^e{}_b\mathbf{f}_e + \tilde{\eta}_{de}\tilde{\boldsymbol{\rho}}^e{}_a\left(\Lambda^d{}_c\mathbf{e}^c\right) - \eta_{ab}\tilde{W}^c\bar{\Lambda}^e{}_c\Lambda^b{}_d\mathbf{f}_e\mathbf{e}^d \\
&= \frac{1}{2}\tilde{S}_a{}^{cd}\bar{\Lambda}^e{}_c\bar{\Lambda}^f{}_d\mathbf{f}_e\mathbf{f}_f - \frac{1}{2}\bar{\Lambda}^b{}_aS_b{}^{ef}\mathbf{f}_e\mathbf{f}_f + \tilde{S}_a{}^c{}_d\bar{\Lambda}^e{}_c\Lambda^d{}_f\mathbf{f}_e\mathbf{e}^f - \bar{\Lambda}^b{}_aS_b{}^c{}_d\mathbf{f}_c\mathbf{e}^d \\
&\quad + \left(\bar{\Lambda}^d{}_e\tilde{\boldsymbol{\rho}}^e{}_a - \bar{\Lambda}^b{}_a\boldsymbol{\rho}^d_b\right)\eta_{cd}\mathbf{e}^c + \left(\bar{\Lambda}^b{}_a\eta_{bd}W^c - \eta_{ab}\Lambda^b{}_d\left(\tilde{W}^e\bar{\Lambda}^c{}_e\right)\right)\mathbf{f}_c\mathbf{e}^d
\end{aligned}
$$

Then

$$\begin{aligned}
\tilde{S}_a^{\ cd} &= \bar{\Lambda}^b_{\ a} S_b^{\ ef} \Lambda^c_{\ e} \Lambda^d_{\ f} \\
\tilde{S}_a^{\ c}{}_{d} &= \bar{\Lambda}^b_{\ a} S_b^{\ e}{}_{f} \Lambda^c_{\ e} \bar{\Lambda}^f_{\ d} \\
\tilde{\boldsymbol{\rho}}^d_{\ a} &= \Lambda^d_{\ e} \boldsymbol{\rho}^e_{\ b} \bar{\Lambda}^b_{\ a} \\
\tilde{W}^a &= W^b \Lambda^a_{\ b}
\end{aligned}$$

In conclusion, we have the Lorentz (rotational) connections

$$\begin{aligned}
\tilde{\boldsymbol{\sigma}}^a_{\ d} &= \Lambda^a_{\ b} \boldsymbol{\sigma}^b_{\ c} \bar{\Lambda}^c_{\ d} - \mathbf{d}_{(x)} \Lambda^a_{\ c} \bar{\Lambda}^c_{\ d} \\
\tilde{\boldsymbol{\gamma}}^a_{\ g} &= \Lambda^a_{\ b} \boldsymbol{\gamma}^b_{\ c} \bar{\Lambda}^c_{\ g} - \bar{\Lambda}^c_{\ g} \mathbf{d}_{(y)} \Lambda^a_{\ c}
\end{aligned}$$

and the tensors

$$\begin{aligned}
\tilde{T}^{ab}_{\ \ c} &= \Lambda^a_{\ d} \Lambda^b_{\ e} T^{de}_{\ \ f} \bar{\Lambda}^f_{\ c} \\
\tilde{T}^a_{\ bc} &= \Lambda^a_{\ b} T^b_{\ de} \bar{\Lambda}^d_{\ b} \bar{\Lambda}^e_{\ c} \\
\tilde{S}_a^{\ cd} &= \bar{\Lambda}^b_{\ a} S_b^{\ ef} \Lambda^c_{\ e} \Lambda^d_{\ f} \\
\tilde{S}_a^{\ c}{}_{d} &= \bar{\Lambda}^b_{\ a} S_b^{\ e}{}_{f} \Lambda^c_{\ e} \bar{\Lambda}^f_{\ d} \\
\tilde{\mu}^a_{\ bc} &= \mu^d_{\ ef} \Lambda^a_{\ d} \bar{\Lambda}^e_{\ b} \bar{\Lambda}^f_{\ c} \\
\tilde{\boldsymbol{\rho}}^d_{\ a} &= \Lambda^d_{\ e} \boldsymbol{\rho}^e_{\ b} \bar{\Lambda}^b_{\ a} \\
\tilde{W}_a &= W_b \bar{\Lambda}^b_{\ a} \\
\tilde{W}^a &= W^b \Lambda^a_{\ b}
\end{aligned}$$

3 Dilatations

Once again, the structure equations in the new basis are

$$\begin{aligned}
\mathbf{d}\boldsymbol{\omega}^a_{\ b} &= \boldsymbol{\omega}^c_{\ b} \boldsymbol{\omega}^a_{\ c} + \Delta^{ah}_{gb} \eta_{hj} \mathbf{e}^j \mathbf{e}^g - \Delta^{ah}_{gb} \eta^{gi} \mathbf{f}_h \mathbf{f}_i + 2\Delta^{ah}_{gb} \Xi^{gi}_{jh} \mathbf{f}_i \mathbf{e}^j + \boldsymbol{\Omega}^a_{\ b} \\
\mathbf{d}\mathbf{e}^a &= \mathbf{e}^b \Theta^{ac}_{db} \boldsymbol{\tau}^d_{\ c} + \frac{1}{2} \eta_{cb} \mathbf{d}\eta^{ac} \mathbf{e}^b + \frac{1}{2} \mathbf{D}\eta^{ae} \mathbf{f}_e + \mathbf{T}^a \\
\mathbf{d}\mathbf{f}_a &= \Theta^{bc}_{da} \boldsymbol{\tau}^d_{\ c} \mathbf{f}_b + \frac{1}{2} \eta^{bc} \mathbf{d}\eta_{ab} \mathbf{f}_c - \frac{1}{2} \mathbf{D}\eta_{ac} \mathbf{e}^c + \mathbf{S}_a \\
\mathbf{d}\boldsymbol{\omega} &= \mathbf{e}^a \mathbf{f}_a + \boldsymbol{\Omega}
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{D}\eta^{ab} &= \mathbf{d}\eta^{ab} + \eta^{cb} \boldsymbol{\tau}^a_{\ c} + \eta^{ac} \boldsymbol{\tau}^b_{\ c} - 2\boldsymbol{\omega}\eta^{ab} \\
&= \mathbf{d}\eta^{ab} + \eta^{cb} \left(\boldsymbol{\alpha}^a_{\ c} + \boldsymbol{\beta}^a_{\ c} \right) + \eta^{ac} \left(\boldsymbol{\alpha}^b_{\ c} + \boldsymbol{\beta}^b_{\ c} \right) - 2\boldsymbol{\omega}\eta^{ab} \\
&= \mathbf{d}\eta^{ab} + \eta^{cb} \boldsymbol{\alpha}^a_{\ c} + \eta^{ac} \boldsymbol{\alpha}^b_{\ c} + \eta^{cb} \boldsymbol{\beta}^a_{\ c} + \eta^{ac} \boldsymbol{\beta}^b_{\ c} - 2\boldsymbol{\omega}\eta^{ab} \\
&= \mathbf{d}\eta^{ab} + 2\eta^{ac} \boldsymbol{\beta}^b_{\ c} - 2\boldsymbol{\omega}\eta^{ab}
\end{aligned}$$

Under conformal transformation, the covariant derivative gives

$$\begin{aligned}
\mathbf{D}\tilde{\eta}^{ab} &= \mathbf{d}\left(e^{2\phi} \eta^{ab} \right) + 2\left(e^{2\phi} \eta^{ac} \right) \tilde{\boldsymbol{\beta}}^b_{\ c} - 2\left(\boldsymbol{\omega} + \mathbf{d}\phi \right) \left(e^{2\phi} \eta^{ab} \right) \\
&= e^{2\phi} \mathbf{d}\eta^{ab} + 2e^{2\phi} \eta^{ab} \mathbf{d}\phi + 2e^{2\phi} \eta^{ac} \tilde{\boldsymbol{\beta}}^b_{\ c} - 2e^{2\phi} \eta^{ab} \boldsymbol{\omega} - 2e^{2\phi} \eta^{ab} \mathbf{d}\phi \\
&= e^{2\phi} \left(\mathbf{d}\eta^{ab} + 2\eta^{ac} \tilde{\boldsymbol{\beta}}^b_{\ c} - 2\eta^{ab} \boldsymbol{\omega} \right) \\
&= e^{2\phi} \mathbf{D}\eta^{ab}
\end{aligned}$$

as long as $\tilde{\boldsymbol{\beta}}^b_{\ c} = \boldsymbol{\beta}^b_{\ c}$.

Now look at the solder form equation before and after a dilatation,

$$\begin{aligned}
\mathbf{d}\mathbf{e}^a &= \mathbf{e}^b \Theta^{ac}_{db} \boldsymbol{\tau}^d_{\ c} + \frac{1}{2}\eta_{cb}\mathbf{d}\eta^{ac}\mathbf{e}^b + \frac{1}{2}\mathbf{D}\eta^{ae}\mathbf{f}_e + \mathbf{T}^a \\
\mathbf{d}\left(e^{\phi}\mathbf{e}^a\right) &= \left(e^{\phi}\mathbf{e}^b\right)\Theta^{ac}_{db}\tilde{\boldsymbol{\tau}}^d_{\ c} + \frac{1}{2}e^{-2\phi}\eta_{cb}\mathbf{d}\left(e^{2\phi}\eta^{ac}\right)e^{\phi}\mathbf{e}^b + \frac{1}{2}e^{2\phi}\left(\mathbf{D}\eta^{ae}\right)e^{-\phi}\mathbf{f}_e + e^{\phi}\mathbf{T}^a
\end{aligned}$$

Subtracting e^{ϕ} times the first from the second,

$$\begin{aligned}
\mathbf{d}\left(e^{\phi}\mathbf{e}^a\right) - e^{\phi}\mathbf{d}\mathbf{e}^a &= e^{\phi}\mathbf{e}^b\Theta^{ac}_{db}\tilde{\boldsymbol{\tau}}^d_{\ c} + \frac{1}{2}e^{-2\phi}\eta_{cb}\mathbf{d}\left(e^{2\phi}\eta^{ac}\right)e^{\phi}\mathbf{e}^b + \frac{1}{2}e^{2\phi}\left(\mathbf{D}\eta^{ae}\right)e^{-\phi}\mathbf{f}_e + e^{\phi}\mathbf{T}^a \\
&\quad - e^{\phi}\mathbf{e}^b\Theta^{ac}_{db}\boldsymbol{\tau}^d_{\ c} - \frac{1}{2}e^{\phi}\eta_{cb}\mathbf{d}\eta^{ac}\mathbf{e}^b - \frac{1}{2}e^{\phi}\mathbf{D}\eta^{ae}\mathbf{f}_e - e^{\phi}\mathbf{T}^a \\
0 &= e^{\phi}\mathbf{e}^b\Theta^{ac}_{db}\tilde{\boldsymbol{\tau}}^d_{\ c} + e^{\phi}\mathbf{e}^a\mathbf{d}\phi - e^{\phi}\mathbf{e}^b\Theta^{ac}_{db}\boldsymbol{\tau}^d_{\ c} + \frac{1}{2}\eta_{cb}\left(2\eta^{ac}\mathbf{d}\phi + \mathbf{d}\eta^{ac}\right)e^{\phi}\mathbf{e}^b - \frac{1}{2}e^{\phi}\eta_{cb}\mathbf{d}\eta^{ac}\mathbf{e}^b \\
&= e^{\phi}\mathbf{e}^b\left(\Theta^{ac}_{db}\tilde{\boldsymbol{\tau}}^d_{\ c} - \Theta^{ac}_{db}\boldsymbol{\tau}^d_{\ c}\right) + e^{\phi}\mathbf{e}^a\mathbf{d}\phi + \mathbf{d}\phi e^{\phi}\mathbf{e}^a + \frac{1}{2}e^{\phi}\eta_{cb}\mathbf{d}\eta^{ac}\mathbf{e}^b - \frac{1}{2}e^{\phi}\eta_{cb}\mathbf{d}\eta^{ac}\mathbf{e}^b \\
&= e^{\phi}\mathbf{e}^b\Theta^{ac}_{db}\left(\tilde{\boldsymbol{\tau}}^d_{\ c} - \boldsymbol{\tau}^d_{\ c}\right)
\end{aligned}$$

so the equation is satisfied with the connection invariant under dilatations.

A corresponding result holds for the co-solder form. Writing the structure equation before and after dilatation,

$$\begin{aligned}
\mathbf{d}\mathbf{f}_a &= \Theta^{bc}_{da}\boldsymbol{\tau}^d_{\ c}\mathbf{f}_b + \frac{1}{2}\eta^{bc}\mathbf{d}\eta_{ab}\mathbf{f}_c - \frac{1}{2}\mathbf{D}\eta_{ac}\mathbf{e}^c + \mathbf{S}_a \\
e^{-\phi}\mathbf{d}\mathbf{f}_a - \mathbf{d}\phi e^{-\phi}\mathbf{f}_a &= \Theta^{bc}_{da}\tilde{\boldsymbol{\tau}}^d_{\ c}e^{-\phi}\mathbf{f}_b + \frac{1}{2}\eta^{bc}\left(-2\mathbf{d}\phi e^{-2\phi}\eta_{ab} + e^{-2\phi}\mathbf{d}\eta_{ab}\right)\mathbf{f}_c - \frac{1}{2}e^{-2\phi}\mathbf{D}\eta_{ac}e^{\phi}\mathbf{e}^c + \tilde{\mathbf{S}}_a
\end{aligned}$$

so subtracting,

$$\begin{aligned}
\mathbf{d}\mathbf{f}_a &= \Theta^{bc}_{da}\boldsymbol{\tau}^d_{\ c}\mathbf{f}_b + \frac{1}{2}\eta^{bc}\mathbf{d}\eta_{ab}\mathbf{f}_c - \frac{1}{2}\mathbf{D}\eta_{ac}\mathbf{e}^c + \mathbf{S}_a \\
e^{-\phi}\mathbf{d}\mathbf{f}_a - \mathbf{d}\phi e^{-\phi}\mathbf{f}_a - e^{-\phi}\mathbf{d}\mathbf{f}_a &= e^{-\phi}\left(\Theta^{bc}_{da}\tilde{\boldsymbol{\tau}}^d_{\ c} - \Theta^{bc}_{da}\boldsymbol{\tau}^d_{\ c}\right)\mathbf{f}_b + \frac{1}{2}\eta^{bc}\mathbf{d}\eta_{ab}e^{-\phi}\mathbf{f}_c - \frac{1}{2}e^{-2\phi}\mathbf{D}\eta_{ac}e^{\phi}\mathbf{e}^c + \tilde{\mathbf{S}}_a \\
&\quad - \eta^{bc}\mathbf{d}\phi\eta_{ab}e^{-\phi}\mathbf{f}_c - \frac{1}{2}e^{-\phi}\eta^{bc}\mathbf{d}\eta_{ab}\mathbf{f}_c + \frac{1}{2}e^{-\phi}\mathbf{D}\eta_{ac}\mathbf{e}^c - e^{-\phi}\mathbf{S}_a \\
-\mathbf{d}\phi e^{-\phi}\mathbf{f}_a &= e^{-\phi}\left(\Theta^{bc}_{da}\tilde{\boldsymbol{\tau}}^d_{\ c} - \Theta^{bc}_{da}\boldsymbol{\tau}^d_{\ c}\right)\mathbf{f}_b - \mathbf{d}\phi e^{-\phi}\mathbf{f}_a + \left(\tilde{\mathbf{S}}_a - e^{-\phi}\mathbf{S}_a\right) \\
0 &= \left(\Theta^{bc}_{da}\tilde{\boldsymbol{\tau}}^d_{\ c} - \Theta^{bc}_{da}\boldsymbol{\tau}^d_{\ c}\right)\mathbf{f}_b
\end{aligned}$$

Now consider the $\mathbf{e}^c$ and $\mathbf{f}_c$ components of the structure equations separately,

3.1 Configuration submanifold

Consider the subparts of the solder form structure equation,

$$\begin{aligned}
\mathbf{d}\mathbf{e}^a &= \mathbf{e}^b\Theta^{ac}_{db}\boldsymbol{\tau}^d_{\ c} + \frac{1}{2}\eta_{cb}\mathbf{d}\eta^{ac}\mathbf{e}^b + \frac{1}{2}\mathbf{D}\eta^{ae}\mathbf{f}_e + \mathbf{T}^a \\
\mathbf{d}_{(x)}\mathbf{e}^a + \mathbf{d}_{(y)}\mathbf{e}^a &= \mathbf{e}^b\boldsymbol{\sigma}^a_{\ b} + \mathbf{e}^b\boldsymbol{\gamma}^a_{\ b} + \frac{1}{2}\eta_{cb}\left(\mathbf{d}_{(x)}\eta^{ac} + \mathbf{d}_{(y)}\eta^{ac}\right)\mathbf{e}^b \\
&\quad + \frac{1}{2}\mathbf{D}\eta^{ae}\mathbf{f}_e + \frac{1}{2}T^a_{\ bc}\mathbf{e}^b\mathbf{e}^c + T^{ab}_{\ \ c}\mathbf{f}_b\mathbf{e}^c
\end{aligned}$$

yielding the two pieces

$$\begin{aligned}\mathbf{d}_{(x)}\mathbf{e}^a &= \mathbf{e}^b\boldsymbol{\sigma}^a_{\ b} + \frac{1}{2}\eta_{cb}\mathbf{d}_{(x)}\eta^{ac}\mathbf{e}^b + \frac{1}{2}T^a_{\ bc}\mathbf{e}^b\mathbf{e}^c \\ \mathbf{d}_{(y)}\mathbf{e}^a &= \mathbf{e}^b\boldsymbol{\gamma}^a_{\ b} + \frac{1}{2}\eta_{cb}\mathbf{d}_{(y)}\eta^{ac}\mathbf{e}^b + \frac{1}{2}\mathbf{D}\eta^{ae}\mathbf{f}_e + T^{ab}_{\ \ c}\mathbf{f}_b\mathbf{e}^c\end{aligned}$$

where any $\mathbf{f}_b\mathbf{f}_c$ part vanishes separately by involution. After gauging,

$$\begin{aligned}\mathbf{d}\left(e^\phi\mathbf{e}^a\right) &= \left(e^\phi\mathbf{e}^b\right)\Theta^{ac}_{db}\tilde{\boldsymbol{\tau}}^d_{\ c} + \frac{1}{2}e^{-2\phi}\eta_{cb}\mathbf{d}\left(e^{2\phi}\eta^{ac}\right)e^\phi\mathbf{e}^b + \frac{1}{2}e^{2\phi}\left(\mathbf{D}\eta^{ae}\right)e^{-\phi}\mathbf{f}_e + e^\phi\mathbf{T}^a \\ e^\phi\mathbf{d}_{(x)}\mathbf{e}^a + e^\phi\mathbf{d}_{(y)}\mathbf{e}^a &= e^\phi\mathbf{e}^b\tilde{\boldsymbol{\sigma}}^a_{\ b} + e^\phi\mathbf{e}^b\tilde{\boldsymbol{\gamma}}^a_{\ b} - e^\phi\mathbf{d}_{(x)}\phi\mathbf{e}^a - \mathbf{d}_{(y)}\phi e^\phi\mathbf{e}^a \\ &\quad+\frac{1}{2}e^{-\phi}\eta_{cb}\left(2\mathbf{d}_{(x)}\phi e^{2\phi}\eta^{ac} + e^{2\phi}\mathbf{d}_{(x)}\eta^{ac} + 2\mathbf{d}_{(y)}\phi e^{2\phi}\eta^{ac} + e^{2\phi}\mathbf{d}_{(y)}\eta^{ac}\right)\mathbf{e}^b \\ &\quad+\frac{1}{2}e^\phi\left(\mathbf{D}\eta^{ae}\right)\mathbf{f}_e + \frac{1}{2}e^\phi T^a_{\ bc}\mathbf{e}^b\mathbf{e}^c + e^\phi T^{ab}_{\ \ c}\mathbf{f}_b\mathbf{e}^c\end{aligned}$$

This also gives two pieces:

$$\begin{aligned}e^\phi\mathbf{d}_{(x)}\mathbf{e}^a &= e^\phi\mathbf{e}^b\tilde{\boldsymbol{\sigma}}^a_{\ b} + \frac{1}{2}e^\phi\eta_{cb}\left(\mathbf{d}_{(x)}\eta^{ac}\right)\mathbf{e}^b + \frac{1}{2}e^\phi T^a_{\ bc}\mathbf{e}^b\mathbf{e}^c \\ e^\phi\mathbf{d}_{(y)}\mathbf{e}^a &= e^\phi\mathbf{e}^b\tilde{\boldsymbol{\gamma}}^a_{\ b} + \frac{1}{2}e^\phi\eta_{cb}\left(\mathbf{d}_{(y)}\eta^{ac}\right)\mathbf{e}^b + \frac{1}{2}e^\phi\left(\mathbf{D}\eta^{ae}\right)\mathbf{f}_e + e^\phi T^{ab}_{\ \ c}\mathbf{f}_b\mathbf{e}^c\end{aligned}$$

and taking the difference of the corresponding pieces, we have

$$\begin{aligned}e^\phi\mathbf{d}_{(x)}\mathbf{e}^a - e^\phi\mathbf{d}_{(x)}\mathbf{e}^a &= e^\phi\mathbf{e}^b\tilde{\boldsymbol{\sigma}}^a_{\ b} + \frac{1}{2}e^\phi\eta_{cb}\left(\mathbf{d}_{(x)}\eta^{ac}\right)\mathbf{e}^b + \frac{1}{2}e^\phi T^a_{\ bc}\mathbf{e}^b\mathbf{e}^c \\ &\quad-e^\phi\mathbf{e}^b\boldsymbol{\sigma}^a_{\ b} - \frac{1}{2}e^\phi\eta_{cb}\mathbf{d}_{(x)}\eta^{ac}\mathbf{e}^b - \frac{1}{2}e^\phi T^a_{\ bc}\mathbf{e}^b\mathbf{e}^c \\ 0 &= e^\phi\mathbf{e}^b\left(\tilde{\boldsymbol{\sigma}}^a_{\ b} - \boldsymbol{\sigma}^a_{\ b}\right) \\ e^\phi\mathbf{d}_{(y)}\mathbf{e}^a - e^\phi\mathbf{d}_{(y)}\mathbf{e}^a &= e^\phi\mathbf{e}^b\tilde{\boldsymbol{\gamma}}^a_{\ b} + \frac{1}{2}e^\phi\eta_{cb}\left(\mathbf{d}_{(y)}\eta^{ac}\right)\mathbf{e}^b + \frac{1}{2}e^\phi\left(\mathbf{D}\eta^{ae}\right)\mathbf{f}_e + e^\phi T^{ab}_{\ \ c}\mathbf{f}_b\mathbf{e}^c \\ &\quad-e^\phi\mathbf{e}^b\boldsymbol{\gamma}^a_{\ b} - \frac{1}{2}e^\phi\eta_{cb}\mathbf{d}_{(y)}\eta^{ac}\mathbf{e}^b - \frac{1}{2}e^\phi\mathbf{D}\eta^{ae}\mathbf{f}_e - e^\phi T^{ab}_{\ \ c}\mathbf{f}_b\mathbf{e}^c \\ &= e^\phi\mathbf{e}^b\left(\tilde{\boldsymbol{\gamma}}^a_{\ b} - \boldsymbol{\gamma}^a_{\ b}\right)\end{aligned}$$

Again we find the components of the spin connection unaltered by dilatation.

3.2 Co-solder structure equation

Repeating for the co-solder equation,

$$\begin{aligned}\mathbf{d}\mathbf{f}_a &= \Theta^{bc}_{da}\boldsymbol{\tau}^d_{\ c}\mathbf{f}_b + \frac{1}{2}\eta^{bc}\mathbf{d}\eta_{ab}\mathbf{f}_c - \frac{1}{2}\mathbf{D}\eta_{ac}\mathbf{e}^c + \mathbf{S}_a \\ \mathbf{d}_{(x)}\mathbf{f}_a + \mathbf{d}_{(y)}\mathbf{f}_a &= \boldsymbol{\sigma}^b_{\ a}\mathbf{f}_b + \boldsymbol{\gamma}^b_{\ a}\mathbf{f}_b + \frac{1}{2}\eta^{bc}\mathbf{d}_{(x)}\eta_{ab}\mathbf{f}_c + \frac{1}{2}\eta^{bc}\mathbf{d}_{(y)}\eta_{ab}\mathbf{f}_c - \frac{1}{2}\mathbf{D}\eta_{ac}\mathbf{e}^c + \mathbf{S}_a\end{aligned}$$

with pieces

$$\begin{aligned}\mathbf{d}_{(x)}\mathbf{f}_a &= \boldsymbol{\sigma}^b_{\ a}\mathbf{f}_b + \frac{1}{2}\eta^{bc}\mathbf{d}_{(x)}\eta_{ab}\mathbf{f}_c - \frac{1}{2}\mathbf{D}\eta_{ac}\mathbf{e}^c + S_{a\ c}^{\ b}\mathbf{f}_b\mathbf{e}^c \\ \mathbf{d}_{(y)}\mathbf{f}_a &= \boldsymbol{\gamma}^b_{\ a}\mathbf{f}_b + \frac{1}{2}\eta^{bc}\mathbf{d}_{(y)}\eta_{ab}\mathbf{f}_c + \frac{1}{2}S_a^{\ bc}\mathbf{f}_b\mathbf{f}_c\end{aligned}$$

and after gauging,

$$\begin{aligned}
\mathbf{d}_{(x)}\left(e^{-\phi}\mathbf{f}_a\right)+\mathbf{d}_{(y)}\left(e^{-\phi}\mathbf{f}_a\right) &= e^{-\phi}\tilde{\boldsymbol{\sigma}}^b{}_a\mathbf{f}_b+\tilde{\boldsymbol{\gamma}}^b{}_a e^{-\phi}\mathbf{f}_b+\frac{1}{2}e^{2\phi}\eta^{bc}\mathbf{d}_{(x)}\left(e^{-2\phi}\eta_{ab}\right)e^{-\phi}\mathbf{f}_c \\
&\quad +\frac{1}{2}e^{2\phi}\eta^{bc}\mathbf{d}_{(y)}\left(e^{-2\phi}\eta_{ab}\right)e^{-\phi}\mathbf{f}_c-\frac{1}{2}e^{-\phi}\mathbf{D}\eta_{ac}\mathbf{e}^c+\tilde{\mathbf{S}}_a
\end{aligned}$$

with independent parts,

$$\begin{aligned}
-\mathbf{d}_{(x)}\phi e^{-\phi}\mathbf{f}_a+e^{-\phi}\mathbf{d}_{(x)}\mathbf{f}_a &= e^{-\phi}\tilde{\boldsymbol{\sigma}}^b{}_a\mathbf{f}_b-\delta^c_a\mathbf{d}_{(x)}\phi e^{-\phi}\mathbf{f}_c+\frac{1}{2}\eta^{bc}\mathbf{d}_{(x)}\eta_{ab}e^{-\phi}\mathbf{f}_c-\frac{1}{2}e^{-\phi}\mathbf{D}\eta_{ac}\mathbf{e}^c+\tilde{S}_{a\ c}^{\ b}\mathbf{f}_b\mathbf{e}^c \\
e^{-\phi}\mathbf{d}_{(x)}\mathbf{f}_a &= e^{-\phi}\tilde{\boldsymbol{\sigma}}^b{}_a\mathbf{f}_b+\frac{1}{2}e^{-\phi}\eta^{bc}\mathbf{d}_{(x)}\eta_{ab}\mathbf{f}_c-\frac{1}{2}e^{-\phi}\mathbf{D}\eta_{ac}\mathbf{e}^c+\tilde{S}_{a\ c}^{\ b}\mathbf{f}_b\mathbf{e}^c \\
-\mathbf{d}_{(y)}\phi e^{-\phi}\mathbf{f}_a+e^{-\phi}\mathbf{d}_{(y)}\mathbf{f}_a &= e^{-\phi}\tilde{\boldsymbol{\gamma}}^b{}_a\mathbf{f}_b-e^{-\phi}\mathbf{d}_{(y)}\phi\mathbf{f}_a+\frac{1}{2}e^{-\phi}\eta^{bc}\mathbf{d}_{(y)}\eta_{ab}\mathbf{f}_c+\frac{1}{2}\tilde{S}_a^{\ bc}\mathbf{f}_b\mathbf{f}_c \\
e^{-\phi}\mathbf{d}_{(y)}\mathbf{f}_a &= e^{-\phi}\tilde{\boldsymbol{\gamma}}^b{}_a\mathbf{f}_b+\frac{1}{2}e^{-\phi}\eta^{bc}\mathbf{d}_{(y)}\eta_{ab}\mathbf{f}_c+\frac{1}{2}\tilde{S}_a^{\ bc}\mathbf{f}_b\mathbf{f}_c
\end{aligned}$$

so the differences give:

$$\begin{aligned}
e^{-\phi}\mathbf{d}_{(x)}\mathbf{f}_a-e^{-\phi}\mathbf{d}_{(x)}\mathbf{f}_a &= e^{-\phi}\tilde{\boldsymbol{\sigma}}^b{}_a\mathbf{f}_b+\frac{1}{2}e^{-\phi}\eta^{bc}\mathbf{d}_{(x)}\eta_{ab}\mathbf{f}_c-\frac{1}{2}e^{-\phi}\mathbf{D}\eta_{ac}\mathbf{e}^c+\tilde{S}_{a\ c}^{\ b}\mathbf{f}_b\mathbf{e}^c \\
&\quad -e^{-\phi}\boldsymbol{\sigma}^b{}_a\mathbf{f}_b-\frac{1}{2}e^{-\phi}\eta^{bc}\mathbf{d}_{(x)}\eta_{ab}\mathbf{f}_c+\frac{1}{2}e^{-\phi}\mathbf{D}\eta_{ac}\mathbf{e}^c-e^{-\phi}S_{a\ c}^{\ b}\mathbf{f}_b\mathbf{e}^c \\
0 &= e^{-\phi}\left(\tilde{\boldsymbol{\sigma}}^b{}_a-\boldsymbol{\sigma}^b{}_a\right)\mathbf{f}_b+\left(\tilde{S}_{a\ c}^{\ b}-e^{-\phi}S_{a\ c}^{\ b}\right)\mathbf{f}_b\mathbf{e}^c
\end{aligned}$$

and

$$\begin{aligned}
e^{-\phi}\mathbf{d}_{(y)}\mathbf{f}_a-e^{-\phi}\mathbf{d}_{(y)}\mathbf{f}_a &= e^{-\phi}\tilde{\boldsymbol{\gamma}}^b{}_a\mathbf{f}_b+\frac{1}{2}e^{-\phi}\eta^{bc}\mathbf{d}_{(y)}\eta_{ab}\mathbf{f}_c+\frac{1}{2}\tilde{S}_a^{\ bc}\mathbf{f}_b\mathbf{f}_c \\
&\quad -e^{-\phi}\boldsymbol{\gamma}^b{}_a\mathbf{f}_b-\frac{1}{2}e^{-\phi}\eta^{bc}\mathbf{d}_{(y)}\eta_{ab}\mathbf{f}_c-\frac{1}{2}e^{-\phi}S_a^{\ bc}\mathbf{f}_b\mathbf{f}_c \\
0 &= e^{-\phi}\left(\tilde{\boldsymbol{\gamma}}^b{}_a-\boldsymbol{\gamma}^b{}_a\right)\mathbf{f}_b+\frac{1}{2}\left(\tilde{S}_a^{\ bc}-e^{-\phi}S_a^{\ bc}\right)\mathbf{f}_b\mathbf{f}_c
\end{aligned}$$

so the spin connection is unchanged and the co-torsion dilates as a weight -1 tensor.

4 Summary of gauge transformations

The metric and Weyl vector transform according to

$$\begin{aligned}
\tilde{\eta}^{ab} &= \left\langle\tilde{\mathbf{e}}^a,\tilde{\mathbf{e}}^b\right\rangle \\
&= e^{2\phi}\left\langle\mathbf{e}^a,\mathbf{e}^b\right\rangle \\
&= e^{2\phi}\eta^{ab} \\
\tilde{\eta}_{ab} &= e^{-2\phi}\eta_{ab} \\
\tilde{W}_\mu &= W_\mu+\partial_\mu\phi \\
\tilde{W}_a &= \tilde{e}_a^{\ \ \mu}\left(W_\mu+\partial_\mu\phi\right) \\
&= e^{-\phi}e_a^{\ \ \mu}\left(W_\mu+\partial_\mu\phi\right) \\
&= e^{-\phi}\left(W_a+\partial_a\phi\right) \\
\tilde{W}^\mu &= W^\mu+\partial^\mu\phi \\
\tilde{W}^a &= \tilde{e}_\mu^{\ \ a}\left(W^\mu+\partial^\mu\phi\right) \\
&= e^{\phi}\left(W^a+\partial^a\phi\right)
\end{aligned}$$

Under Lorentz transformations, we have the connections

$$\begin{aligned}
\tilde{\boldsymbol{\sigma}}^a{}_d &= \Lambda^a_b \boldsymbol{\sigma}^b{}_c \bar{\Lambda}^c{}_d - \mathbf{d}_{(x)} \Lambda^a{}_c \bar{\Lambda}^c{}_d \\
\tilde{\boldsymbol{\gamma}}^a{}_g &= \Lambda^a_b \boldsymbol{\gamma}^b{}_c \bar{\Lambda}^c{}_g - \bar{\Lambda}^c{}_g \mathbf{d}_{(y)} \Lambda^a{}_c
\end{aligned}$$

and the tensors

$$\begin{aligned}
\tilde{T}^{ab}{}_c &= \Lambda^a_d \Lambda^b{}_e T^{de}{}_f \bar{\Lambda}^f_c \\
\tilde{T}^a_{bc} &= \Lambda^a_b T^b_{de} \bar{\Lambda}^d_b \bar{\Lambda}^e_c \\
\tilde{S}_a^{cd} &= \bar{\Lambda}^b{}_a S_b{}^{ef} \Lambda^c{}_e \Lambda^d_f \\
\tilde{S}_a{}^c{}_d &= \bar{\Lambda}^b{}_a S_b{}^e{}_f \Lambda^c{}_e \bar{\Lambda}^f_d \\
\tilde{\mu}^a{}_{bc} &= \mu^d{}_{ef} \Lambda^a_d \bar{\Lambda}^e_b \bar{\Lambda}^f_c \\
\tilde{\boldsymbol{\rho}}^d_a &= \Lambda^d{}_e \boldsymbol{\rho}^e{}_b \bar{\Lambda}^b{}_a \\
\tilde{W}_a &= W_b \bar{\Lambda}^b{}_a \\
\tilde{W}^a &= W^b \Lambda^a_b
\end{aligned}$$

The dilatational gauge transformations are given by:

$$\begin{aligned}
\tilde{T}^a_{bc} &= e^{-\phi} T^a_{bc} \\
\tilde{T}^{ab}{}_c &= e^{\phi} T^{ab}{}_c \\
\tilde{S}_b^{cd} &= e^{\phi} S_b^{cd} \\
\tilde{S}_b{}^c{}_d &= e^{-\phi} S_b{}^c{}_d
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\sigma}^a{}_{bc} &= e^{-\phi} \left(\sigma^a{}_{bc} + 2\Theta^{ad}_{cb} \partial_d \phi \right) \\
\tilde{\mu}^a{}_{ce} &= e^{-\phi} \mu^a{}_{ce} \\
\tilde{\mu}^a{}_{bc} &= e^{-\phi} \mu^a{}_{bc} \\
\tilde{\gamma}^a_c{}^b &= e^{\phi} \left(\gamma^a_c{}^b + 2\Theta^{ab}_{ec} \partial^e \phi \right) \\
\tilde{\rho}^d_b{}^c &= e^{\phi} \rho^d_b{}^c
\end{aligned}$$