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Examining the Impact of Different Virtual Manipulative Types on the Nature of Students' Small-Group Discussions: An Exploratory Mixed-Methods Case Study of Techno-Mathematical Discourse

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EXAMINING THE IMPACT OF DIFFERENT VIRTUAL MANIPULATIVE TYPES ON THE NATURE OF STUDENTS' SMALL-GROUP DISCUSSIONS: AN EXPLORATORY MIXED-METHODS CASE STUDY OF TECHNO-MATHEMATICAL DISCOURSE

by

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ABSTRACT

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by

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Utah State University, 2014

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This study examined the influence of different virtual manipulative types on the nature of students' techno-mathematical discourse (TMD) when working with a partner. The research used a concurrent mixed-methods design using identical samples to compare and synthesize the results. For this study, six fifth-grade students participated in nine sessions of mathematics instruction using virtual manipulatives. The study compared three virtual manipulative types: combined (multiple representations, open environment), pictorial (single visual representation, open environment), and tutorial (multiple representations, structured environment). Students' levels of discourse in generalization, justification, and collaboration were measured as well as students' use of physical and computer gestures while working with each virtual manipulative type.

One-way ANOVAs indicated statistically significant differences in quality of student discourse when using the different virtual manipulative types. When working with combined virtual manipulatives, students' discussions reflected consistently higher levels of discourse than when working with pictorial or tutorial virtual manipulatives. When working with tutorial and pictorial virtual manipulatives, students' discussions reflected consistently lower levels of discourse. However, pictorial virtual manipulatives were associated with the largest amount of discussion among student pairs and the highest frequency of gesture use.

The results of this study suggest that in order to encourage meaningful TMD, teachers should choose technology tools (e.g., virtual manipulatives) that combine multiple representations (i.e., combined virtual manipulative type) and provide the opportunity to engage in cognitively demanding tasks. The results of this study indicate that tutorial virtual manipulatives did not encourage meaningful mathematical discourse with these student pairs. This means that the tutorial virtual manipulative type may be better suited for the practice of mathematics concepts or for individual learning than for partner work. The patterns and trends identified in this study contribute to the existing literature on the complex issues that surround mathematical discourse and the use of technology in the classroom.

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Katie L. Anderson-Pence

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CHAPTER I

INTRODUCTION

Reform efforts in mathematics education have called for classrooms where all students have access to engaging mathematics and high-quality instruction. Mathematical discourse and technology have emerged as key components in developing this type of instruction and in how students learn mathematics. Closely connected to the communication, proof and reasoning, and representations standards in *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics [NCTM], 2000), such classroom discourse has the potential to significantly influence the depth to which students understand important mathematics. NCTM (2007) highlights the importance of discourse in the Standards for Teaching and Learning Mathematics:

> The discourse of the mathematics class reflects messages about what it means to know mathematics, what makes something true or reasonable, and what doing mathematics entails. It is central to both what students learn about mathematics and how they learn it. Therefore, the discourse of the mathematics class should be founded on mathematical ways of knowing and ways of communicating. (p. 54)

Additionally, uses of technology are becoming more prevalent in today's classrooms as compared to years past (Gray, Thomas, & Lewis, 2010). The NCTM (2000) technology principle stated, "Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning" (p. 24). Technology tools, such as calculators and computers, have the potential to influence classroom mathematical discourse and shape how students understand mathematical concepts. The types of conversations students have when using a particular application of technology called virtual manipulatives (Moyer, Bolyard, & Spikell, 2002) provided the focus for this study.

Background and Problem Statement

The purpose of this study was to describe, categorize, and interpret students' conversations as they worked with different types of virtual manipulatives. By so doing, the researcher intended to develop theory on the interactions among partner discourse, virtual manipulatives, and mathematical tasks. The interpretations emerging from this study give direction to researchers and educators on the nature of students' mathematical discussions when different types of virtual manipulatives are used with student pairs and suggest ways that they may be used in the classroom.

Over the past two decades virtual manipulatives have emerged as a tool for teachers to use in their instruction of mathematics and support student learning (see Figure 1). Moyer and colleagues (2002) defined a virtual manipulative as "an interactive, Web-based visual representation of a dynamic object that present opportunities for constructing mathematical knowledge" (p. 373). Virtual manipulatives offer unique affordances that support students' learning of mathematics. Most notable is their ability to simultaneously link symbolic and pictorial representations, and thereby provide tools to increase students' conceptual understanding of mathematics (Moyer-Packenham & Westenskow, 2013). Early collections of virtual manipulatives that still exist today include Shodor Interactivate Activities (originally created in 1994, www.shodor.org/ interactivate/activities/), the National Library of Virtual Manipulatives (originally created in 1999, nlvm.usu.edu/), and Illuminations: Activities (originally created in 2000,

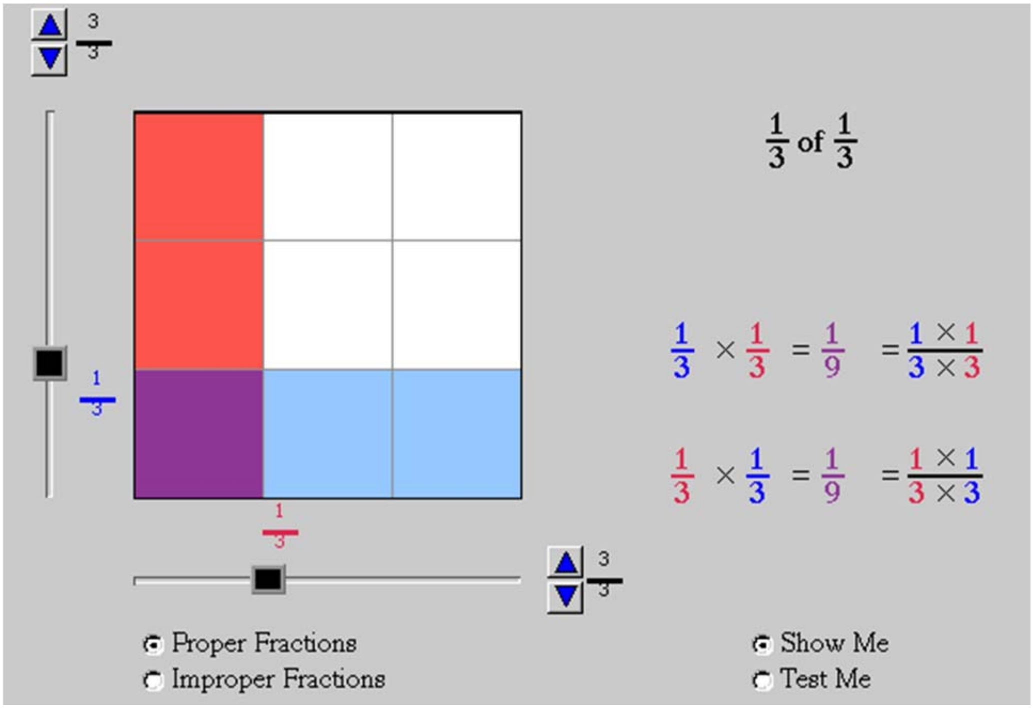


Figure 1. Virtual manipulative applet for fraction multiplication (nlvm.usu.edu).

illuminations.nctm.org/ActivitySearch.aspx).Since their inception, these websites have grown to offer more than 300 online virtual manipulative tools collectively. Since then, other developers have offered online tools and games geared toward learning mathematics (e.g., www.mathplayground.com, www.abcya.com). In recent years, textbook companies (e.g., McGraw-Hill, Glencoe) have also developed virtual manipulatives as resources for their most recent editions of elementary mathematics textbooks.

With this increase of available online resources, the types and educational quality of virtual manipulatives vary widely (see Bolyard & Moyer, 2007; Kay & Knaack, 2008). The design of virtual manipulatives varies in the degree to which the virtual manipulative reflects the students' cognitive processes, the degree to which the virtual manipulative adheres to effective instructional practices, and even the degree to which the virtual manipulative represents mathematical concepts correctly (Moyer-Packenham, Salkind, & Bolyard, 2008; Zbiek, Heid, Blume, & Dick, 2007). Virtual manipulatives also vary in the type and amount of feedback they give to the students (Steen, Brooks, & Lyon, 2006). Given these variations, questions arise regarding how teachers incorporate different virtual manipulative types into their classroom instruction and the role of different virtual manipulative types in students' learning experiences. Many possibilities exist for research examining the impact of different virtual manipulative types on students' learning (e.g., whole-group use, individual use, focus on procedural and/or conceptual knowledge).

By categorizing the descriptions of students' discussions while they are using different virtual manipulative types, this study developed theory on the nature of interactions among partner discourse, virtual manipulatives, and mathematical tasks.

Significance of Study

This study focused on the intersection of two aspects of instruction: the effectiveness and use of virtual manipulatives as an instructional tool and the nature of mathematical discourse. A growing body of research exists on the first aspect—the effectiveness and use of virtual manipulatives as an instructional tool in supporting students' learning of mathematics concepts. A recent meta-analysis identified 32 studies comparing virtual manipulatives with physical manipulatives as an instructional treatment. Twelve of these studies reported significant differences in favor of virtual manipulatives. The other studies reported mixed results, no significant differences, or did not report on the presence of significance (Moyer-Packenham & Westenskow, 2013). Overall, research indicates that virtual manipulatives positively contribute to students' understanding of mathematics concepts. However, few studies on virtual manipulatives include any examination of discourse.

Extensive research has been conducted on the second aspect—the nature of mathematical discourse (e.g., Cobb, Wood, & Yackel, 1993; Mendez, Sherin, & Louis, 2007; Moschkovich, 2007; Nathan & Knuth, 2003). Walshaw and Anthony (2008) reviewed a large collection of studies on classroom mathematical discourse and grouped the findings according to several constructs: (a) participating rights and obligations, (b) differentiating between responses and supporting students' thinking, (c) fine-tuning mathematical thinking through language, and (d) shaping mathematical argumentation. They note that according to current research, the creation of meaningful discourse in mathematics classrooms is a complex and delicate activity. However, few studies include the impact of technology on classroom mathematical discourse practices, particularly in small-group settings.

More and more classrooms are using technology (Gray et al., 2010), and students are learning mathematics as they interact with the technology and with each other. However, we know very little about these interactions students have with each other when also interacting with technology to complete mathematical tasks. This study represents an intersection of the two research fields of virtual manipulatives and classroom discourse and adds to the research literature on the impact of technology on classroom mathematical discourse.

Research Questions

The purpose of this research study was (a) to describe and categorize the nature of students' mathematical discourse as they worked with various virtual manipulative types and (b) to develop theory on the interactions among partner discourse, virtual manipulatives, and mathematical tasks. The over-arching research question and subquestions guiding the study were:

1. In what ways do different virtual manipulative types influence the nature of students' mathematical discourse?
    a. How do different virtual manipulative types influence the level of generalization in students' mathematical discourse?
    b. How do different virtual manipulative types influence the level of justification in students' mathematical discourse?
    c. How do different virtual manipulative types influence the level of collaboration in students' mathematical discourse?
    d. How do different virtual manipulative types influence physical and computer-based gestures when students are engaged in mathematical discourse?

Summary of Research Study Design

In order to explore the nature of students' mathematical discourse as they worked with different types of virtual manipulatives and to develop theory on the interactions among classroom discourse, technology tools, and mathematical tasks, this study employed a mixed methods case study design (Creswell & Plano Clark, 2011; Luck, Jackson, & Usher, 2006) utilizing both qualitative and quantitative methods to analyze students' mathematical discussions. Three pairs of fifth-grade students participated in the study. Each pair of students was considered a separate case for analysis as they discussed mathematics while using different types of virtual manipulatives over time. The data for this study were collected during a 4-month time period in one public school. Data were collected from field observations, student task sheets, and video recordings of the participating students' mathematical discussions and interactions with the virtual manipulatives. Data analysis included within- and cross-case analyses and focused on describing students' conversations, categorizing the descriptions, interpreting the categories, and developing theory.

Assumptions and Scope of Study

The researcher made several assumptions at the onset of this study. First, it was assumed that the participating students would talk to each other about mathematics and that the video recordings would capture those discussions. It was also assumed that how students interact with technology tools such as virtual manipulatives would affect the discourse during mathematics lessons, and that identifiable dimensions of discourse would be present in the data.

Since little research had been conducted on the nature of students' mathematical discourse while using different virtual manipulative types, this study was designed as an exploratory investigation. The researcher recognized the unique characteristics of the participating students. The sample was limited to fifth-grade students who were accustomed to working with technology on a regular basis. The study was not designed for wide generalization to multiple populations (Lesh & Lovitts, 2000) such as students of different ages, ethnic groups, and socioeconomic status. The focus of this study was specifically on how different virtual manipulative types influenced students' mathematical discourse. Effects of students' achievement, familiarity with the virtual manipulative, and perceptions of the virtual manipulative types were beyond the scope of this study. Additionally, this study focused on the specific interactions of small groups of two students each. Because of the focus on student-student interactions, the role of the teacher in initiating or influencing the mathematical discussion was not addressed in this study. Nor was this study interested in the development and validation of lessons designed to promote mathematical discourse. Investigation of these factors—other demographic groups, student characteristics, the teacher's involvement, and lesson development—was beyond the scope of this study. More studies will be needed to investigate these factors more fully.

Definition of Terms

The following terms are defined for this study.

Classroom mathematical discourse (discussion): The ways of representing, thinking, talking, agreeing, and disagreeing about mathematical ideas (NCTM, 2007, p. 46); concerns both the process and content of communicating mathematical ideas in a classroom setting (Sherin, 2002).

Techno-mathematical discourse: Discourse in which students use technological representations (e.g., virtual manipulatives) to mediate discussion while engaging in worthwhile mathematical tasks.

Virtual manipulative: Interactive, web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge (Moyer et al., 2002, p. 373).

Affordance: A design feature that determines how the object will be used (Norman, 1988).

Feedback: Specific messages (verbal or nonverbal) communicating the correctness of a student's mathematical choices.

Generalization: The capacity to form connections among related ideas (Mendez et al., 2007).

Justification: A logical, warranted argument or proof of mathematical processes (Mendez et al., 2007).

Collaboration: A process in which two or more individuals work together to integrate information in order to enhance learning (Mendez et al., 2007).

Gesture: The movements of hands and arms that we see when people talk (McNeill, 1992); hand and arm movements that are interpreted by others as part of what a person says (Roth, 2001).

CHAPTER II

LITERATURE REVIEW

This chapter reviews the theoretical literature and empirical research related to the current study. The first section examines research literature on classroom mathematical discourse. The second section examines research literature on uses and the affordances of virtual manipulatives as tools for classroom instruction in mathematics. The third section examines research literature on worthwhile mathematical tasks and representations. The fourth section examines existing research studies related to the impact of technology on classroom mathematical discourse, and identifies an emerging construct among these studies—*techno-mathematical discourse* (TMD). Finally, the fifth section identifies areas of needed research and the contributions of this study to the field of educational research.

Classroom Mathematical Discourse

In this study, classroom mathematical discourse was defined as "the ways of representing, thinking, talking, agreeing, and disagreeing about mathematical ideas" (NCTM, 2007, p. 46). Mathematical discussion, as described in this definition, reflects current mathematics education reform efforts, which call for classroom communities focused on the communication of mathematical ideas (NCTM, 2000). Here, the terms "discourse" and "discussion" are used interchangeably. Educational researchers have identified multiple characteristics of classroom mathematical discourse, including explanation, argumentation, and defense of mathematical ideas (Walshaw & Anthony, 2008). Sherin (2002) described two elements of classroom mathematical discourse:

process and content.

> The *process* of mathematical discourse refers to the way that the teacher and students participate in class discussions. This involves how questions and comments are elicited and offered, and through what means the class comes to consensus. In contrast, the *content* of mathematical discourse refers to the mathematical substance of the comments, questions, and responses that arise. (p. 206)

Educational, linguistic, and psychological researchers have studied the nature and characteristics of classroom mathematical discourse widely. The following subsections review: (a) the theoretical underpinnings and empirical research related to classroom mathematical discourse and (b) the use of gesture in communicating mathematical ideas.

Theoretical Underpinnings of Classroom Mathematical Discourse

The use of discourse and communication as a means to learn new concepts is emphasized by Vygotsky (1978) in his seminal work in sociocultural theory, *Mind in society: The development of higher psychological processes*. Here, he described learning as a socially constructed phenomenon and asserts three major tenets: (a) higher mental processes are determined by how and when they occur, (b) higher mental processes first occur on the social plane (i.e., between people), and then occur on the individual psychological plane, and (c) higher mental processes are mediated by cultural tools and signs (e.g., symbols, speech, and writing). Therefore, students develop understanding as they interact with other individuals through verbal or nonverbal communications or written words. This study considers technological representations (i.e., virtual manipulatives) as mediating cultural tools for mathematical learning.

Sfard (2007) referred to the socially constructed phenomenon of learning as *commognition*—a combination of communication and cognition. She further asserted that thinking can be defined as “the individualized form of the activity of communicating, that is, as communication with oneself” (p. 569). Therefore, in order to deeply understand complex concepts, some form of discussion must take place—even if that conversation occurs within one individual. Piccolo, Harbaugh, Carter, Capraro, and Capraro (2008) described rich meaningful communication in the classroom setting as consisting of “interactive and sustained discourses of a dialogic nature between teachers and students aligned to the content of the lesson that addresses specific student issues” (p. 378). In other words, meaningful classroom discourse contributes to students’ understanding by promoting effective communication and articulation of thought.

The culture of a classroom also plays a considerable role in shaping classroom mathematical discourse. Sociomathematical norms (Yackel & Cobb, 1996) develop within a classroom and constitute what interactions are valued and what counts as an acceptable mathematical explanation. Through these interactions, students analyze and evaluate the mathematical thinking and strategies of others and deepen their own mathematical understanding. Students must organize and consolidate their mathematical thinking in order to communicate effectively with their classmates and with the teacher Chapin, O’Conner, & Anderson, 2009; Cobb et al., 1993; Huang, Normandia, & Greer, 2005; Imm & Stylianou, 2012; Piccolo et al., 2008; Sfard, 2007).

Empirical Research on Classroom Mathematical Discourse

Extensive research has been conducted on the nature and use of discourse in mathematics classrooms. The next subsections review portions of this body of research relevant to the current study. The first section gives an overview of classroom discourse communities encompassing whole-class discussions. The second section focuses on research with small-group discourse, particularly on students' roles and how they differ from whole-class discourse.

Whole-class classroom discourse. The majority of research on mathematical discourse examines interactions during whole-class discussions. The following sections elaborate on four main themes in this body of research: (a) explaining mathematical thinking, (b) structure and flow of mathematical discourse, (c) teacher- versus student-centrality, and (d) the impact of teachers' instructional decisions.

Explaining mathematical thinking. Multiple studies have examined the process of mathematical explanation and reasoning. In an effort to examine components of effective mathematical discourse, Mendez and colleagues (2007) developed a framework for robust mathematical discussion (RMD). RMD categorizes students' comments along two dimensions: *mathematics* and *discussion*. The *mathematics* dimension addresses three aspects of mathematical argumentation: representation, generalization, and justification. The *discussion* dimension examines three aspects of discourse: engagement, intensity, and building on others' ideas. The researchers used the RMD framework as they observed and analyzed 66 lessons over the course of one calendar year in one teacher's eighth-grade mathematics class. In their analysis, discourse seemed to be most effective in developing understanding when students' explanations included reasoning and justification and when students valued each other's comments (i.e., when classified as high in each of the RMD dimensions).

Other researchers have found that high expectations for discourse coupled with a relaxed and nonthreatening atmosphere promote students' willingness to share their strategies. For example, Zolkower and Shreyar (2007) conducted a 9-month study of the discourse patterns in a sixth-grade urban classroom. The teacher in this classroom consistently verbalized her own mathematical thinking and expected her students to do the same. Based on the transcripts provided in the study report, some students seemed to contribute to the discussion while they were still processing the mathematics and were, therefore, supported by other students' verbalizations (see also Lau, Singh, & Hwa, 2009).

Structure and flow of mathematical discourse. The study of the structure and flow of mathematical discourse in the classroom has generally taken a sociocultural or sociolinguistic perspective, and many researchers have developed different ways to conceptualize and classify how discourse occurs in classrooms (e.g., Evans, Feenstra, Ryon, & McNeill, 2011; Gresalfi, Martin, Hand, & Greeno, 2008; Herbel-Eisenmann & Wagner, 2010; Iiskala, Vauras, Lehtinen, & Salonen, 2011; Imm & Stylianou, 2012; Nathan & Knuth, 2003; Sinclair, 2005; Webb, Nemer, & Ing, 2006; Wood & Kalinec, 2012; Zolkower & Shreyar, 2007). In their research on intersubjectivity in the mathematics classroom, Nathan, Eilam, and Kim (2007) identified four common discourse sequences: initiation-open or -closed event (I), response (R), demonstration or solution (D), and evaluation or elaboration (E). In addition to the typical IRE pattern (i.e., initiation-response-evaluation) most associated with closed-ended "I" events (Mehan, 1979; Sinclair & Coulthard, 1975), their analysis revealed a prevalence of an IDE sequence (i.e., initiation-demonstration-evaluation) most regularly associated with open-ended “I” events. The richness of the ensuing discourse took place mainly during the evaluation or elaboration phases when students expressed agreement or attempted to reconcile conflicting ideas. Interestingly, the IDE sequences tended to “chain” onto one another, perpetuating the existing discourse. In contrast, the few IRE sequences showed no “chaining” patterns. From this study, the researchers contend that the IDE sequences provide the underlying structure of effective discourse.

Other researchers have taken a broader view of the flow of classroom discourse. Truxaw and DeFranco (2007, 2008) studied the classroom discourse practices of seven accomplished middle-grades mathematics teachers. Through a fine-grained analysis of the discourse in these classrooms, the researchers identified three distinct models of teaching: inductive (building meaning, exploring, hypothesizing), deductive (transmitting meaning, presenting definitions and/or procedures, applying to individual problems), and mixed (a hybrid bordering on the deductive model). Discourse sequence maps revealed that most inductive model lessons tended toward dialogic discourse involving both teacher and students in active communication. However, discourse in deductive model lessons contained universally univocal discourse, requiring little feedback from students. Interestingly, the type of discourse depended exclusively on the teacher’s intentions and use of verbal assessment moves.

In a similar analysis, Piccolo and colleagues (2008) examined the various pathways to understanding taken by students and teachers during discourse. Their resulting Dynamic Student-Teacher Communications Pathways map identified multiple teacher- and student-initiated interactions leading to understanding. Each of these pathways involved active student participation and focused teacher facilitation, and is mediated by the types of questions and responses generated by both teachers and students. These findings support the work of Nathan and colleagues (2007) and Truxaw and DeFranco (2007, 2008) in that teachers' verbal moves to involve students' active participation lead to richer and more robust discussions. Sherin (2002) conceptualized the flow of discourse differently by describing iterative processes of idea generation, comparison and evaluation, and filtering to generate the next cycle of new ideas. The many ways researchers have described the flow of these discussions illustrate the complexities of classroom mathematical discourse.

Teacher- versus student-centrality. Teacher- versus student-centrality in whole-class discourse is a common underlying theme in many of the studies on classroom discourse (e.g., Brodie, 2011; Hunter, 2010; Mendez et al., 2007; Mueller, 2009; Truxaw & DeFranco, 2008; White, 2003). The framework developed by Hufferd-Ackles, Fuson, and Sherin (2004), Levels of the Math-Talk Learning Community: Action Trajectories for Teacher and Student, articulated the movement from a teacher-centered traditional mathematics classroom (Level 0) to a student-centered classroom with meaningful and collaborative discourse (Level 3). This framework supports other research indicating that student-led discussion improves when the teacher removes herself from the center of discussion (Lau et al., 2009; Nathan et al., 2007; Nathan & Knuth, 2003).

Impact of teachers' instructional decisions. Ultimately, the type of mathematical discourse present in a classroom is determined by the teacher's instructional decisions.

The teacher is responsible for setting the tone and culture of the classroom and for orchestrating discourse throughout the lesson. Imm and Stylianou (2012) emphasized the important relationship between cognitively demanding tasks and mathematical talk. In their analysis of the discourse practices of five middle school teachers, they observed that a cognitively demanding, open-ended task could fail to produce rich discussion if the teacher accompanied the task with univocal talk. In this way, “a conceptually rich task could be refashioned as a set of procedures in disguise” (p. 138). Therefore, the task itself and how the teacher chooses to present the task impact the quality of students’ mathematical discussions (see also Truxaw & DeFranco, 2008). Of course, the teacher’s work does not stop with the introduction of the task. Appropriate verbal moves throughout the lesson serve to lead the students to a deeper understanding and reflection of their own knowledge (Brodie, 2011; Gresalfi et al., 2008; Hunter, 2010; Nathan et al., 2007; White, 2003).

During a lesson, teachers also make decisions based on informal assessments and observations. By paying close attention to what students are thinking and saying, the teacher can know which students to call on to provoke meaningful discourse. He or she can also sequence particular strategies or ideas to build collective class knowledge or assist individual students in making connections (Mendez et al., 2007; Truxaw & DeFranco, 2008). Another issue arises when students’ contributions contain misconceptions or when a student struggles with particular concepts. The teacher’s response to students’ comments greatly influences the discussion path and the classroom culture (Baxter & Williams, 2010). Many researchers (e.g., Brodie, 2011; Huang et al., 2005; Hufferd-Ackles et al., 2004) agreed with the practice of allowing students to work through the mathematics instead of directly addressing their misconceptions. Clearly, this practice requires patience on the part of the teacher. Mueller (2009), Piccolo and colleagues (2008), and Zolkower and Shrayer (2007) each described teachers in their studies who did not explicitly evaluate their students' contributions to the discussion. Instead, they used student strategies and error as opportunities to learn and integrated them into their previously planned instruction.

Small-group classroom discourse. While a large corpus of literature exists on whole-class discourse, relatively little research has been conducted specifically on the discourse of students in small groups or partnerships. The studies highlighted here describe aspects of student collaboration in mathematics classrooms and provide different views on the effectiveness of such collaborations.

Effective small-group collaborations. One key aspect of small-group peer interactions is "co-construction," defined by Mueller (2009) as a "form of collaboration in which an argument is built simultaneously by the learners from conception" (p. 141). This collaboration is characterized by the back and forth nature of its discourse. In her analysis of classroom videos, Mueller noted three ways in which students built upon each other's ideas: expanding, redefining, and reiterating. Very often students made their own solutions stronger by integrating the ideas of others—a strong indicator of a well-functioning intellectual community. Mueller also contended that open-ended task design and collaborative classroom environments promote student-student mathematical discussion around multiple representations and solutions.

Similarly, by applying the Math-Talk Learning Community framework (Hufferd-Ackles et al., 2004) to small groups, Bruce and Flynn (2011) observed a mathematical debate between two first-grade students. The researchers concluded that the sustained math-talk enabled the students to present ideas and to question one another in productive ways. As one student explained her mathematical thinking the other student took responsibility for her own learning by listening intently and asking clarifying questions. Interestingly, although the teacher did not take an active role in the discussion between these students, her one interjection into the conversation propelled them on to greater mathematical understanding.

Another important aspect of small-group peer interaction is "socially shared metacognition," defined by Iiskala and colleagues (2011) as "the consensual monitoring and regulation of joint cognitive processes in demanding collaborative problem-solving situations" (p. 379). Within pairs of high-achieving 10-year-old students, researchers identified episodes of socially shared metacognition, and then categorized those episodes by their function (facilitate or inhibit) and focus (situation model, operation, or incidental matters). A positive relationship emerged between the problem difficulty and the amount of shared metacognition among students. Easy problems did not require the students to gauge their thinking; they just solved the problem. More difficult problems provided opportunities for students to jointly problematize the mathematics and formulate a general solution model. This research suggested that in order for pairs of students to engage in productive mathematics discourse, the cognitive demand of the given task should be sufficient enough to support meaningful conversation. However, this study examined the discourse practices of only high-achieving students. Questions remain regarding the generalizability of these findings to groups of students with different academic dispositions.

Challenges to small-group student collaborations. Despite the previously discussed research, other studies do not find peer interactions to be as effective in building mathematical knowledge. For example, Sinclair (2005) reported mixed results in the effectiveness of paired groupings. She identified a number of teacher-like roles assumed by students in such small-group settings—leading, showing and telling, shepherding, checking, reinforcing, inviting, blocking, enculturating, modeling, praising, and rug-pulling—and concluded that peer interactions in isolation could sometimes interfere with the development of mathematical understanding (see also Grant, 2009).

Likewise, other researchers indicate that small-group discussions tend to stray from the mathematical tasks at hand. Wood and Kalinec (2012) analyzed one case of three fourth-grade students working together on a mathematics problem. They categorized the students' comments by the type of on- or off-task behaviors and by the content of the utterances. For this particular group of students, the discussion consisted mostly of off-task behavior and comments—only about 10% of utterances were devoted to mathematizing even with attempted interventions from the teacher. However, some degree of learning seemed to occur despite the generally off-task nature of the discussion. The researchers suggest that quality of mathematical discourse plays a greater role in learning than the quantity of mathematical discourse.

Many factors may account for the lack of mathematical engagement as described in Wood and Kalinec's (2012) study. For example, Kotsopoulos (2010) noted that when students vocalize their thought processes, either as a clarification of their thinking or as an expression of confusion, it is not always acknowledged by their peers as attempted communication. Therefore, even though a student may initiate mathematical talk by offering his or her thoughts, if other students are not listening attentively, they will be unable to respond appropriately. Additionally, research suggests that students generally have difficulty in identifying peers' misconceptions and/or effectively helping others to understand mathematical concepts (Webb et al., 2006).

The reasons for such mixed results on the effectiveness of small-group student collaborations remain unclear. Factors such as classroom culture, teacher's expectations, and students' own mathematical identity most likely influence peer collaborations. Further research may be required to ascertain key elements and determine to what extent these factors impact productive small-group mathematical discourse.

Use of Gestures in Classroom Mathematical Discourse

Gestures, defined as "hand and arm movements that are interpreted by others as part of what a person says" (Roth, 2001, p. 369), play an important role in communication in mathematics classrooms. According to Nuñez (2007), "Mathematics is perhaps the most abstract conceptual system that we can think of, but even this it is ultimately embodied in the nature of our bodies, language, and cognition" (p. 152). Thus, conceptual understanding of mathematical ideas involves physical action, whether materially enacted or mentally construed. Students utilize physical actions as they use gestures to communicate their mathematical thinking (see also Nemirovsky, Rasmussen, Sweeney, & Wawro, 2012). Their use of gestures while engaging in classroom mathematical discourse serves to enhance the communication of mathematical ideas (Cutica & Bucciarelli, 2008).

Many typologies exist for classifying gestures (e.g., Ekman & Friesen, 1969; Kendon, 1988; Rime & Schiaratur, 1991; Wundt, 1973). However, most educational researchers used the typology identified in McNeill's seminal work *Hand and Mind: What Gestures Reveal About Thought* (McNeill, 1992). McNeill identified four basic types of gesture: *iconic* (resembling a concrete object), *metaphoric* (resembling an abstract idea), *beat* (repetitive movements indicating significance), and *deictic* (pointing movements). Early experimental research on gestures suggests that gestures reveal knowledge not expressed in speech and that they may even show evidence of emergent knowledge—a precursor to more sophisticated cognitive development (Goldin-Meadow, 1999; Roth, 2001).

Recent research on the use of gestures while learning mathematics indicated that cospeech gestures help to retain knowledge (Cook, Mitchell, & Goldin-Meadow, 2008), make abstract concepts accessible to young children (Kim, Roth, & Thom, 2011), lighten cognitive load (Yoon, Thomas, & Dreyfus, 2011a, 2011b), and improve the richness of mathematical discourse (Cutica & Bucciarelli, 2008). Others have noted that particular gestures convey specific types of mathematical thinking (Bjuland, Cestari, & Borgersen, 2008; Edwards, 2009). For example, building on McNeill's (1992) typology of gestures, Alibali and Nathan (2012) observed that when individuals use deictic or beat gestures, their thinking tends to be grounded in the physical environment. When individuals use iconic or metaphoric gestures, their mathematical thinking tends to incorporate perceptions and action. However, gestures are not inherently mathematical. They must be interpreted in the context of the mathematics being discussed (de Freitas & Sinclair, 2012) and within the social context of the learning situation (Kim et al., 2011; Williams, 2009). When gestures are interpreted within the context of mathematics, then they can be used to communicate ideas, and to encourage experimentation with new ideas (Yoon et al., 2011a, 2011b).

Little research has been conducted on the use of gesture when learning mathematics with computer technologies (e.g., Botzer & Yerushalmy, 2006, 2008; Bruce, McPherson, Sabeti, & Flynn, 2011; Evans et al., 2011). It remains to be seen what gestures learners use while interacting with a computer screen, how those gestures correspond with mouse or touchpad gestures, and if the activities presented by the computer program influence the gestures used by learners.

Virtual Manipulatives

Over the past few decades, technology has developed new ways to think about and to represent mathematics (Moreno-Armella, Hegedus, & Kaput, 2008). These new "cognitive technology tools" (Pea, 1985) enhance the learning of mathematics concepts by expanding representational possibilities and by amplifying and reorganizing students' approaches to problem solving. An elaboration of the NCTM technology principle stated;

> Electronic technologies—calculators and computers—are essential tools for teaching, learning, and doing mathematics. They furnish visual images of

> mathematical ideas, they facilitate organizing and analyzing data, and they compute efficiently and accurately…. When technological tools are available, students can focus on decision making, reflection, reasoning, and problem solving. (NCTM, 2000, p. 24)

Recent research also reiterated the role of technology in mathematics education. For example, Manches, O'Malley, and Benford (2010) observed 65 children (aged 4-8 years) solve problems using varying levels of technology. They noted that particular characteristics of the technology interface determined students' choice of solution strategies. The following subsections review (a) the theoretical underpinnings for the use of virtual manipulatives as instructional tools, and (b) empirical research related to virtual manipulatives.

Theoretical Underpinnings of Virtual Manipulatives

With the advancement of computer capabilities, virtual manipulatives have emerged as cognitive technology tools for use in mathematics classrooms. A virtual manipulative is defined as "an interactive, web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge" (Moyer et al., 2002, p. 373). Based on this dynamic nature, virtual manipulatives seem to be a combination of *manipulative models* (e.g., base-ten blocks, fraction bars, counting bears), which allow for concrete examples of mathematical relationships and operations and *static pictures*, which provide an image for a learner to internalize (Lesh, Post, & Behr, 1987). These "computer based renditions of common mathematics manipulatives and tools" (Dorward, 2002, p. 329) provide teachers and students with expanded tools for thinking about mathematics concepts.

Computer-based representations vary in their level of cognitive fidelity (Zbiek et al., 2007). Some representations offer manipulative tools that truly reflect the user's actions and choices without dictating solution paths. Other representations include concept tutorials (with or without manipulative tools) to guide students to a conceptual or procedural understanding of the mathematics. Still, others present an electronic figure, either static or in motion, very similar to a textbook or worksheet, and Kay (2012) identifies some virtual manipulatives as open-ended and others as structured. A recent meta-analysis of research on virtual manipulatives (Moyer-Packenham & Westenskow, 2013) identified five affordances offered by specific design features and elements of different virtual manipulatives: focused constraint, creative variation, simultaneous linking, efficient precision, and motivation. These varying features of virtual manipulatives have implications for their instructional use.

The simultaneous linking of representations afforded by virtual manipulatives is based on dual coding theory (Clark & Paivio, 1991; Paivio, 2007; Sfard, 1991; Skemp, 1987), which posits that learning occurs in two different manners: verbal and nonverbal. The verbal system deals with learning modes related to the linear functions of language (i.e., spoken word, written words and symbols). The nonverbal system deals with nonlinguistic learning modes and mental imagery (i.e., geometric figures, diagrams). Each system alone can process a limited amount of information at a time (Clark, Nguyen, & Sweller, 2006). However, the combination of verbal and nonverbal inputs results in increased ability to make connections between representations and to attend to more complex ideas. For example, students may use the Fractions—Rectangle Multiplication applet from the National Library of Virtual Manipulatives (http://nlvm.usu.edu) to visualize the meaning of multiplying fractions. This applet enables students to manipulate the size of each factor and to observe simultaneous changes in pictorial (nonverbal) and symbolic (verbal) representations of the product. Virtual manipulatives have the capacity to combine representations from both systems, and thereby, increase working memory capacity. In these ways, technology tools enhance the mathematical content presented to students.

Empirical Research on Virtual Manipulatives

Research on virtual manipulatives has emerged over the past 25 years, and provides a foundation for future studies. The next subsection gives an overview of research on the effectiveness of virtual manipulatives. Thereafter, two affordances of virtual manipulatives are discussed: simultaneous linking of representations and immediate feedback.

Effectiveness of virtual manipulatives. In 2013, Moyer-Packenham and Westenskow conducted a meta-analysis to assess the overall effectiveness of virtual manipulatives on student learning. Their analysis of 82 effect sizes from 32 studies yielded a moderate effect size (0.35) for virtual manipulatives when compared to other methods of instruction. Twelve studies reported significant differences in favor of virtual manipulatives; eight studies reported no significant differences; seven studies reported mixed results; and five studies did not report any analysis of significance. This collection of studies indicates that the use of virtual manipulatives can be just as effective in mathematics as other instructional tools. For example, in examining the effect of concrete manipulatives as compared with the effect of virtual manipulatives on third- and fourth-grade students' learning, Burns and Hamm (2011) found that both types of manipulatives effectively reinforce mathematics concepts. They found no significant difference in student achievement among groups using the different manipulative types (see also Crossley, 2003; Haistings, 2009; Mendiburo & Hasselbring, 2011; Smith, 2006; Steen et al., 2006; Trespalacios, 2010; Whitmire, 2006).

Other research indicates that characteristics of the virtual manipulative may determine the strategy types a student chooses to use. Manches and colleagues (2010) conducted task-based interviews with 65 children (ages 4-8) focusing on part-part-whole relationships. They observed that young students working with virtual manipulatives to complete the tasks favored compensation strategies (i.e., adding or subtraction an amount from a previous solution to solve a problem). Students working with physical manipulatives to complete the same tasks favored strategies utilizing the commutative property. This finding suggests that different actions with the materials may influence how students explore important structural relationships in mathematics. Manches and colleagues asserted, "If different actions lead to different ideas, it is important to consider how a particular interface can foster the desired ideas we want children to grasp" (p. 639). Controlling what actions can be made by the virtual manipulative may also affect the mathematical ideas students develop. Additionally, Galbraith (2006) identified four ways technology is used in mathematics education: as a master, as a servant, as a partner, and as an extension of self. Students engage in meaningful learning of mathematics when they use technology as a partner or as an extension of self (McDougall & Karadag, 2009).

As a partner, the technology supports the students' cognitive activities by storing information in databases or correcting calculation mistakes. As an extension of self, the technology enables students to improve their own cognitive capabilities and to extend previous understanding. These uses are impacted by different characteristics and affordances of the virtual manipulative.

Simultaneous linking of representations provided by virtual manipulatives. One of the five affordances revealed by the meta-analysis by Moyer-Packenham and Westenskow (2013)—simultaneous linking—has particular relevance to this study. Simultaneous linking refers to a feature of some virtual manipulatives to present different representations of mathematical concepts at the same time. As students work with the manipulative, the linked representations change simultaneously to connect the mathematical concepts. Multiple studies highlight the impact of this feature of virtual manipulatives. For example, Bolyard and Moyer-Packenham (2012) investigated the impact on 99 sixth-graders' learning of integers with three different virtual manipulatives. Students made significant gains in achievement regardless of which virtual manipulative they used. The study also found that students used one or more representational forms to self-evaluate and support their work (see also Izydorczak, 2003; Suh & Moyer, 2007). Other studies highlight how the simultaneous linking of representations helps to support students at varying achievement levels (e.g., Suh & Moyer-Packenham, 2008; Suh, Moyer, & Heo, 2005). The affordance of virtual manipulatives to simultaneously link representations affords students the opportunity to make connections between different representations and deepen their understanding of mathematical concepts.

Potential feedback provided by virtual manipulatives. Another feature in some virtual manipulatives is the potential to provide immediate feedback to students as they learn mathematics concepts. Some virtual manipulatives provide step-by-step instructions in a tutorial fashion to guide students' understanding (e.g., Fractions–Adding on nlvm.usu.edu). In contrast, other virtual manipulatives allow students to explore freely with the mathematical concept (e.g., Fractions–Rectangle Multiplication on nlvm.usu.edu). Each of these types of virtual manipulatives gives feedback to students in different ways. The tutorial virtual manipulatives give very direct feedback, while the open-ended virtual manipulatives give indirect or inferential feedback (Bolyard & Moyer, 2007). Multiple studies have noted the importance of feedback as students work with virtual manipulatives. For example, Reimer and Moyer (2005) found that students preferred the immediate feedback provided with the virtual manipulatives as opposed to feedback from the teacher when using paper and pencil methods. This type of feedback has also been found to motivate students in their work (Steen et al., 2006; Suh & Moyer, 2007). Additionally, Suh and colleagues (2005) noted that the nonjudgmental feedback given by the virtual manipulatives can encourage students to test hypotheses and persist in problem solving tasks.

The impact of such feedback also depends on the cognitive, pedagogical, and mathematical fidelity (Dick, 2007) of the virtual manipulative tool. Cognitive fidelity concerns the level to which the representations of the mathematics match the cognitive processes of the student. A virtual manipulative with low cognitive fidelity would direct a students' thinking in a particular direction without considering the students' own thought processes. Pedagogical fidelity concerns the level to which the virtual manipulative maintains effective instructional design. Mathematical fidelity concerns the level to which the virtual manipulative correctly represents mathematical concepts. Whether tutorial or open-ended in nature, effective virtual manipulatives should maintain high levels in each of these three domains.

Mathematical Tasks

The mathematical content and tasks presented in a lesson significantly affect the richness of classroom discourse. In order for rich discussions to take place, students must be presented with tasks worth talking about (Iiskala et al., 2011; Lack, 2010; Mendez et al., 2007; Sherin, 2002). Worthwhile mathematical tasks, as defined by NCTM (2007) promote communication, engage students' intellect, develop mathematical understandings and skills, represent mathematics as an ongoing human activity, and embed mathematics in meaningful contexts. For example, instead of having students simply memorize multiplication facts or mathematical vocabulary, worthwhile tasks embed the multiplication facts and vocabulary in "meaningful contexts that help students see the need for definitions and terms as they learn new concepts" (NCTM, 2007, p. 33).

A worthwhile mathematical task is one that engages students' intellect and calls for problem solving and mathematical reasoning. According to Smith and Stein (1998), tasks vary in their level of cognitive demand. Tasks with lower levels of cognitive demand involve reproduction of memorized facts and algorithmic procedures with no connection to the concepts underlying the procedures. They have clear solution paths and require no explanation of mathematical thinking beyond a description of the procedure used. Tasks with higher levels of cognitive demand (i.e., worthwhile mathematical tasks) involve multiple solution paths and/or multiple possible solutions. Students must analyze the task and present solutions in multiple representational forms. Smith and Stein note that tasks with higher levels of cognitive demand likely produce anxiety for some students due to the uncertain and unpredictable nature of the problem. This anxiety is a sign of cognitive disequilibrium experienced by students as they come to understand new concepts (Piaget, 1952). By presenting non-routine problems that require students to actively engage in mathematics (as opposed to mindlessly following procedures), worthwhile mathematical tasks represent mathematics as an "ongoing human activity" (NCTM, 2007, p. 33) and provide opportunities for students to make deep connections between mathematical ideas.

Other research has been conducted on the engagement potential of mathematical tasks. For example, English (1998) interviewed third-, fifth-, and seventh-grade students from classrooms implementing worthwhile mathematical tasks (i.e., reform-based practices) about their perceptions of the quality of different types of tasks posed in their classrooms. Overall, the students felt that interesting, meaningful, and relevant tasks provided them with the best learning experiences. They rated tasks as more engaging if the tasks involved deductive reasoning and provided some type of structural support for problem-solving (e.g., hints, diagrams). Students also rated spatial reasoning tasks and tasks based in a real-world context as more engaging overall. However, the perceived mathematical complexity of the problem seemed to have more influence on the students' ratings of engagement than any of the previously identified factors. If the task seemed either too simple or too complex, students were reluctant to rate it as engaging. Together with Smith and Stein's (1998) framework for levels of cognitive demand, these findings suggest that the effectiveness of mathematical tasks greatly depend on designing the task to reflect students' interests and experiences, to provide sufficient representational support, and to match the mathematical complexity to students' ability levels. Worthwhile mathematical tasks play a key role in students' mathematical discussions and in the development of students' mathematical understandings and skills.

Techno-Mathematical Discourse

The convergence of classroom discourse, technology, and worthwhile mathematical tasks gives rise to TMD (see Figure 2). This discourse is unique in that students use technological representations (e.g., virtual manipulatives) to mediate discussion while engaging in worthwhile mathematical tasks (Vygotsky, 1978). As an emerging construct, the TMD framework provides a means for analyzing and interpreting aspects of social learning with technology during mathematics instruction.

In TMD, technology enhances the communication of mathematical ideas and supports students' learning of mathematics concepts. When learning mathematics concepts with technology in a discourse community, students have access to multiple modalities of mathematical representations. First, technology tools, such as virtual manipulatives, provide dynamic pictorial and symbolic representations of mathematics concepts. Second, the dynamic visual displays serve as common experiences about which

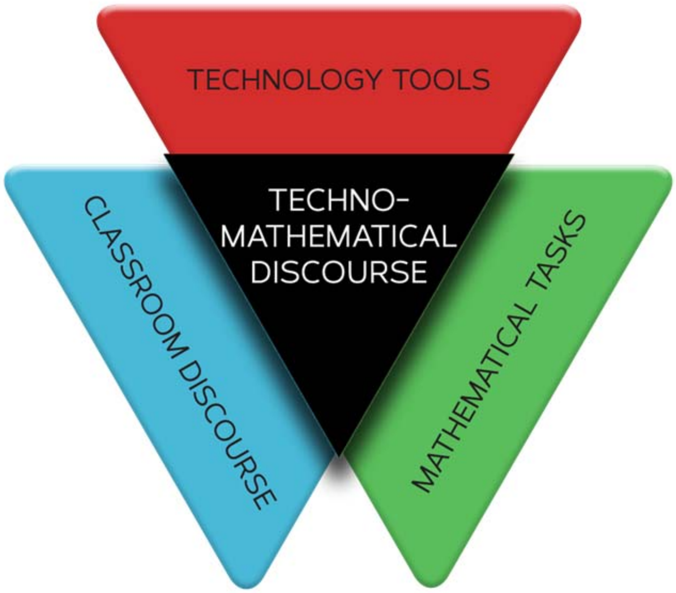


Figure 2. Theoretical framework of techno-mathematical discourse (TMD).

students can engage in meaningful classroom discussions incorporating both verbal and gestural (i.e., embodied) interactions. Students' understanding of mathematical concepts is strengthened when they make connections among representations in pictorial, symbolic, verbal, and embodied modalities (Clark & Paivio, 1991). Of course, the strength of TMD is influenced by the affordances of the available technology tools, by the quality of the worthwhile mathematical tasks used for instruction, and by the teacher's ability to orchestrate the classroom discourse.

Empirical Research on Techno-Mathematical Discourse

Research on the impact of technology in education has expanded over the past few decades. Technological developments constantly emerge presenting opportunities to improve classroom practices and learning. A great deal of research has been conducted in an attempt to verify the usefulness of such technologies. This synthesis of research findings focuses on the use of technology in elementary through high school classrooms as a representational tool for developing mathematical concepts. Major themes related to classroom discourse with technology will be discussed in three sections: (a) the impact of dynamic representations on the content and nature of mathematical discourse, (b) the impact of computer feedback on student collaborations, and (c) shifts in the teacher's role as facilitator of mathematical discourse.

Impact of dynamic representations on classroom discourse. Technology has the potential to produce dynamic representations of mathematics concepts. The dynamic nature of these representations has a profound impact on the level of classroom mathematical discourse. For example, Ares, Stroup, and Schademan (2008) described a lesson using networked classroom technology—a wireless network of graphing calculators that collects students' solutions and displays them collectively on a screen at the front of the room. In this particular lesson, students used their calculators to "maneuver an elevator" by determining how many levels it would move up or down in one-second intervals. The collective resulting position-time graphs were then displayed on the front screen. Different tasks throughout the lesson gave specific parameters causing the students to focus on different mathematical relationships (e.g., end on the –2 floor using any combination of movements, the fourth movement must be to go up three floors). The researchers noted that the collective representation encouraged students to interact with each other and comment on the various solutions. Students focused on the mathematics represented dynamically on the visual display and used it as a basis for their mathematical discussions. Additionally, the visual display mediated a shift in the discourse from conceptual to more formal language (e.g., “they all go up at the same time” to “each line has the same slope, so they are all parallel to each other).

Similarly, Sinclair (2005) and González and Herbst (2009) each reported on studies with dynamic interactive geometry software (Geometer’s Sketchpad and Cabri Geometry, respectively). In Sinclair’s study, students worked in pairs with Geometer’s Sketchpad to complete a sequence of tasks on proving congruency (e.g., applications of reflection and rotation). The dynamic nature of the software enabled the students to test conjectures and receive immediate feedback. Just as observed by Ares and colleagues (2008), the students in Sinclair’s study used the visual representations to fuel their mathematical discussions. However, these students displayed varying degrees of effectiveness in their discussions. As noted above, they engaged in productive discourse by explaining their thinking and asking thoughtful questions. But at other times, students’ discourse actually hindered the development of mathematical ideas. Due to this variation in productivity, Sinclair emphasizes the need for follow-up classroom sessions after time spent in the computer lab to solidify understanding and to ensure that all students have appropriate opportunities to learn the content.

González and Herbst (2009) reported a more positive view of student discourse when working with dynamic interactive geometry software. Students in this study also completed a sequence of tasks to investigate congruency. However, instead of applying transformations (as in the previous study) these tasks required them to experiment with midpoints and angles. The measuring and dragging features of the Cabri Geometry software enabled students to quickly and accurately assess the results of their experiments. The interactive features of the software tools supported all students' learning in the lesson. In whole-class discussions, advanced students described how they used the tools to prove their conjectures and pointed out new ideas. At the same time, other students who did not fully understand the technical terms for the geometrical relationships could still participate in discussions because of the support of the technological representations. Therefore, this study confirms previous findings (e.g., Ares et al., 2008; Sinclair, 2005) that interacting with dynamic representations enables and encourages students to talk deeply about mathematics.

Impact of computer feedback on student collaborations. As mentioned briefly in the previous section, the ability for technology to give dynamic feedback to students, either verbally or nonverbally, contributes to the level of classroom mathematical discourse. Studies have shown that valuable visual feedback provided by graphing software programs, among other technologies, prompt productive problem-solving student discourse. For example, Gibbs (2006) documented students' attempts to graph particular quadratic functions with varying scales. When the computer-produced graph did not visually match the graphs students had previously drawn, discussions ensued regarding the discrepancies and how to reconcile them. Likewise, other studies report positive effects on problem-solving discussions as a result of feedback from dynamic computer diagrams (González & Herbst, 2009; White, 2006).

Evans and colleagues (2011) conducted a study comparing effects of virtual and physical tangram puzzles on student discourse. Using a multimodal approach (speech, gesture, gaze, and actions) to analyze the discourse of 7- to 8-year-old children, the researchers identified more coreferences (i.e., shared reference points) among the students when using the virtual manipulative tangrams. They determined that discourses associated with the virtual manipulatives tended to be of a more collaborative nature, perhaps due to a forced focus on a common screen and having to negotiate control of the mouse. However, students using the physical tangram pieces had the option to handle the pieces individually without permission from the rest of the group. The focus on a common display to promote active mathematical discourse aligns with previous findings (Ares et al., 2008; White, 2006).

Shifts in teacher's role as facilitator of mathematical discourse. With the addition of technology to the classroom environment, the role of the teacher in facilitating mathematical discourse shifts slightly. During whole-class discussions, the teacher becomes responsible for orchestrating students' interactions with the technology as well as interactions with each other (Ares et al., 2008; González & Herbst, 2009). Furthermore, the teacher's modeling of appropriate discourse practices becomes even more imperative as students work in small-group collaborations on the computers (Sinclair, 2005; White, 2006). During these small-group collaboration sessions, the teacher's roles of intervening when necessary (Baxter & Williams, 2010), and questioning to extend students' thinking (e.g., Brodie, 2011; Gibbs, 2006; Hufferd-Ackles et al., 2004; Piccolo et al., 2008) become even more imperative as students work with dynamic technological representations.

Summary of Research on Techno-Mathematical Discourse

The research on the impact of technology on classroom mathematical discussions begins to shed light on how particular technologies influence students' mathematical discussions and the development of mathematical understanding. Most of the studies in this area have examined a single technology tool (e.g., networked graphing calculators, Geometer's Sketchpad, Cabri Geometry) and have been conducted with secondary students in a small number of classrooms. These studies used a variety of methods and analysis, including case studies (Gibbs, 2006; Sinclair, 2005), a discourse analysis (González & Herbst, 2009), and a mixed methods microgenetic analysis (Ares et al., 2008; see Siegler, 2006). Only one study (Evans et al., 2011) employed an experimental design and examined elementary students' discussions while using different types of tools (e.g., virtual tangrams compared to physical tangrams). More research on the effects of other technology tools (e.g., different types of virtual manipulatives) with various school settings and age groups will strengthen the knowledge base in this field of research.

Unique Contributions of the Current Study

A large corpus of research exists on the nature of whole-class discourse and the teacher's role in orchestrating discussions. However, relatively little research has focused on the unique social dynamics of two-person collaborations, and the research that does exist has produced mixed results. Additionally, the relatively new research base on the instructional use of virtual manipulatives has focused on their impact on student achievement and on how characteristics (i.e., simultaneous linking, feedback, etc.) of the virtual manipulatives affect student learning. However, little research has been conducted on how virtual manipulatives as technology tools facilitate mathematical discussions. By using the TMD framework, the current research study contributes to the field by examining specific relationships between the nature of small-group mathematical discourse and virtual manipulative types.

The study serves to inform future research on other aspects of TMD, such as the impact of different classroom discussion formats, different types of worthwhile mathematical tasks, and different technology tools. The study is significant because more and more classrooms are incorporating technology into mathematics instruction. Teachers and researchers need to understand how the technology facilitates classroom interactions and how to best leverage technology tools to enhance students' learning of mathematics.

CHAPTER III

METHODS

The purpose of this research study was (a) to describe and categorize the nature of students' mathematical discourse as they worked with various virtual manipulative types and (b) to develop theory on the interactions among partner discourse, virtual manipulatives, and mathematical tasks.

This study employed a mixed methods case study design (Creswell & Plano Clark, 2011; Luck et al., 2006) utilizing both qualitative and quantitative methods to analyze students' mathematical discussions. The bounded case was the placeholder in which qualitative and quantitative data were collected (Ellinger, Watkins, & Marsick, 2005; Luck et al., 2006). In this study, a case was defined as one pair of students as they discussed mathematics over time while using different types of virtual manipulatives. Onwuegbuzie and Collins (2007) suggested that mixed methods sampling designs should be characterized by the time coordination of qualitative and quantitative phases (i.e., whether they occur concurrently or sequentially) and by the samples used for qualitative and quantitative data collection (i.e., identical, nested, parallel, or multilevel). Qualitative and quantitative data for this study involved exactly the same sample members and was collected simultaneously. Therefore, this mixed methods case study utilized a concurrent design using identical samples.

The data for this case study were collected over the course of four months in one public school. While engaged in researcher-developed lessons, fifth-grade students participated in instructional activities incorporating different virtual manipulative types that were based on objectives from the Common Core State Standards (Common Core State Standards Initiative, 2010). Data were collected from field observations, student task sheets, and video recordings of the participating students' mathematical discussions and interactions with the virtual manipulatives. Data analysis included within- and cross-case analyses and focused on describing students' conversations, categorizing the descriptions, interpreting the categories, and developing theory. The over-arching question and subquestions guiding this study were as follows.

1. In what ways do different virtual manipulative types influence the nature of students' mathematical discourse?
   a. How do different virtual manipulative types influence the level of generalization in students' mathematical discourse?
   b. How do different virtual manipulative types influence the level of justification in students' mathematical discourse?
   c. How do different virtual manipulative types influence the level of collaboration in students' mathematical discourse?
   d. How do different virtual manipulative types influence physical and computer-based gestures when students are engaged in mathematical discourse?

The following sections outline the setting, participants, procedures, data sources, instruments, and data analysis for this study.

Setting and Participants

One elementary school in the western United States was chosen as a data collection site for this study. This school was chosen as a criterion-based purposive sample (Onwuegbuzie & Collins, 2007) because its administration has invested in classroom technology integration for several years. As a result, the students in the school were very familiar with using computer technology in the classroom for mathematics learning. Student laptops were used regularly in the upper-grade classrooms, and every two fifth-grade students shared one laptop regularly for classroom projects. Additionally, students could check them out to work on school projects at home. When used during mathematics class, the computers usually served as tools to take online assessments and to practice fluency with mathematics concepts. The school was located in an upper-middle class neighborhood, and students at the school were mostly Caucasian (98%). The school typically scored well above the district and state average on end-of-year exams. Teachers reported that parents regularly participate in school activities and volunteer in classrooms.

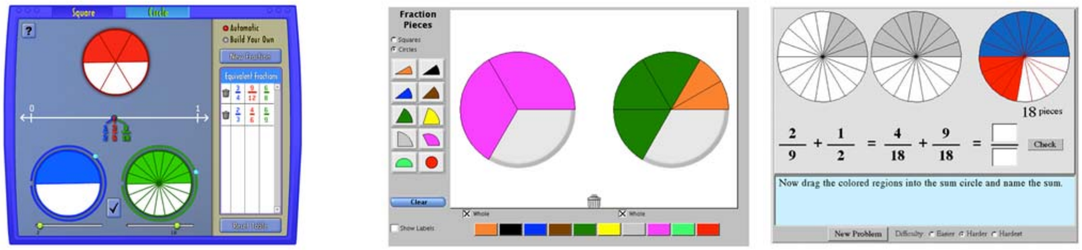
Three pairs of students participated in this study. All six student participants were Caucasian, and each pair consisted of one female and one male student, ages 10–11 years. The six student participants in this study came from three fifth-grade classrooms in the school chosen as a data collection site. The criteria for the selection of the participating students were that (a) they had demonstrated a tendency to process their thinking verbally and (b) they were paired with another student with whom they had a positive interaction and with whom they could easily converse. Mathematics achievement was not a deciding factor when selecting students for this study.

Procedures

The study was conducted in two stages: preimplementation and implementation. In the preimplementation stage, the researcher compiled a list of virtual manipulatives representing three different types (see Figure 3) and conferenced with the classroom teachers to select the virtual manipulatives to be used in the instructional activities. Next, the researcher developed lesson plans for the selected virtual manipulatives, created a website to facilitate the distribution of the lesson plans, obtained appropriate Institutional Review Board (IRB) and district approval (see Appendix A), and conducted a pilot study. Classroom teachers completed a demographic questionnaire on their typical use of technology and students' classroom discussion in mathematics lessons. In the implementation stage, each pair of participating students engaged in instructional activities led by the classroom teacher, and data were gathered from student task sheets, observation field notes, and video recordings of the participating students. The following sections detail the preimplementation and implementation phases.

Preimplementation

The researcher compiled a list of examples of each type of virtual manipulative (see Appendix B) and conferenced with the classroom teachers to select three of each type of virtual manipulative to be used in the study: combined, pictorial, and tutorial (see Figure 3). During the conference, the classroom teachers expressed interest in lesson topics not covered by the researcher's initial compilation. Consequently, the researcher expanded the search and identified additional virtual manipulatives that were acceptable



Combined Pictorial Tutorial

Figure 3. Examples of types of virtual manipulative apps used during lessons.

to the teachers, as well as to the research study. The same virtual manipulatives were used in each classroom with each pair of students. Selection of virtual manipulatives was based on the classroom teachers' planned instructional units (see Appendix B). This selection ensured that the virtual manipulatives used in the study aligned with the concepts that students would be learning at that point in the school year.

Next, the researcher developed lesson plans and student task sheets for the selected virtual manipulatives (see Appendices E and F). The lessons were based on objectives from the Common Core (Common Core State Standards Initiative, 2010) and incorporated different virtual manipulative types. These researcher-developed lesson plans were designed to promote discourse so that the researcher could observe how different virtual manipulative types influenced students' mathematical discussions. The classroom teachers reviewed and provided feedback on the lesson plans prior to implementation. The lesson plans, student task sheets, and virtual manipulative links were uploaded to a website developed by the researcher (see Figure 4). This website provided teachers and students with easy access to the resources needed to participate in the study. It also provided answers to frequently asked questions related to the study.

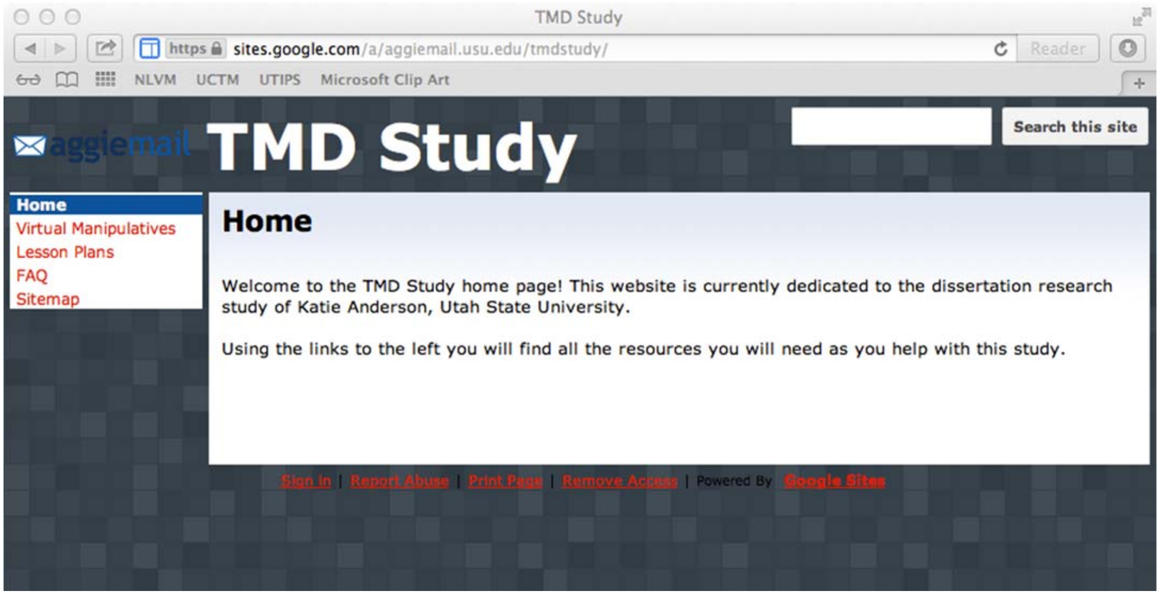


Figure 4. Study website.

Prior to the implementation of the lessons, the appropriate approval from the IRB and from the local district office was obtained. Finally, teachers completed a demographic questionnaire, which included items such as: number of students in the class; frequency of computer use by students; frequency of computer use by students for mathematics, and for what purpose; and the amount of mathematics instructional time used for group work, individual work, and teacher demonstrations (see Appendix C). The purpose of the demographic questionnaire was to provide a context for the students' mathematical discussions.

The structure of lessons designed for this study was based on a widely used three-stage guided-inquiry model (e.g., Hendrickson, Hilton, & Bahr, 2010; Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006), which included a 5- to 10-minute introduction of the learning task, a 20- to 30-minute small-group exploration of mathematical tasks and concepts, and a 20- to 30-minute whole-class discussion to examine students' solution strategies, highlight key mathematical concepts, and conclude the lesson. This lesson format has also been used in other research studies on virtual manipulatives (i.e., learning objects) in the field of instructional technology (Kay, 2012). Tasks for each virtual manipulative were adapted from tool-specific lesson explorations suggested by the illuminations.nctm.org and nlvm.usu.edu websites.

The design of each lesson began by introducing the mathematical concept (i.e., activating students' prior mathematical knowledge), demonstrating the virtual manipulative, and orienting the students to the task sheet that would guide their small-group explorations and discussions. The introduction was designed to last 5-10 minutes. The task sheets guiding the exploration consisted of two parts, Part A and Part B. Part A oriented students to the particular features of the virtual manipulative. This orientation gave them hands-on experience with the workings of the virtual manipulative before using it to explore specific mathematical concepts. Part B guided students' learning by posing specific tasks and discussion questions focused on key mathematical concepts. The exploration was designed to last 20-30 minutes. The design of each lesson concluded with an element requiring the teacher to conduct a whole-class discussion to debrief students' answers and observations while working with the virtual manipulative. The conclusion was designed to last 20-30 minutes. The purpose of this lesson design structure was to support students' exploration of mathematical concepts and to prompt their mathematical discussions. The lesson design structure is summarized in Table 1.

Lesson Implementation

The classroom teachers of the three participating pairs of students provided

Table 1

Lesson Design Structure

Stage	Duration	Activity	Purpose
Introduction	5-10 minutes	Demonstration	Introduce learning task
Exploration	20-30 minutes	Completion of student task sheet	Part A: Orientation to virtual manipulative Part B: Concept development and application
Conclusion	20-30 minutes	Whole-class discussion	Examine students' solution strategies, highlight key mathematical ideas

instruction according to the researcher-developed lessons during the implementation stage of the study. During the lessons, students worked in pairs, sharing a laptop computer while they worked through the assigned tasks. Data collection on students' work with the virtual manipulatives took place in three cycles. In each cycle, data were collected on students' discussions during three lessons—one lesson for each of the virtual manipulative types. During the second and third cycles, the students experienced the virtual manipulative types in a different order than in the first cycle (see Appendix B). Therefore, the three student pairs participated in nine lessons using the virtual manipulatives—a total of 27 observed lessons.

The participating pair of students in each classroom used a research study computer that was enabled to record video using a built-in camera. The use of built-in video recording capabilities eliminated the need for external video cameras and allowed the data collection to proceed with minimal intrusion on the authentic classroom experience. The video recording took place during the exploration stage of each lesson. It began when the teacher finished with the introduction stage of the lesson and ended when the teacher began the conclusion stage of the lesson. The purpose for recording the exploration stage of the lesson was to capture details of the participating students' mathematical discussions as they worked with the virtual manipulatives in pairs. In this stage of the lesson, students had the opportunity to discuss the mathematics with their partner.

Data Sources and Instruments

There were three sources of data in this study: video recordings, observation field notes, and student task sheets. The primary source of data was the video recordings. The purpose of the data collected in the form of observation field notes and student task sheets was to verify the data collected from the video recordings. In this way, multiple data sources were used to describe the nature of students' mathematical discussions for each type of virtual manipulative (Creswell & Plano Clark, 2011). The following sections describe the instruments used to collect each type of data.

Classroom Video Recordings

When examining the nature of classroom mathematical discussion, many researchers recommend using video recordings to collect data (e.g., Baxter & Williams, 2010; Brodie, 2011; Gresalfi et al., 2008; Hufferd-Ackles et al., 2004; Lau et al., 2009; Piccolo et al., 2008). When using video recording as a means of data collection, Hall (2000) recommended a wide framing that purposefully deletes or foregrounds particular aspects of the chosen activities depending on the theoretical lens used for the study. One way to accomplish this wide framing is to “selectively [attend] to different aspects of human activity” (Hall, 2000, p. 658). This study was designed to examine students’ interactions with the virtual manipulatives and with each other. In order to capture these two types of interactions, two different video perspectives were recorded (Lesh & Lehrer, 2000): face-capture and screen-capture. The first perspective, face-capture, utilized the built-in camera located at the top and center of the computer screen and made a video and audio recording of the students’ mathematical discussions, their facial expressions, and their gestures. The second perspective, screen-capture, recorded what the students did with the virtual manipulatives. This screen-capture included mouse movement, mouse clicks (a circle appeared around the pointer whenever it was clicked), and external audio. Both of these perspectives were recorded simultaneously using Quicktime Player.

By using this procedure, two videos were recorded during each lesson’s exploration stage—a total of 54 video files. For each lesson, the face-capture and screen-capture video files were imported into a video-editing program—Adobe Premiere Pro. Using a picture-in-picture feature, the two video files were combined to create a new video file with the face-capture recording superimposed on the screen-capture recording (see Figure 5). This combination allowed the researcher to view the students’ discussions, including facial expressions, and gestures, simultaneously with their actions with the virtual manipulative. The two videos were synced by matching up the audio recording that was included in the recording of each perspective. The video files were backed-up on a series of external hard drives: a working hard drive, a back-up hard drive, and an archive hard drive. Each video file was labeled with the type of virtual manipulative used

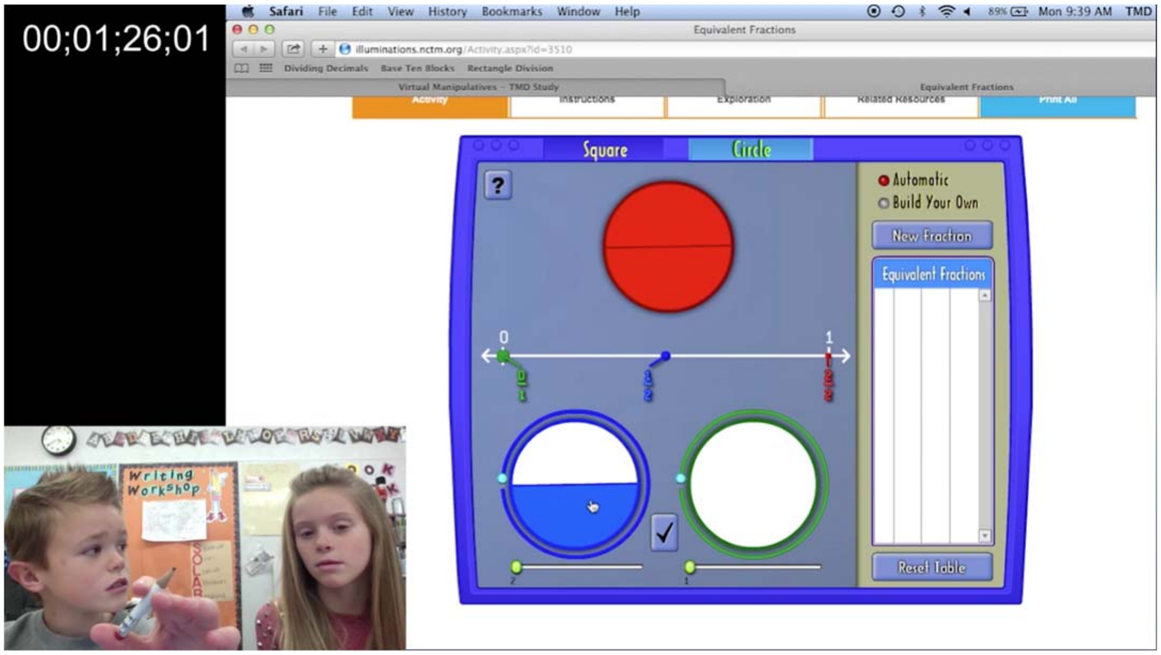


Figure 5. Sample of simultaneous video perspectives.

(signified as C, P, or T), the cycle number (signified as 1, 2, or 3), the student pair (signified as A, B, or C), and the date (e.g., P2C).

Observation Field Notes and Student Task Sheets

Merriam (2009) suggested three main sources when collecting data for qualitative analysis: observations, documents, and interviews. This study utilized two of these data sources (observations and documents) to provide multiple perspectives and to capture aspects of the learning environment not evident from the video recorded data. Lesh and Lehrer (2000) noted, “Every time a videocamera focuses on one thing, it tends to de-emphasize or ignore something else, and, in general, videotapes are poorly suited to record certain types of information” (p. 671). Many factors influence students’ mathematical discussions (e.g., lesson expectations, properly functioning technology, classroom environment), and some gestures may not be within the frame of the video camera. Thus, the researcher's observation field notes and students' task sheets provided valuable data on the nature of students' mathematical discussions as they worked with different virtual manipulatives that would have otherwise been missed in the video recording.

During each lesson, the researcher recorded written observation field notes on the targeted students' mathematical discussions. These observations were guided by a researcher-designed lesson observation protocol (see Appendix D). One focus of these observations was to track external and affective factors present in the classroom that were not captured by the video recording. Students' task sheets were also collected as evidence of mathematical concepts discussed during the lesson. At the conclusion of each lesson, the researcher shared the observation field notes and student task sheets with the classroom teacher to member-check the validity of the observations (Glesne, 2010).

Pilot Study

Prior to this study, a lesson plan was piloted in one classroom. The purpose of this pilot was to test the video recording equipment (particularly the scope of the built-in camera), the lesson design structure, the student task sheet questions, and the observation protocol. During the pilot study, the video recording equipment was able to capture the mathematical discussion and screen movement of the participating students. The pacing of the lesson allowed a sufficient amount of time for students to work in pairs, and the student task sheet questions effectively guided the students' exploration. The lesson observation protocol elicited valuable information.

In addition, two unexpected issues arose. First, the participating students were sitting near the front of the classroom, so the video recording captured much of the discussions from other pairs of students and other students' interactions with the teacher. At times, these occurrences distracted from the recording of the participating students' discussion. To avoid this issue during the study, the pair of participating students were seated near the back of the room. Second, the participating students were not sitting directly in front of the computer's camera. When their attention was directed toward the computer, the camera was able to capture their discussion without difficulty. However, when the participating students turned their attention to recording their answers on the task sheet, they leaned outside of the scope of the camera. Their discussion could be heard, but their facial expressions were not discernable. To avoid this during the current study, the researcher instructed the participating students to position their chairs so that they were both directly in front of the computer's camera.

Validity and Reliability

Establishing validity and reliability in a mixed method case study is achieved through rigorous practices of checking the quality and the interpretation of the data (Creswell & Plano Clark, 2011). Two common methods used to support validity in mixed method case studies are member-checking and data triangulation (Creswell & Plano Clark, 2011; Merriam, 2009; Yin, 2009). Member-checking involves sharing the preliminary findings of the study with the participants who provide feedback on the accuracy of the researcher's interpretations. Triangulation of the data from video recordings, student task sheets, and observation field notes strengthened this study by providing multiple sources of evidence from which to develop interpretations and theory (Imm & Stylianou, 2012; Huang et al., 2005; Mendez et al., 2007; Nathan & Knuth, 2003; Sherin, 2002; Truxaw & DeFranco, 2008). In order to support the construct validity of the study, previously established frameworks were used to code the video recorded discussions (McNeill, 1992; Mendez et al., 2007). These frameworks have been used in multiple studies to describe students' classroom discussions (e.g., Bjuland et al., 2008; Botzer & Yerushalmy, 2006; Mendez, 1998). A commonly used method for ensuring reliability is establishing intercoder agreement (Creswell & Plano Clark, 2011). One PhD-level researcher with over 30 years of experience in education and multiple research publications in mathematics education separately coded 10% of the video recorded discussions using the established coding frameworks resulting in 81% agreement.

Data Analysis

Data analysis procedures for this case study were designed to describe the mathematical discussions, categorized the descriptions, interpreted the categories, and developed theory on how virtual manipulative types (combined, pictorial, and tutorial) influenced students' TMD. Merriam (2009) recommended that case-study analyses begin by compiling and organizing all of the information about each case so that the data are easily retrievable. This compilation included coding results for each video recorded discussion, student task sheets, and observation field notes. The data were organized electronically using Microsoft Excel. Once the data were compiled and organized, analyses of the cases proceeded in two stages: within- analysis and cross-case analysis

(Merriam, 2009). The following sections outline the data analysis procedures.

Within-Case Analyses

Data analysis for each individual case (i.e., discussions by a pair of students) commenced immediately after the first cycle of lessons. The purpose of the within-case analyses was to provide an in-depth description, categorization, and interpretation of the discourse of each pair of students. First, the video recorded discussions were transcribed and coded. The analysis of the video data focused on quantitizing the case data (Tashakkori & Teddlie, 2010) with descriptive and inferential statistics of the students' use of gesture and discourse levels of generalization, justification, and collaboration. Second, the researcher analyzed the data (i.e., video recordings, transcripts, student task sheets, and observation field notes) using open and axial data coding to identify patterns and trends in students' discussions not identified in the research literature (Stake, 1995; Strauss & Corbin, 1998). Open and axial coding resulted in the development of categories and identification of significant events. Finally, the researcher interpreted the categories and formulated support for developing theory. The steps of transcribing, coding, categorizing, and interpreting were repeated for the second and third cycles of data collection. This process resulted in within-case analyses for three individual cases (i.e., pairs of students). The following sections detail the transcription and coding procedures.

Unitization and transcription. Studies of classroom discussions commonly use unitization schemes to organize data into segments for analysis (Nathan et al., 2007). In this study, each video recorded lesson was labeled as an episode, and students' individual contributions to the discussion were labeled as speaking turns. Gee (2005) suggested using stanzas (similar to paragraphs) to further organize the units that make up the discussion:

> Each stanza is a group of lines about one important event, happening, or state of affairs at one time and place, or it focuses on a specific character, theme, image, topic, or perspective. When the time, place, character, event, or perspective changes, we get a new stanza. (p. 109)

In this study, an episode was defined as the discussion from one instructional session. Each episode was labeled for the virtual manipulative type used, the cycle, and the student pair (e.g., C2A, T1B, P3C, etc.) The researcher transcribed the speaking turns in each episode and delineated stanzas within the episode. This transcription was then exported to a Microsoft Excel spreadsheet for coding of generalization, justification, and collaboration, and gestures.

Coding of video recorded data. First, each episode was coded along three dimensions of discourse: level of generalization, level of justification, and level of collaboration (Mendez et al., 2007). Each speaking turn in the episodes was coded for each dimension according to one of three categories representing increasingly more sophisticated understandings. Each speaking turn received three separate scores for its level of sophistication in each of the dimensions (0: not codable, 1: lowest level of sophistication, 2: middle level of sophistication, 3: highest level of sophistication). This coding scheme was adapted from Mendez and colleagues' framework for robust mathematical discussion. For levels of generalization, speaking turns were coded as concrete (limited to one specific context), comparison (between two concepts), or generalization (recognition of a pattern or making comparisons over multiple contexts).

For levels of justification, speaking turns were coded as statement (void of explanation), explanation (including how the answer was obtained), or proof (a logical argument for why the solution is correct). For levels of collaboration, speaking turns were coded as unrelated idea (no link to earlier comments), response (agreement or disagreement with other student's thinking without elaboration), or build (elaboration on other student's thinking or additional support).

Percentages and frequencies of codable speaking turns for each level on each dimension were calculated. These calculations provided a measure of the quantity of discourse in each episode. The leveled codes were also used to calculate separate composite scores for each dimension. The composite scores for each dimension consisted of a summation of the codes for each speaking turn within the episode for that dimension divided by the total number of codable speaking turns for that dimension, and then multiplied by 100. For example, a discussion with 100 total speaking turns coded for justification—60 turns as statement (level 1), 30 turns as explain (level 2), and 10 turns as proof (level 3)—would yield a composite score of $\left(\frac{(60\times 1)+(30\times 2)+(10\times 3)}{100}\right)100 = 150$ for the dimension of justification. The calculation of the composite scores provided a measure of the quality of discourse in each episode.

Second, each episode was coded for the frequency and type of gestures used during the discussion. First, each gesture used during the discussion was coded as either physical- or computer-based. Then, each gesture was coded according to McNeill's (1992) basic types of gestures: iconic (resembling a concrete object), metaphoric (resembling an abstract idea), beat (repetitive movements indicating significance), or deictic (pointing movements).

Therefore, for each discussion, the resulting video data were (a) percentages and frequencies of each discourse level and a composite score for each dimension of discourse: generalization, justification, and collaboration; (b) percentages and frequencies of physical gesture types; and (c) percentages and frequencies of computer-based gesture types. Table 2 provides an overview of the video data coding frameworks and analyses for each research subquestion.

Table 2

Video Data Analysis Overview

Research questions	Coding framework	Data analysis
Sub (a): How do different virtual manipulative types influence the level of ***generalization*** in students' mathematical discourse?	Code each speaking turn for level of generalization: • Concrete (1) • Comparison (2) • Generalization (3)	Percentages and frequencies Composite scores Graphical analysis One-way ANOVA
Sub (b): How do different virtual manipulative types influence the level of ***justification*** in students' mathematical discourse?	Code each speaking turn for level of justification: • Statement (1) • Explanation (2) • Proof (3)	Percentages and frequencies Composite scores Graphical analysis One-way ANOVA
Sub (c): How do different virtual manipulative types influence the level of ***collaboration*** in students' mathematical discourse?	Code each speaking turn for level of collaboration: • Unrelated idea (1) • Response (2) • Build (3)	Percentages and frequencies Composite scores Graphical analysis One-way ANOVA
Sub (d): How do different virtual manipulative types influence physical and computer-based ***gestures*** when students are engaged in mathematical discourse?	Code each physical and computer-based gesture: • Iconic • Metaphoric • Beat • Deictic	Percentages and frequencies Graphical analysis

Cross-Case Analyses

Once the analysis of each individual case was completed, data analysis across cases commenced. The purpose of the cross-case analyses was to examine the influence of virtual manipulative types on students' mathematical discussions and to build abstractions across cases (Merriam, 2009). These analyses provided an in-depth description, categorization, and interpretation of students' discourse when using different virtual manipulative types. The cross-case analysis followed a similar procedure as the within-case analysis (Merriam, 2009). First, the researcher re-examined the video recorded data by grouping discussions based on the type of virtual manipulative used. A side-by-side comparison (Creswell & Plano Clark, 2011) of students' discourse was displayed numerically and graphically (for example, see Figure 8 in the following chapter). Second, the researcher identified categories using open and axial coding (Stake, 1995; Strauss & Corbin, 1998). Finally, the researcher interpreted the categories and formulated support for developing theory.

A one-way ANOVA on the amount of coded speaking turns was conducted for each dimension of discourse to compare the quantity of discourse for each virtual manipulative type. A one-way ANOVA on composite scores was also conducted for each dimension of discourse to compare the quality of discourse for each virtual manipulative type. This process resulted in cross-case analyses of students' discussions for each of the three virtual manipulative types used in the study (i.e., combined, pictorial, and tutorial).

CHAPTER IV

RESULTS

The purpose of this study was to (a) describe and categorize the nature of students' mathematical discourse as they worked with various virtual manipulative types and (b) to develop theory on the interactions among student-led discourse, virtual manipulatives, and mathematical tasks. This study used both quantitative and qualitative analyses to answer the research questions. The research question guiding this study was: In what ways do different virtual manipulative types influence the nature of students' mathematical discourse? Four subquestions focusing on aspects of discourse further delineated the focus of the study: (a) How do different virtual manipulative types influence the level of *generalization* in students' mathematical discourse? (b) How do different virtual manipulative types influence the level of *justification* in students' mathematical discourse? (c) How do different virtual manipulative types influence the level of *collaboration* in students' mathematical discourse? (d) How do different virtual manipulative types influence physical and computer-based *gestures* when students are engaged in mathematical discourse? The results presented in the sections that follow are based on 54 videos from 27 lessons with three student pairs.

First, a within-case analysis was conducted to provide an in-depth description, categorization, and interpretation of the discourse of each pair of students. Quantitative and qualitative analyses were used to describe and characterize the students' use of gesture and levels of generalization, justification, and collaboration. Second, a cross-case analysis was conducted to examine the influence of virtual manipulative type on students' mathematical discussions. Analyses focused on determining differences and identifying patterns and trends in the levels of generalization, justification, and collaboration among different virtual manipulative types. The following sections describe the results of the within-case and cross-case analyses.

Within-Case Analysis of Student Pairs

Three student pairs from three different classrooms participated in this study. Each student pair consisted of one boy and one girl. The results that follow present the coded speaking turns and the non-coded speaking turns from their mathematical discussions. Coded speaking turns include those reflecting students' mathematical thinking. Non-coded speaking turns include those reflecting off-task behavior, students' reading of the instructions, or discussion of the requirements of the assignment. The following sections provide a description, categorization, and interpretation of the discourse of each pair of students. Quantitative results supported by qualitative descriptions are presented for each student pair.

Quantitative Comparison of Student Pairs

Amount of discourse. A comparison of the amount of discourse for each pair of students reveals similarities and differences as they worked with the virtual manipulatives. Table 3 summarizes the totals and means for episode duration, speaking turns, speaking turns per minute, and coded versus noncoded speaking turns for each student pair.

Table 3

Frequency of Coded and Noncoded Speaking Turns for Student Pairs

	Aaron and Abbie		Brandon and Bonnie		Colton and Callie	
	Total	M^a	Total	M^a	Total	M^a
Duration	177m 11s	19m 41s	157m 06s	17m 27s	187m 56s	20m 53s
Speaking turns	1840	204.44	1027	114.33	960	106.67
Speaking turns per minute	10.38		6.54		5.11	
Generalization						
Coded	845 (45.92)	93.89	587 (57.05)	65.22	539 (56.15)	59.89
Noncoded	995 (54.08)	110.56	440 (42.95)	48.89	421 (43.85)	46.78
Justification						
Coded	819 (44.51)	91.00	568 (55.20)	63.11	522 (54.38)	58.00
Noncoded	1022 (55.49)	113.44	459 (44.80)	51.00	438 (45.63)	48.67
Collaboration						
Coded	1145 (62.23)	127.22	679 (65.99)	75.44	640 (66.67)	71.11
Non-coded	695 (37.77)	77.22	348 (34.01)	38.67	320 (33.33)	35.56

[a] $N = 9$ episodes

As Table 3 shows, discussion between the first pair of students, Aaron and Abbie, was characterized by a considerably higher number of speaking turns in their discussions in relation to the other participating pairs of students, averaging 204.44 speaking turns per episode with an average episode lasting 19m 41s, approximately 10.38 speaking turns per minute. Similarly, Colton and Callie's discussions lasted, on average, 20m 53s. However, for a similar amount of time, Colton and Callie had nearly half the amount of speaking turns (106.67) in their discussions as Aaron and Abbie, resulting in approximately 5.11 speaking turns per minute. Brandon and Bonnie's discussions (17m 27s) were considerably shorter than those of the other two pairs and had a similar number of speaking turns (114.33) as Colton and Callie's discussions, resulting in approximately 6.54 speaking turns per minute. Therefore, the rate of speaking turns per minute was considerably higher for Aaron and Abbie's discussions in relation to both Brandon and Bonnie's and Colton and Callie's and discussions.

Aaron and Abbie's discussions had the lowest percentage of coded speaking turns across all dimensions of discourse. This low percentage of coded speaking turns indicates the amount of off-task discussion that occurred with this student pair. For all student pairs, approximately half of the speaking turns were coded for generalization and justification. However, about two thirds of the speaking turns were coded for collaboration.

A one-way ANOVA comparison of the number of speaking turns indicated a statistically significant overall difference among the student pairs at the 95% level, $F(2, 24) = 15.393$, $p < 0.001$. This corresponded to an effect size of $\eta^2 = .56$; that is, about 56% of the variance in number of speaking turns was predictable from the student pair. This is a moderate effect. Individual post hoc comparisons using Tukey's HSD indicated a statistically significant difference between Aaron and Abbie and each of the other two student pairs, $p < 0.001$, based on their speaking turns. There was not a statistically significant difference between Brandon and Bonnie and Colton and Callie.

Levels of discourse. Further comparison of the levels of discourse also reveals similarities and differences among the student pairs. Table 4 summarizes the composite scores and totals and averages for coded speaking turns at each level to answer the specific research subquestion on generalization for each student pair.

As Table 4 shows, for the dimension of generalization, all three pairs of students had similar composite scores, with Colton and Callie's average score (118.53) being slightly higher in relation to the other two pairs. Aaron and Abbie had the lowest percentage of speaking turns coded at the highest level of generalization (4.97%). The other two pairs of students had similar percentages of speaking turns coded at this level (5.96% and 6.49%). Overall, a large majority of speaking turns coded for the generalization levels occurred at the lowest level, concrete, with Brandon and Bonnie's discussions having the highest percentage of speaking turns at this level (90.46%). A one-way ANOVA comparison of the composite scores for generalization indicated that the overall difference among the student pairs was not statistically significant.

Table 4

Composite Scores and Frequency of Speaking Turns for Each Level of Generalization

	Aaron and Abbie		Brandon and Bonnie		Colton and Callie	
Level of generalization	Total $n = 845$	M^a	Total $n = 587$	M^a	Total $n = 539$	M^a
Composite score	1044.88	116.10	1048.80	116.53	1066.79	118.53
Concrete (lowest level)	746 (88.28)	82.89	531 (90.46)	59.00	468 (86.83)	52.00
Comparison	57 (6.75)	6.33	21 (3.58)	2.33	36 (6.68)	4.00
Generalization (highest level)	42 (4.97)	4.67	35 (5.96)	3.89	35 (6.49)	3.89

[a] $N = 9$ episodes.

Table 5 summarizes the composite scores and totals and averages for coded speaking turns at each level of justification for each student pair, the second research subquestion.

As Table 5 shows, for the dimension of justification, Aaron and Abbie's average composite score (117.17) was considerably lower in relation to Brandon and Bonnie's score (127.45) and Colton and Callie's score (124.27). Their low score resulted from the low percentage of speaking turns in their discussions coded as proof (2.69%). The other two pairs of students had similar percentages of speaking turns in their discussions coded at this highest level of justification (4.93% and 4.21%). Brandon and Bonnie had the fewest percentage of speaking turns coded for statement (78.17%), and the highest percentage of speaking turns coded for explanation and proof (16.90% and 4.93%). This combination contributed to Brandon and Bonnie having the highest composite score for justification of the three pairs of students. This shows that Brandon and Bonnie's

Table 5

Composite Scores and Frequency of Speaking Turns for Each Level of Justification

	Aaron and Abbie		Brandon and Bonnie		Colton and Callie	
Level of justification	Total $n = 818$	M^a	Total $n = 568$	M^a	Total $n = 522$	M^a
Composite score	1054.50	117.17	1147.06	127.45	1118.39	124.27
Statement (lowest level)	693 (84.62)	76.89	444 (78.17)	49.33	419 (80.27)	46.56
Explanation	104 (12.70)	11.56	96 (16.90)	10.67	81 (15.52)	9.00
Proof (highest level)	22 (2.69)	2.44	28 (4.93)	3.11	22 (4.21)	2.44

[a] $N = 9$ episodes

discussions reflected higher levels of justification than the other two pairs of students. Overall, a large majority of speaking turns coded for justification occurred at the lowest level, statement, with Aaron and Abbie's discussions having the highest percentage of speaking turns at this level (84.62%). A one-way ANOVA comparison of the composite scores for justification indicated that the overall difference among the student pairs was not statistically significant.

Table 6 summarizes the composite scores and totals and averages for coded speaking turns at each level of collaboration for each student pair, the third research subquestion.

For the dimension of collaboration, Colton and Callie's discussions had the highest composite score (169.80). They had the lowest percentage of speaking turns coded as unrelated idea (52.66%), and the highest percentage of speaking turns coded as build (23.13%). This shows that Colton and Callie's discussions were highly collaborative. Brandon and Bonnie's discussions had the highest percentage of speaking

Table 6

Composite Scores and Frequency of Speaking Turns for Each Level of Collaboration

	Aaron and Abbie		Brandon and Bonnie		Colton and Callie	
Level of collaboration	Total *n* = 1145	M^a	Total *n* = 879	M^a	Total *n* = 640	M^a
Composite score	1427.78	158.64	1346.28	149.59	1528.24	169.80
Unrelated idea (lowest level)	615 (53.71)	68.33	424 (62.44)	47.11	337 (52.66)	37.44
Response	366 (31.97)	40.67	170 (25.04)	18.89	155 (24.22)	17.22
Build (highest level)	164 (14.32)	18.22	85 (12.52)	9.44	148 (23.13)	16.44

[a] N = 9 episodes.

turns coded as unrelated ideas (62.44%). Aaron and Abbie's discussions had the highest percentage of speaking turns coded as response (31.97%). Overall, just over half of the speaking turns coded for collaboration occurred at the lowest level, unrelated idea.

Discourse maps. This section presents one discourse map from one episode for each pair of students to provide a graphic representation of students' conversations. The sample discourse maps (see Figures 6, 7, and 8) illustrate the differences in each pair's levels of collaboration. Gaps in the map (i.e., unconnected points) indicate speaking turns not coded for collaboration. Figures 6, 7, and 8 show the levels of each speaking turn coded for collaboration throughout the course of an episode for the three different student pairs. The discourse map of Aaron and Abbie's discussion is tight and concentrated, reflecting a high number of speaking turns overall (see Figure 6). The discourse map of Brandon and Bonnie's discussion is more sparse, reflecting fewer speaking turns overall and limited speaking turns coded at the highest level, build (see Figure 7). The discourse map of Colton and Callie's discussion (see Figure 8) shows a similar number of speaking turns coded at the highest level as Aaron and Abbie's discourse map, yet has fewer speaking turns coded overall. This reflects the highly collaborative nature of Colton and Callie's discussions.

A one-way ANOVA comparison of the composite scores for collaboration indicated a statistically significant overall difference among the student pairs at the 95% level, $F(2, 24) = 3.645$, $p = 0.041$. This corresponded to an effect size of $\eta^2 = .23$; that is, about 23% of the variance in collaboration composite scores was predictable from the student pair. This is a small effect size. Individual post hoc comparisons using Tukey's

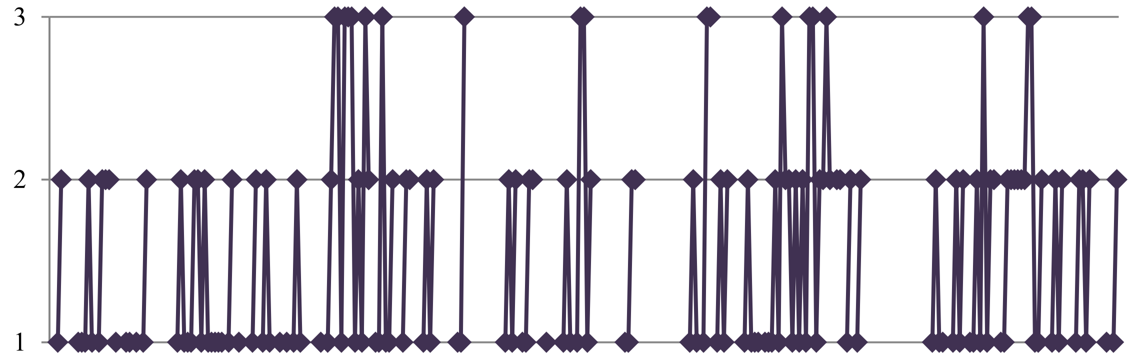


Figure 6. Discourse map of speaking turns coded for levels of collaboration: Aaron and Abbie, episode P3A.

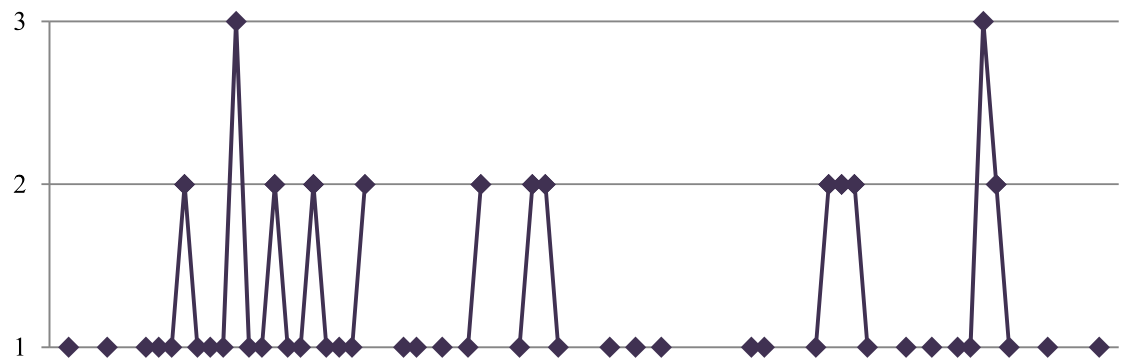


Figure 7. Discourse map of speaking turns coded for levels of collaboration: Brandon and Bonnie, episode P3B.

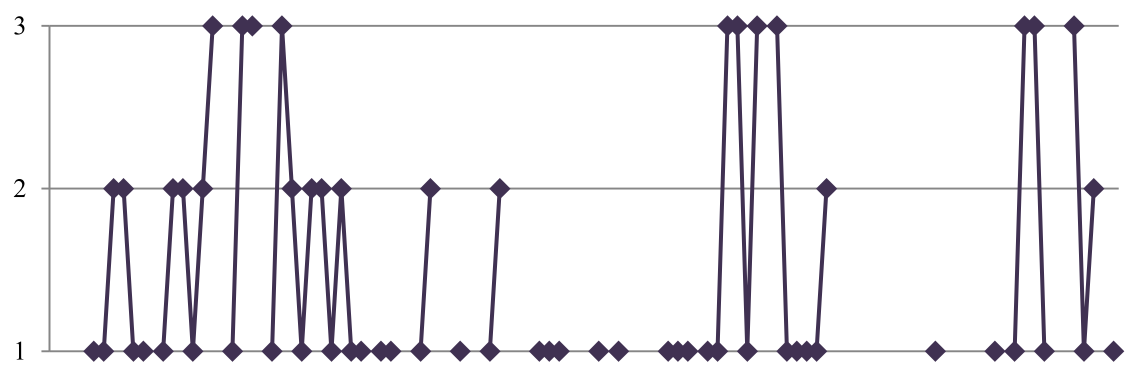


Figure 8. Discourse map of speaking turns coded for levels of collaboration: Colton and Callie, episode P3C.

HSD indicated a statistically significant difference between Brandon and Bonnie’s and Colton and Callie’s collaboration scores, $p = 0.033$. There was not a statistically significant difference between Aaron and Abbie’s and Brandon and Bonnie’s collaboration scores or between Aaron and Abbie’s and Colton and Callie’s collaboration scores.

Aaron and Abbie: Fast-Paced and Divergent

Aaron and Abbie’s discussions were characterized by enthusiastic fast-paced comments (see Figure 6). They enjoyed working together and supported each other in making sense of the mathematics. Most of the time, their discussions were productive and related to the assigned learning task. In the following excerpt from episode P1A, Aaron and Abbie are using a pictorial virtual manipulative, Shape Sorter, to sort seven triangles according to a series of rules (see Figure 9). This exchange lasted 2 minutes 5 seconds, and illustrates a typical fast-paced interaction between Aaron and Abbie.

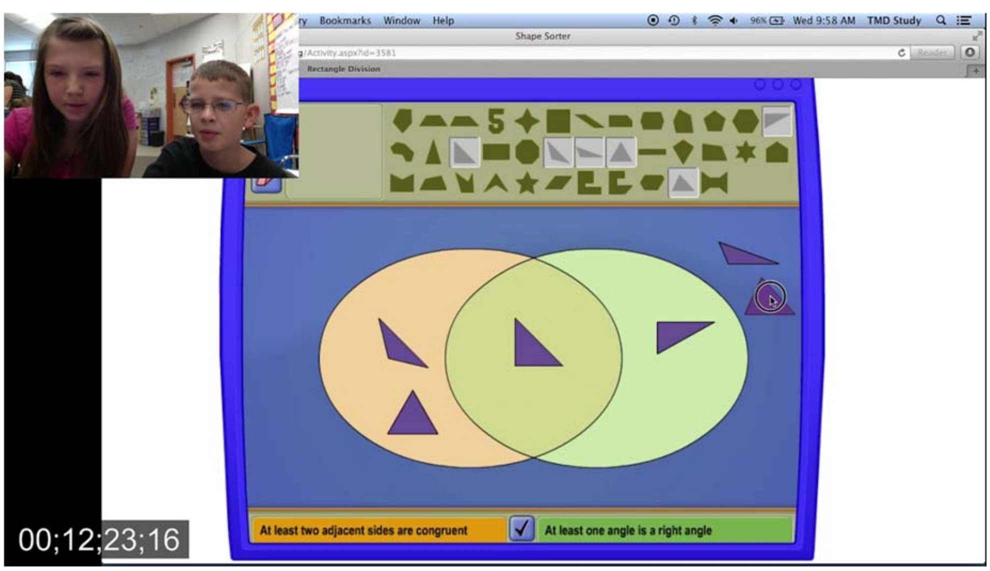


Figure 9. Aaron and Abbie working with shape sorter, episode P2A.

62	Aaron	"With the orange select a rule menu, select 'at least 2 adjacent sides are congruent.'" at least two… "With the green select a rule, 'at least one angle is a right angle.'"
63	Abbie	So at least one angle is a right angle?
64	Aaron	Whoa! That is so cool!
65	Abbie	Ok, now…
66	Aaron	"Talk to your partner and drag all 7 triangles to the correct region on the diagram. Check your work with the check mark icon. If any are incorrect, try to move them to the correct region." Ok. So we need to find some… ok
67	Abbie	…that have both. So why don't we have a right angle…
68	Aaron	Ok let's… hey, remember when I said, this one has a right angle!
69	Abbie	Yeah, but…
70	Aaron	And these have…
71	Abbie	Are they congruent sides?
72	Aaron	Yeah. This one and this one.
73	Abbie	No, they're not.
74	Aaron	See these are the same.
75	Abbie	But is that one?
76	Aaron	At least *two.*
77	Abbie	Ok. Ok now.
78	Aaron	This one has two adjacent, but it has no right angle. Right?
79	Abbie	Yeah.
80	Aaron	This one…
81	Abbie	…is outside.
82	Aaron	Now this one is here. That's 4. This has a right angle, but no sides are adjacent.
83	Abbie	So…
84	Aaron	And then this one. No right angle and no adjacent side. And I think…that's none of them are the same.
85	Abbie	Wait. Unless you tilted it to the side…
86	Aaron	I don't think that counts. Right angle! Right, right, right, right angle! Right?
87	Abbie	No.
88	Aaron	No, that's obtuse.
89	Abbie	Wait, what's the…
90	Aaron	4, 5, 6… no that's not it. No, no, no. No, there's no right angle.
91	Abbie	No, there is no right angle.
92	Aaron	2, 5, 6… and then this one goes… (places last triangle) Let's try!
93	Abbie	And then…
		(VM feedback: all correct)
94	Aaron	Oh! First guesses!

In episode P2A, Aaron and Abbie’s discourse consisted of fast-paced interactions. Many times these students interrupted each other (e.g., lines, 67, 70, 71, 81). Sometimes Aaron and Abbie’s discussions were off task, diverging from the task at hand either to talk about an incident at recess or to push the boundaries of the virtual manipulative. However, they quickly returned to the task at hand and completed the assignment in a timely manner. Aaron typically initiated these tangents, while Abbie attempted to focus him on the assigned task. In the following excerpt from episode C2A, Aaron and Abbie are using a combined virtual manipulative, Cubes, to determine the volume of a 3 x 5 x 7 box (see Figure 10). This exchange illustrates the tendency to diverge from the task at hand.

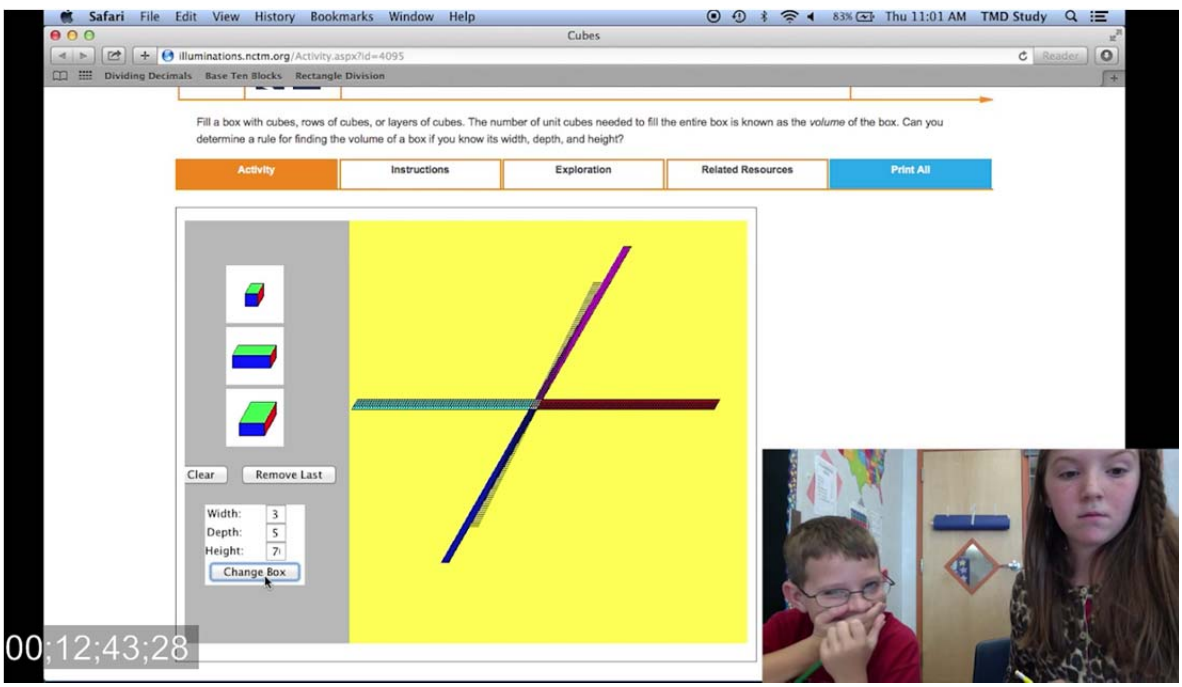


Figure 10. Aaron and Abbie working on cubes, episode C2A.

168	Aaron	What's 3 by what?
169	Abbie	1, 2, 3…
170	Aaron	It's 3 times 5.
171	Abbie	1, 2, 3, 4, 5.
172	Aaron	15.
173	Abbie	So 15, and then…
174	Aaron	15, 30…let's do this. 15, 30…45…60.
175	Abbie	50? No…
176	Aaron	45, 60.
177	Abbie	60.
178	Aaron	75…90…105, no 100.
179	Abbie	No, because 80.
180	Aaron	Wait, no! Even better! What's 15 times 7? (Teacher interrupts for whole-class announcement.)
181	Aaron	(Changes height dimension from 7 to 700) Haha! I did a box that's 700 high. Wait. Look at this. This is 70 height.
182	Abbie	You can't…
183	Aaron	It would take forever. Ok, so let's change this back to 7. Sorry. I wanted to…. That's what happens when it's 70 high.
184	Abbie	Ok, so.
185	Aaron	I did the math. 15 times 7 equals 105.
186	Abbie	So it would be…
187	Aaron	105 teeny boxes.
188	Abbie	105…unit…blocks.

Although Aaron and Abbie tended to include off-task topics in their conversation (line 181), the fast-paced nature of their discussion enabled them to still be productive in completing the assigned tasks in the same amount of time as the other student pairs (see Table 3).

Brandon and Bonnie: Minimal and Disengaged

Brandon and Bonnie's discussions were characterized by brief, to-the-point interactions interspersed by periods of observed disengagement from each other and from the assigned task. Most of the time, their discussions reflected the minimum amount of interaction required to complete the task. In the following excerpt from episode C3B, Brandon and Bonnie are using a combined virtual manipulative, Equivalent Fractions, to create circle area models of equivalent fractions (see Figure 11). This exchange illustrates the tendency to want to perform the least amount of work as possible.

59	Bonnie	Ok. Next one. New fraction. Should we do this one? Six-eighths.
60	Brandon	Let's see. (clicks 'New Fraction' repeatedly)
61	Bonnie	Get one that's easier.
62	Brandon	…There!
63	Bonnie	One-third. Ok. That's easy.… Make that one two-sixths. Make that two-sixths. That one…
64	Brandon	…would be…
65	Bonnie	…9. Make that a 9. 9…. Make 4…. No…3, yeah. 9. It's the same.
66	Brandon	So the number 2…
67	Bonnie	Blue was two-sixths…. Red was one-third…. And three-ninths…. Number 3.
68	Brandon	Ooo! I want to make it one whole… Ooo! One half, even better.

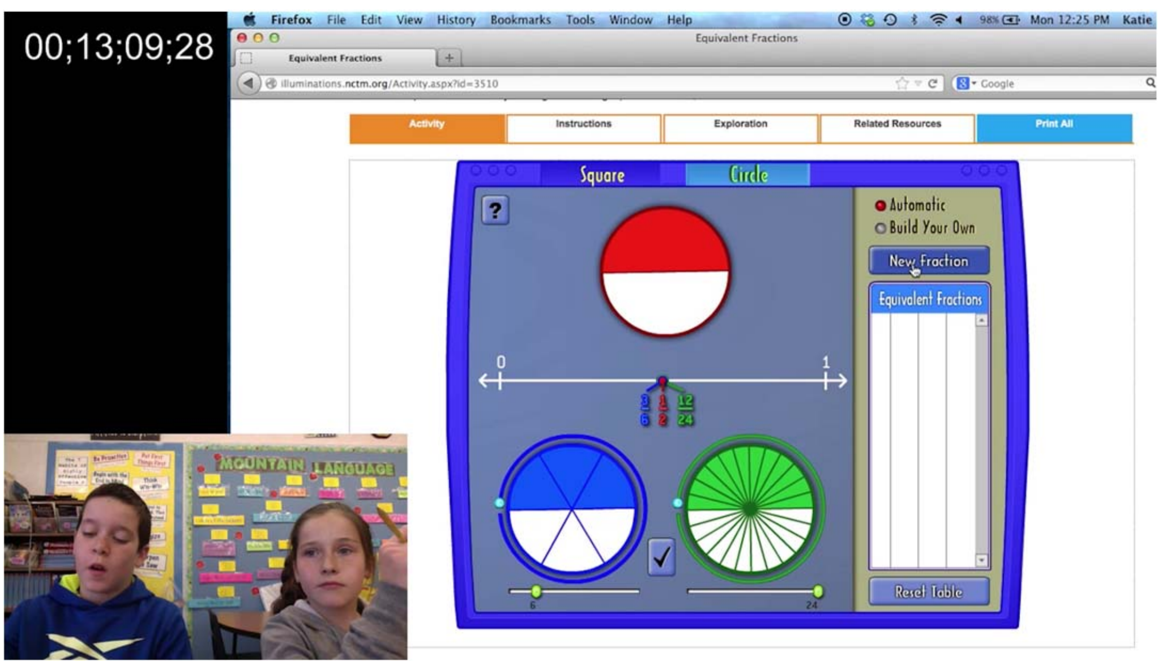


Figure 11. Brandon and Bonnie working on equivalent fractions, episode C3B.

In episode C3B, Brandon and Bonnie avoided fractions that appeared complicated and only selected “easier” fractions (line 61). Brandon and Bonnie’s discussions while working with the virtual manipulatives were also marked with frequent moments of disengagement from each other. For example, Brandon made a comment about the task while Bonnie was not paying attention to what he was saying. At other times, both Brandon and Bonnie were reasoning through the task, but neither was listening to the other. They each verbalized their own thought processes without considering the thinking of the other. This lack of interaction coupled with the tendency to want get through the task quickly typically resulted in brief discussions (see Table 3). In the following excerpt from episode C2B, Brandon and Bonnie are using a combined virtual manipulative described previously, Cubes, to determine the volume of a 3 x 5 x 7 box, and of a 5 x 7 x 3 box (see Figure 12). This exchange illustrates the tendency of this student pair to not attend to the other’s comments and to observably disengage from each other.

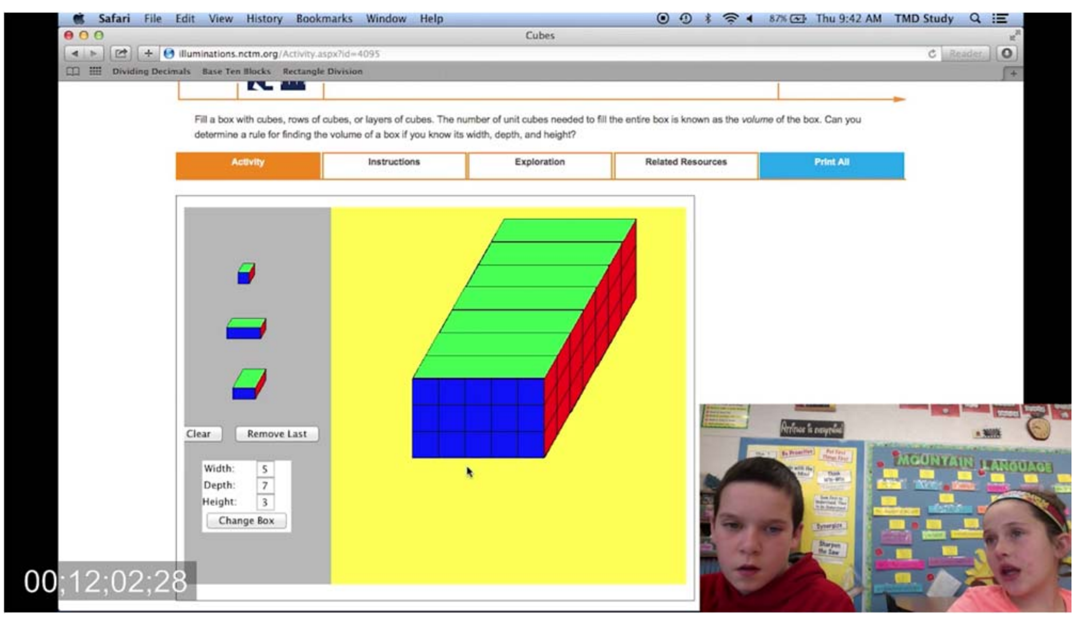


Figure 12. Brandon and Bonnie working on cubes, episode C2B.

42	Bonnie	"Use the virtual manipulatives to solve the following problems. Talk to your partner and record your answers. How many blocks will fill a box with dimensions width 3, depth 5 and height 7?" …(types dimensions) And change box. Whoa!
43	Brandon	Ok. How many blocks will fill…? How many…?
44	Bonnie	So…
45	Brandon	Well it takes 7…. It takes 7 layers….
46	Bonnie	So 1, 2, 3, 4, 5, 6, 7…. 1, 2, 3, 4, 5…. 5 times 3. That equals 15.
47	Brandon	…Times 7. So 15 times 7…. 15…35…70…105.
48	Bonnie	1, 2, 3, 4…
49	Brandon	It's 105.
50	Bonnie	So 1, 2, 3, 4, 5, 6. One more layer.
51	Brandon	It takes 105.
52	Bonnie	So 15 times 7… 35…equals…105 unit-blocks…to… fill…the… box.
53	Brandon	Ok. 5, 7, 3. (types new dimensions) Ok. Change box. So it would be….
54	Bonnie	Should we do row-blocks?
55	Brandon	Wait, we need to know how many it would take to fill all of it. So it would be…. 1, 2, 3, 4, 5. 1, 2, 3, 4, 5, 6, 7. 5 times 7 is 35, times….
56	Bonnie	Ok, what is the volume of a box…? Wait. No.
57	Brandon	What's the height…? 35 times 3. It would be 35 times 3.
58	Bonnie	1, 2, 3, 4….
59	Brandon	15.
60	Bonnie	…5, 6, 7 times 3!
61	Brandon	It's 105 again…. It's 105 again, 105.
62	Bonnie	It takes 7 times 3. That equals…21.
63	Brandon	I did…
64	Bonnie	21. 21 blocks.
65	Brandon	21 times 5.
66	Bonnie	'Cause there's 7 in each of these. Then you times it by 3 and that equals…. You get 21.
67	Brandon	Times…. I got 105 because 7 times 5 is 35, times 3 is 105.
68	Bonnie	by 35? How did you get 35? (Brandon removes blocks from the model) You could have just hit clear.
69	Brandon	1, 2, 3, 4, 5. 1, 2, 3, 4, 5, 6, 7. 7 times 5 is 35. And you need 3 layers to fill it. So 35 times 3 is 105.
70	Bonnie	…ok…times…3…5, but then you add it with the row-blocks. That would be 21. If you do it with row-blocks it would be 21. If you do it with unit-blocks it would be 105….
71	Brandon	It says, "How many unit-blocks will fill a box with dimensions width 3, depth 5, height 7? What about a box with dimensions

> width 5, depth, 7, height 3?" So we're still talking about unit blocks.

As shown in episode P2B, Brandon and Bonnie typically worked out the mathematics for the assigned problems independently. However, near the end of the exploration, they typically verified and explained their answers to each other (lines 66–71). This tendency explains why Brandon and Bonnie had the highest composite score for justification (see Table 5).

Colton and Callie: Executive and Collaborative

Colton and Callie's discussions were characterized by executive, task-oriented, and collaborative interactions. They worked together to understand the assigned tasks, and then proceeded to accomplish the tasks effectively and efficiently. Their discussions reflected on-task behavior throughout each of the recorded episodes. As a result of the executive nature of their discussions, Colton and Callie averaged the lowest number of speaking turns (106.67) of all of the participating student pairs. However, the duration of their discussions (20m 53s) averaged as the highest of all of the participating student pairs. This was because their discussions contained long pauses in which they were reflecting on each other's comments or verifying that they had completed the assigned tasks. In the following excerpt from episode P1C, Colton and Callie are using a pictorial virtual manipulative, Base 10 Blocks, to solve a story problem involving division of a decimal fraction by a whole number (see Figure 13). This exchange illustrates a typical executive interaction between Colton and Callie.

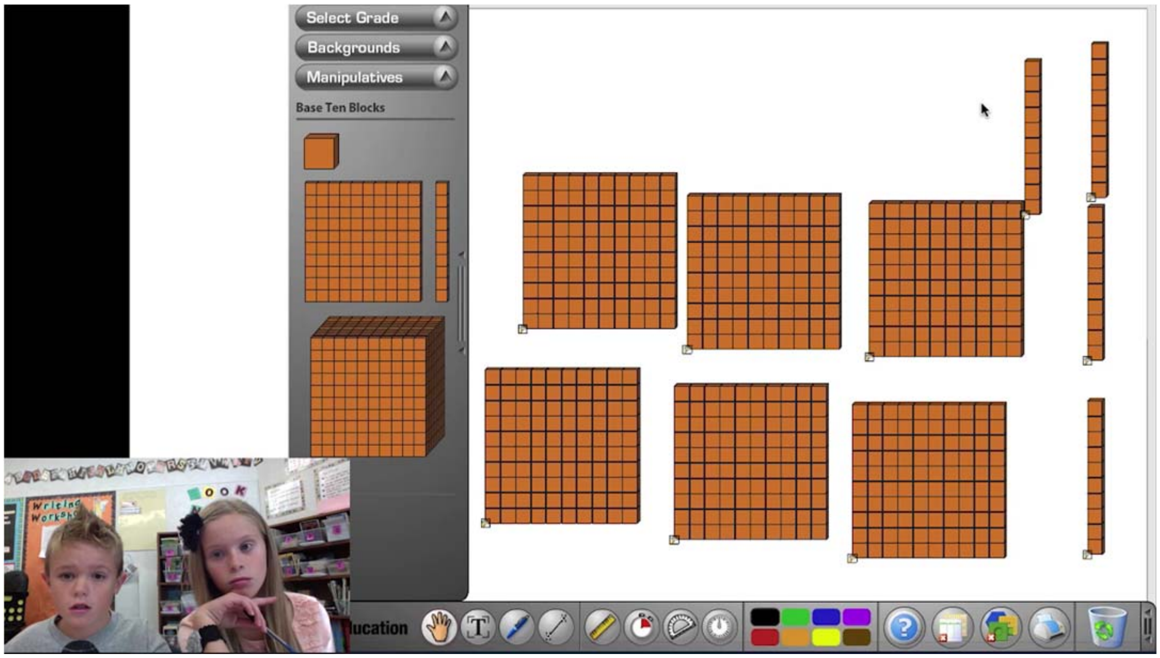


Figure 13. Colton and Callie working on base 10 blocks, episode P1C.

43	Colton	"Nancy's poster for the school council election covers a space of 6 point 4 square meters. She wants to divide the poster into 4 equal sections for her slogan. How much space will be in each section? Hint: one 10 by 10 square represents 1 square meter. Talk with your partner about how to solve this problem. Write down your answer and explain your thinking."
44	Callie	So…. Let's clear. So…
45	Colton	So 64 divided by 4 equals what?
46	Callie	Yeah, so we know that we have 6 point 4 square meters. And a 10 by 10 equals one square meter. So if we have six, then we need six 10 by 10s. (Colton moves blocks) ...Then it says "Talk with your partner about…." So we know that we have 6 whole pieces. But then what about the other…?
47	Colton	We need point 4. So that would be the tens.
48	Callie	Yeah. So we need 4 of those…. Ok, so then it says "Write down your answer and explain your thinking."
49	Colton	So we need to split it into 4 equal sections. So that's…
50	Callie	You can do it. (Gives control of mouse to Colton)
51	Colton	1, 2, 3…. One and a half…. Wait. One and a half. Then add one of these. That's 1 point 6 meters. And then…. Yeah. It would be 1…. Wait.
52	Callie	We need 1 point 1 if we take those two.
53	Colton	Yeah. And then you add the 2 left over….
54	Callie	Yeah. Two 10 by 10s.

55	Colton	And then 1 divided by 4 would be…. No, 2 divided by 4 would be a half. So you would add another point 5.
56	Callie	So…
57	Colton	That would be 1 point…6.
58	Callie	6. Yeah.
59	Colton	So it would be 1 point 6 square meters.

In episode P1C, Colton and Callie built on each other's ideas as they solved the problem (lines 52-55). High levels of collaboration also characterized Colton and Callie's discussions. They worked together to clarify misconceptions and built on each other's ideas. For example, in one lesson Colton and Callie were trying to find a simplified fraction for five-ninths. After multiple failed attempts at creating a model equivalent to five-ninths, Colton reasoned that it was already in its simplest form because "nothing can go into five AND nine except for one." In the following excerpt from episode C2C, Colton and Callie are using a combined virtual manipulative, Cubes, to determine the volume of a 3 x 5 x 7 box, of a 5 x 7 x 3 box, and of a 7 x 3 x 5 box (see Figure 14). This exchange illustrates the collaborative nature of their discussions.

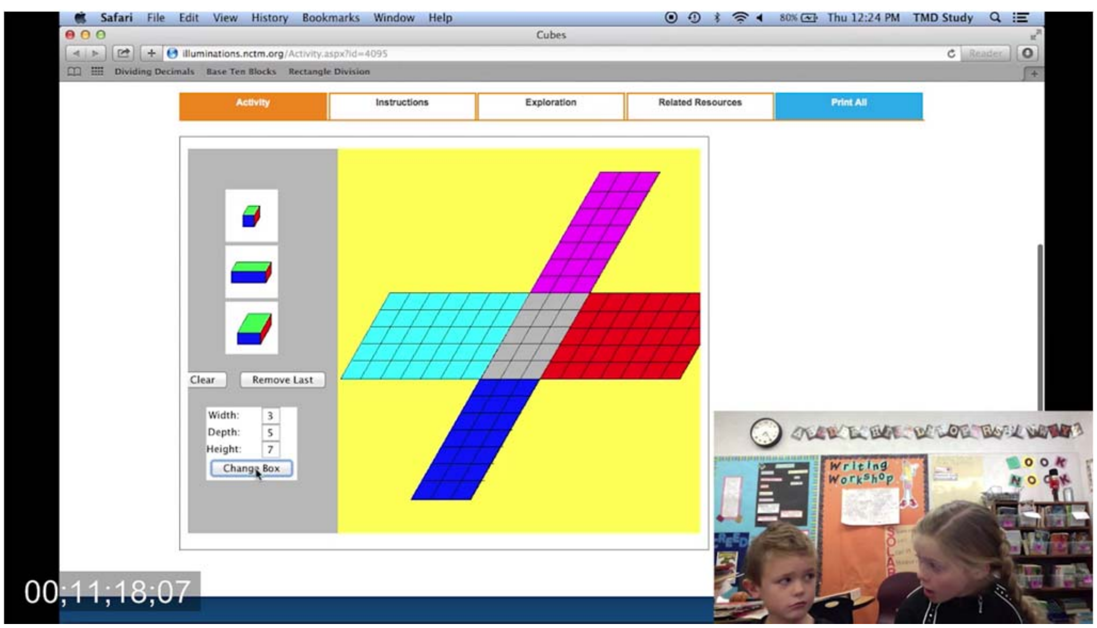


Figure 14. Colton and Callie working on cubes, episode C2C.

64	Colton	"Use the virtual manipulative to solve the following problems. Talk with your partner and record your answers. How many unit blocks will fill a box with dimensions width 3, depth 5, height 7?"
65	Callie	Ok, so how many unit blocks to fill the box?
66	Colton	20….
67	Callie	…So 6, no that's 5 blocks….
68	Colton	1, 2, 3, 4, 5, 6, 7. 20 times 7 is 140. So it should be 140 blocks.
69	Callie	…For the whole thing, yeah. 140 blocks.
70	Colton	20 times 7 is 140. I'll just write 140 blocks…. So change it to 5. What's the next number?
71	Callie	Width 5, depth 7, height 3. So 3 times….
72	Colton	Wait. Hold on. Just a second. Let's go back to that one. It didn't have 20 on the bottom. We need to go back.
73	Callie	Yeah it did.
74	Colton	3…5…7. Look, 1, 2, 3, 4, 5 times 3. So it's…
75	Both	…15…
76	Colton	…times 7…. Not 20.
77	Callie	15…105.
78	Colton	Ok. All right. Now we do this….
79	Callie	So the next one is width 5, depth 7, height 3. So….
80	Both	1, 2, 3, 4, 5, 6, 7…
81	Callie	…times 5.
82	Colton	…times 1, 2, 3, 4, 5. Yep.
83	Callie	So 35 times…
84	Colton	…3. I got…
85	Both	…105.
86	Callie	Again! …Ok "What is the volume of a box with width 7, depth 3,…"
87	Colton	It's just changing the numbers up. So I think it will be 105.
88	Callie	"…height, 5." Let's double check to see if it is 105 blocks. So…
89	Colton	1, 2, 3.
90	Callie	7 times 3. 21 times…
91	Colton	…5?
92	Callie	5. So…105.
93	Colton	Yep…. Ok

As shown in this example, Colton and Callie collaborated well and were efficient in their problem solving strategies. Their language (e.g., let's, we, etc.) reflected a joint effort (lines 72, 78, 88). They quickly recognized errors in their work and made the necessary adjustments (line 76).

In summary, the discussions of each participating student pair had unique characteristics that impacted the overall quality of their discourse. Aaron and Abbie's discussions were fast-paced and enthusiastic. Yet, at times they diverged from the task at hand to talk about topics not related to the assigned tasks. Brandon and Bonnie's discussions were minimal and directed at finishing the assignment as quickly as possible. They typically did not pay attention to what the other could contribute to the task at hand. Colton and Callie's discussions were executive in nature and very task-oriented. They collaborated well and used their time efficiently.

Cross-Case Analysis of Virtual Manipulative Types

The overarching research question for this study was: In what ways do different virtual manipulative types influence the nature of students' mathematical discourse? To answer this research question, a cross-case analysis of students' discourse was conducted. Data were collected during 27 separate instructional sessions as the three student pairs engaged with virtual manipulatives—nine of each virtual manipulative type. The results that follow present only the data from the coded speaking turns. Non-coded speaking turns have been removed from these analyses. The following sections provide a description, categorization, and interpretation of the students' discourse when working with each virtual manipulative type. Quantitative results supported by qualitative descriptions are presented for each dimension of discourse. Thus, quantitative and qualitative data were mixed in this portion of the cross-case analysis to determine how different virtual manipulative types influenced students' mathematical discourse.

Quantity of Discourse

A one-way ANOVA indicated no statistically significant differences among virtual manipulative types in the number of speaking turns, $F(2, 24) = 3.258$, $p = .056$. However, the number of speaking turns covered a large range—from 47 when using a tutorial virtual manipulative to 182 when using a pictorial virtual manipulative (see Figure 15). Discussions associated with the pictorial virtual manipulatives had the highest number of speaking turns ($M = 111.22$, $SD = 42.92$), and the tutorial virtual manipulatives had the lowest number of speaking turns ($M = 69.67$, $SD = 19.61$).

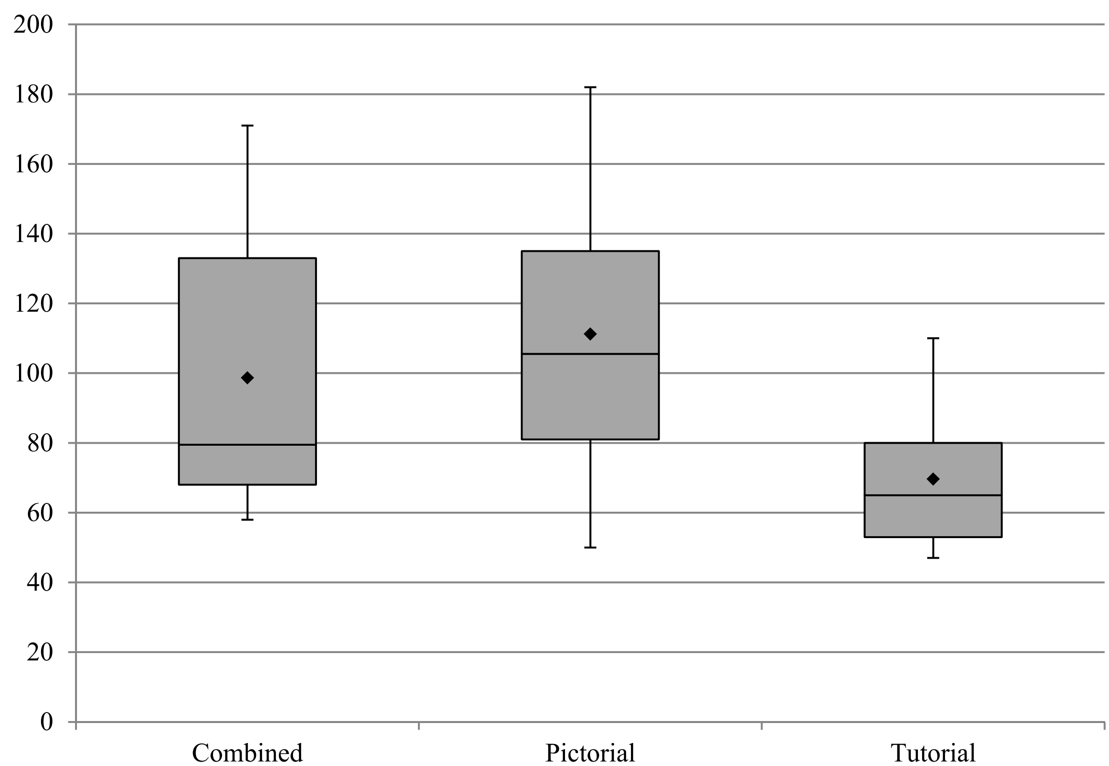


Figure 15. Box-plots comparing number of speaking turns across virtual manipulative type.

Research Subquestion (a): Generalization

Research subquestion (a) was: How do different virtual manipulative types influence the level of *generalization* in students' mathematical discourse? Generalization is defined as the capacity to form connections among related ideas (Mendez et al., 2007). Overall, students engaged in higher levels of generalization when working with combined virtual manipulatives than with pictorial or tutorial virtual manipulatives. Figure 16 displays the range and means of generalization composite scores for each virtual manipulative type. Combined virtual manipulatives had the highest average composite score (M = 128.52, SD = 15.56), followed by pictorial (M = 115.26, SD = 5.80) and tutorial (M = 107.39, SD = 13.37).

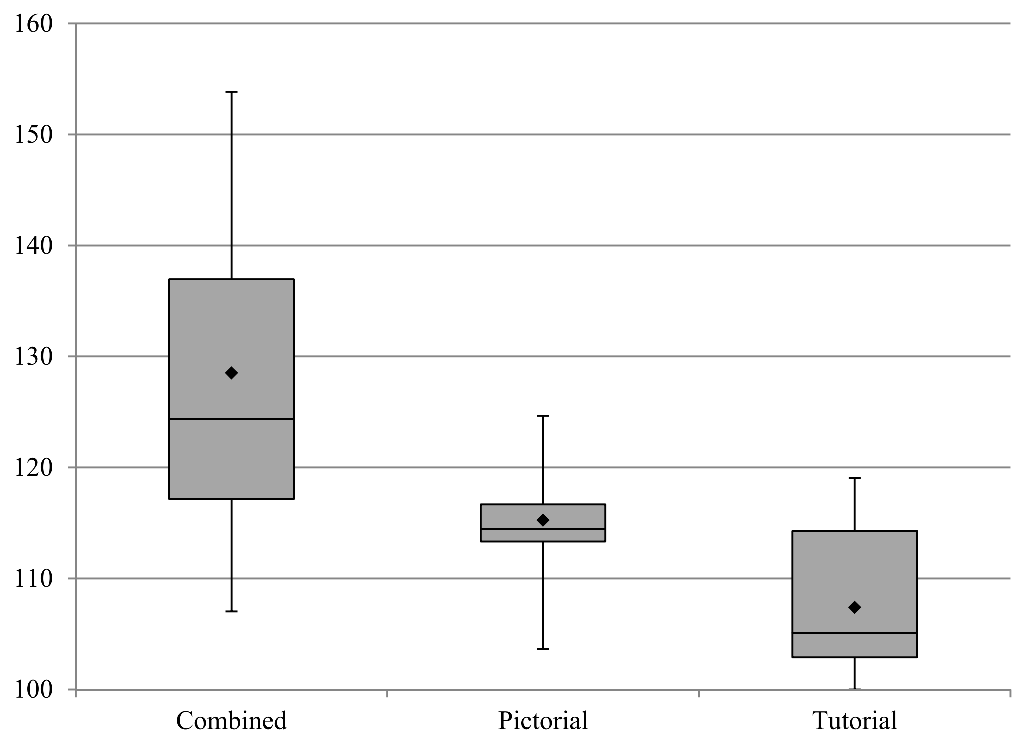


Figure 16. Box-plots comparing composite scores for generalization across virtual manipulative type.

A one-way ANOVA comparison of generalization composite scores indicated a statistically significant overall difference among the virtual manipulative types at the 95% level, F (2, 24) = 9.460, p = 0.001. This corresponded to an effect size of η^2 = .44; that is, about 44% of the variance in generalization composite scores was predictable from the type of virtual manipulative. This is a moderate effect. Individual post hoc comparisons using Tukey's HSD indicated a statistically significant difference between the combined and pictorial virtual manipulative types, p = 0.033, and between the combined and tutorial virtual manipulative types, p = .001. There was not a statistically significant difference between the pictorial and tutorial virtual manipulative types.

Differences in composite scores resulted from the frequency of different levels of generalization in students' discussions. Table 7 displays the number of speaking turns at each level of generalization for each virtual manipulative type.

When working with the combined virtual manipulative type, 17.18% of students' speaking turns were at the two higher levels of generalization—nearly twice as much as

Table 7

Frequencies of Speaking Turns at each Generalization Discourse Level for Virtual Manipulative Types

	Combined		Pictorial		Tutorial	
Level of generalization	Total n = 675	M^a	Total n = 766	M^a	Total n = 530	M^a
Concrete (lowest level)	559 (82.81)	62.11	684 (89.30)	76.00	502 (94.72)	55.78
Comparison	53 (7.85)	5.89	43 (5.61)	4.78	18 (3.40)	2.00
Generalization (highest level)	63 (9.33)	7.00	39 (5.09)	4.33	10 (1.89)	1.11

[a] N = 9 episodes

when working with pictorial virtual manipulatives (10.70%) and three times as much as when working with tutorial virtual manipulatives (5.29%). Levels of generalization rarely rose above the concrete level when students worked with tutorial virtual manipulatives.

In summary, these results suggest that students typically had lower levels of generalization in their discussions when working with the tutorial virtual manipulative type. Students typically had higher levels of generalization when working with the combined virtual manipulative type.

Research Subquestion (b): Justification

Research subquestion (b) was: How do different virtual manipulative types influence the level of *justification* in students' mathematical discourse? Justification is defined as a logical, warranted argument or proof of mathematical processes (Mendez et al., 2007). Overall, students engaged in higher levels of justification when working with combined virtual manipulatives than with pictorial or tutorial virtual manipulatives. Figure 17 displays the range and means of justification composite scores for each virtual manipulative type. Combined virtual manipulatives had the highest average composite score ($M = 135.00$, $SD = 14.78$), followed by pictorial ($M = 122.20$, $SD = 6.15$) and tutorial ($M = 113.15$, $SD = 9.35$).

A one-way ANOVA comparison of justification composite scores indicated a statistically significant overall difference among the virtual manipulative types at the 95% level, $F\ (2, 24) = 9.459$, $p = 0.001$. This corresponded to an effect size of $\eta^2 = .44$; that is, about 44% of the variance in justification composite scores was predictable from the type of virtual manipulative. This is a moderate effect size. Individual post hoc

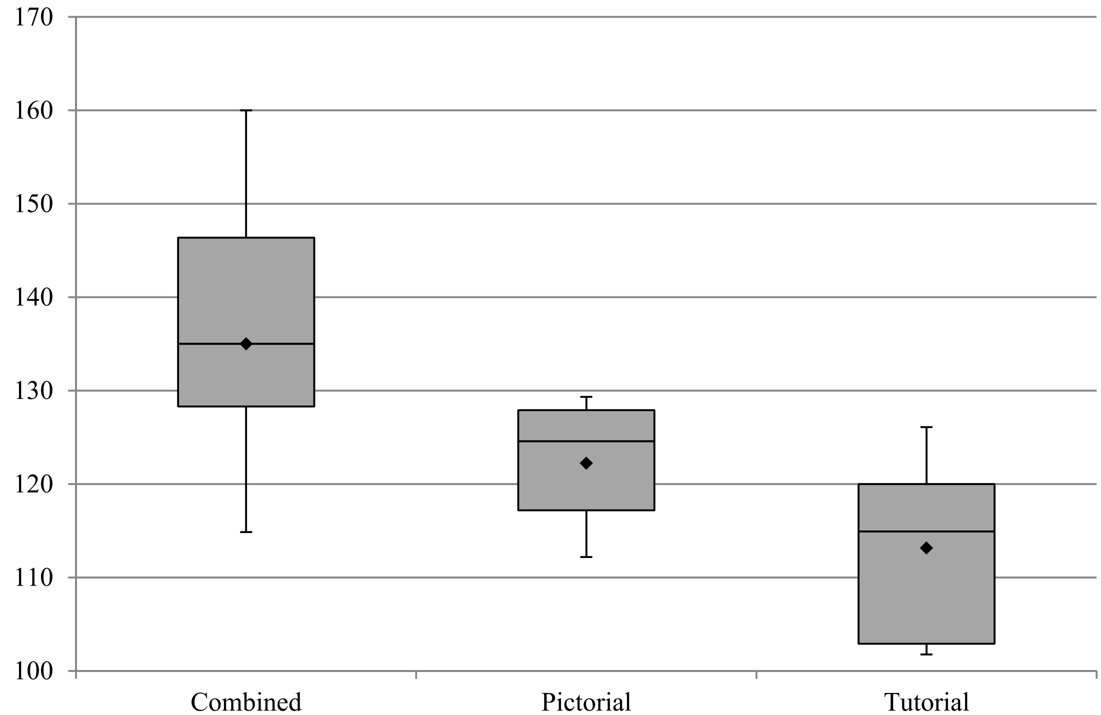


Figure 17. Box-plots comparing composite scores for justification across virtual manipulative type.

comparisons using Tukey's HSD indicated a statistically significant difference between the combined and pictorial virtual manipulative types, p = 0.046, and between the combined and tutorial virtual manipulative types, p = .001. There was not a statistically significant difference between the pictorial and tutorial virtual manipulative types.

Differences in composite scores resulted from the frequency of different levels of justification in students' discussions. Table 8 displays the number of speaking turns at each level of justification for each virtual manipulative type.

When working with the combined virtual manipulative type, 7.20% of students' speaking turns were at the highest level of justification—nearly twice as much as when working with pictorial virtual manipulatives (3.44%) and nearly four times as much when

Table 8

Frequencies of Speaking Turns at Each Justification Discourse Level for Virtual Manipulative Types

	Combined		Pictorial		Tutorial	
Level of justification	Total $n = 653$	M^a	Total $n = 726$	M^a	Total $n = 536$	M^a
Statement (lowest level)	491 (75.19)	54.56	592 (81.54)	65.78	473 (88.25)	52.56
Explanation	115 (17.61)	12.78	109 (15.01)	12.11	57 (10.63)	6.33
Proof (highest level)	47 (7.20)	5.22	25 (3.44)	2.78	6 (1.12)	0.67

[a] $N = 9$ episodes.

working with tutorial virtual manipulatives (1.12%).

In summary, these results suggest that students typically had lower levels of justification in their discussions when working with the tutorial virtual manipulative type. Students typically had higher levels of justification when working with the combined virtual manipulative type.

Research Subquestion (c): Collaboration

Research subquestion (c) was: How do different virtual manipulative types influence the level of *collaboration* in students' mathematical discourse? Collaboration is defined as a process in which two or more individuals work together to integrate information in order to enhance learning (Mendez et al., 2007). Overall, students engaged in higher levels of collaboration when working with combined virtual manipulatives than with pictorial or tutorial virtual manipulatives. Figure 18 displays the range and means of collaboration composite scores for each virtual manipulative type. Combined virtual

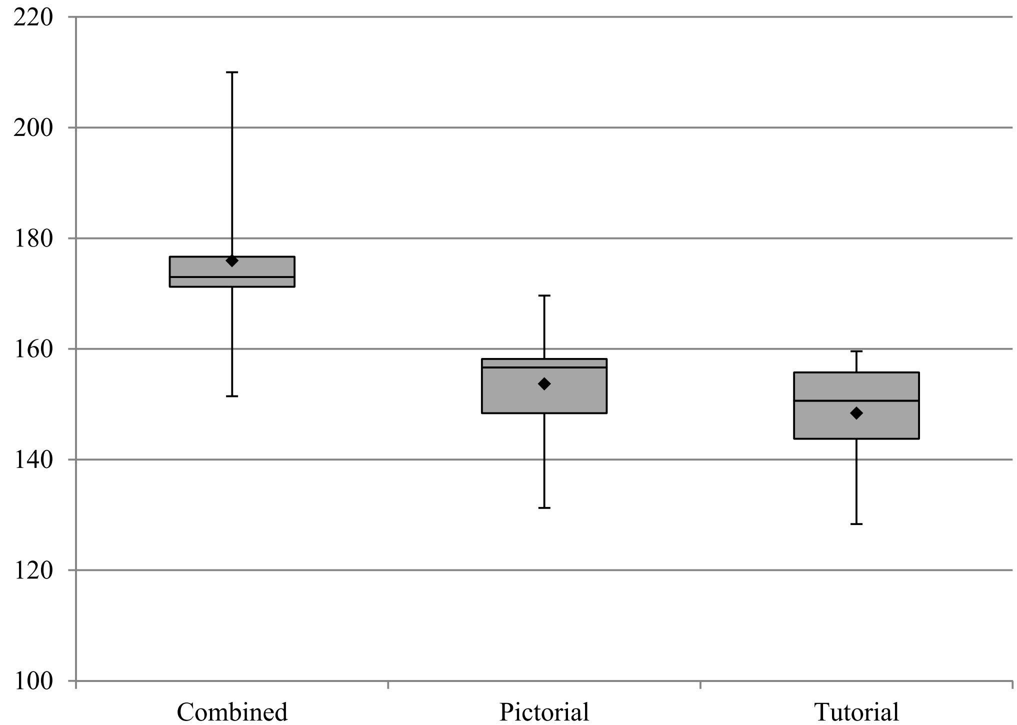


Figure 18. Box-plots comparing composite scores for collaboration across virtual manipulative type.

manipulatives had the highest average composite score (*M* = 175.99, *SD* = 16.57), followed by pictorial (*M* = 153.69, *SD* = 11.62) and tutorial (*M* = 148.40, *SD* = 9.72).

A one-way ANOVA comparison of collaboration composite scores indicated a statistically significant overall difference among the virtual manipulative types at the 95% level, F (2, 24) = 11.476, $p < 0.001$. This corresponded to an effect size of $\eta^2 = .49$; that is, about 49% of the variance in collaboration composite scores was predictable from the type of virtual manipulative. This is a moderate effect size. Individual post hoc comparisons using Tukey's HSD indicated a statistically significant difference between the combined and pictorial virtual manipulative types, $p = 0.004$, and between the combined and tutorial virtual manipulative types, $p < .001$. There was not a statistically significant difference between the pictorial and tutorial virtual manipulative types.

Differences in composite scores resulted from the frequency of different levels of collaboration in students' discussions. Table 9 displays the number of speaking turns at each level of collaboration for each virtual manipulative type.

When working with the pictorial and tutorial virtual manipulative types, students' discourse reflected similar levels of collaboration (14.81% and 12.80% at the highest level). However, students' collaboration when working with the combined virtual manipulatives was considerably higher (20% at the highest level).

In summary, these results suggest that students typically had lower levels of collaboration in their discussions when working with the pictorial and tutorial virtual manipulative types. Students typically had higher levels of collaboration when working with the combined virtual manipulative type.

Table 9

Frequencies of Speaking Turns at Each Collaboration Discourse Level for Virtual Manipulative Types

	Combined		Pictorial		Tutorial	
Level of collaboration	Total $n = 860$	M^a	Total $n = 979$	M^a	Total $n = 625$	M^a
Unrelated idea (lowest level)	393 (45.70)	43.67	583 (59.55)	64.78	400 (64.00)	44.44
Response	295 (34.30)	32.78	251 (25.64)	27.89	145 (23.20)	16.11
Build (highest level)	172 (20.00)	19.11	145 (14.81)	16.11	80 (12.80)	8.89

[a] $N = 9$ episodes.

Research Subquestion (d): Gestures

Research subquestion (d) was: How do different virtual manipulative types influence physical and computer-based *gestures* when students are engaged in mathematical discourse? Gestures are defined as the hand and arm movements that are interpreted by others as part of what a person says (McNeill, 1992; Roth, 2001). Table 10 summarizes the frequencies of physical gestures used when students were working with each virtual manipulative type. Gestures were coded as iconic when the movement resembled an actual object (e.g., motioning the side of a shape). Gestures were coded as metaphoric when the movement indicated an abstract idea (e.g., same, different, higher, lower). Gestures were coded as beat when the movement added emphasis to what the student was saying. Gestures were coded as deictic when the movement served to point out a mathematical idea or keep track of counting a quantity.

As Table 10 shows, deictic (i.e., pointing) gestures made up the majority of

Table 10

Frequencies of Physical Gestures for Virtual Manipulative Types

	Combined		Pictorial		Tutorial	
Gesture type	Total $n = 352$	M^a	Total $n = 382$	M^a	Total $n = 201$	M^a
Iconic	14 (3.98)	1.56	15 (3.93)	1.67	17 (7.14)	1.89
Metaphoric	40 (11.36)	4.44	45 (11.78)	3.00	31 (13.03)	3.44
Beat	148 (42.05)	16.44	96 (25.13)	10.67	59 (24.79)	6.56
Deictic	150 (42.61)	16.67	226 (59.16)	25.11	131 (55.04)	14.56

[a] $N = 9$ episodes.

physical gestures used for all virtual manipulative types. When using the combined and pictorial virtual manipulative types, students used approximately the same number of gestures overall (*n* = 352 and *n* = 382, respectively). However, when using the tutorial virtual manipulative types, students used considerably fewer gestures (*n* = 201). Students used deictic gestures the most when working with the pictorial virtual manipulative type (*n* = 226). Students used beat gestures the most when working with combined virtual manipulative type (*n* = 148). When working with the tutorial virtual manipulative type, students used the lowest amount of metaphoric, beat and deictic gestures. However, students' amount of use of iconic gestures was approximately the same for all virtual manipulative types. This shows that students were more inclined to point to mathematical representations when working with the pictorial and combined virtual manipulative types than with the other virtual manipulative types. They were less inclined to use gestures when working with the tutorial virtual manipulative type.

Table 11 summarizes the frequencies of computer-based gestures (i.e., mouse movements) used when students were working with each virtual manipulative type. Mouse movements were coded as iconic when the movement resembled an actual object (e.g., outlining a triangle) or motion (e.g., moving a point up or down). Mouse movements were coded as metaphoric when the movement indicated an abstract idea (e.g., higher or lower). Mouse movements were coded as beat when the movement showed emphasis on a particular part of the virtual manipulative (e.g., repeated circles around a shape or number). Mouse movements were coded as deictic when the movement served to point out a mathematical idea or keep track of counting a quantity.

Table 11

Frequencies of Computer Gestures for Virtual Manipulative Types

	Combined		Pictorial		Tutorial	
Gesture type	Total n = 99	M^a	Total n = 146	M^a	Total n = 86	M^a
Iconic	11 (11.11)	1.22	9 (6.16)	1	11 (12.79)	1.22
Metaphoric	3 (3.03)	.33	12 (8.22)	1.33	6 (6.98)	0.67
Beat	17 (17.17)	1.89	28 (19.18)	3.11	25 (29.07)	2.78
Deictic	68 (68.69)	7.56	97 (66.44)	10.78	44 (51.16)	4.89

[a] N = 9 episodes.

As Table 11 shows, fewer computer-based gestures were coded overall than for physical gestures (n = 935 vs. n = 302). Deictic (i.e., pointing) gestures made up the majority of computer-based gestures used for all virtual manipulative types. The highest frequency of computer-based gestures occurred when students were using pictorial virtual manipulatives (n = 146). The lowest frequency of computer-based gestures occurred when students were using tutorial virtual manipulatives (n = 86). Similar to the results of the physical gestures, the results of the computer-based gestures show that students were less inclined to use the mouse to gesture when working with the tutorial virtual manipulative type than with the other virtual manipulative types. Students used deictic gestures the most when working with the pictorial virtual manipulative type (n = 97). This shows that students were more inclined to use the mouse to point to mathematical representations when working with the pictorial virtual manipulative type.

Quartile Analysis of Discourse Levels

In the final analysis, the data were examined for levels of discourse over the course of the students' interactions. This analysis indicates differences in the progression of discussions among virtual manipulative types. In order to compare the discourse progressions of discussions of varying lengths, each discussion was divided into quartiles according to the number of speaking turns. Then, for each quartile, the number of speaking turns coded for each level was calculated. The following sections report the results of this quartile analysis.

Generalization. Figures 19, 20, and 21 compare levels of generalization across the three virtual manipulative types. For combined virtual manipulatives, the highest level of generalization occurred steadily throughout the course of the discussions (see Figure 19). However, it occurred most frequently in the last quartile of the discussions. The second level of generalization—comparison—occurred in similar proportions in the first and second quartiles (14.10% and 14.20%), and then decreased for the third and fourth quartiles (1.71% and 5.26%). For pictorial virtual manipulatives, the two highest levels of generalization occurred most during the last quartile of the discussion (see Figure 20). For tutorial virtual manipulatives, discussion remained at the most basic level, concrete throughout the discussion (see Figure 21). More statements were coded for the second level, comparison, in the first two quartiles of the discussions than for the last two quartiles of the discussions. Speaking turns coded at the highest level accounted for less than 1% of the first and fourth quartiles of discussions with tutorial virtual manipulatives.

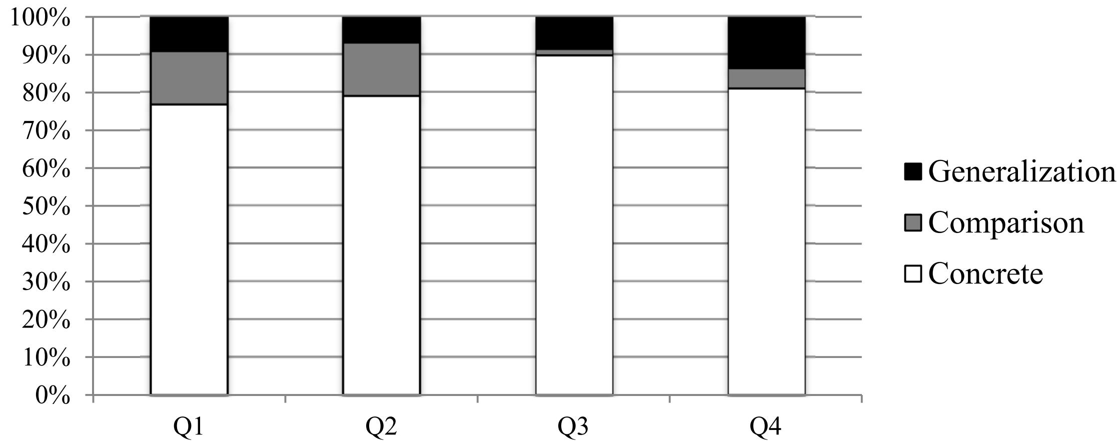


Figure 19. Quartile analysis of generalization for combined virtual manipulatives.

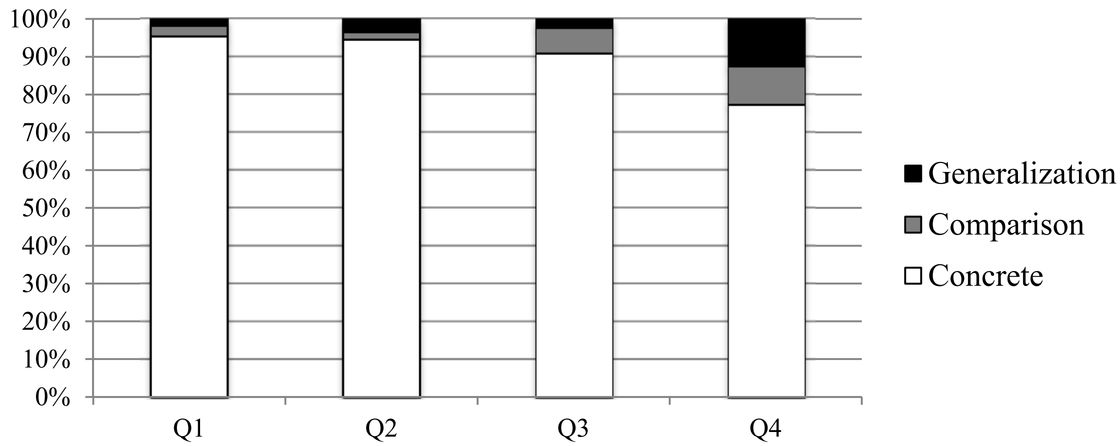


Figure 20. Quartile analysis of generalization for pictorial virtual manipulatives.

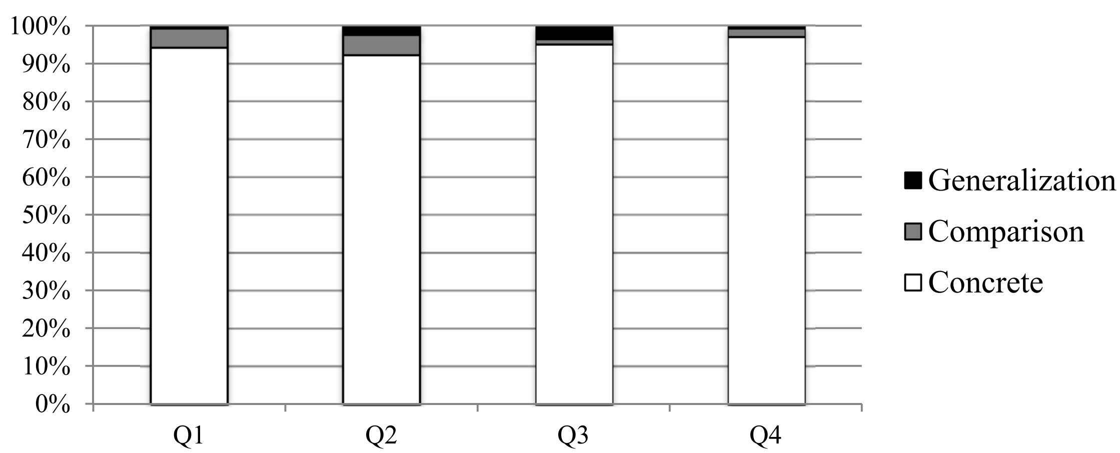


Figure 21. Quartile analysis of generalization for tutorial virtual manipulatives.

Justification. Figures 22, 23, and 24 compare levels of justification across the three virtual manipulative types. For combined virtual manipulatives, levels of justification increased as the discussions progressed (see Figure 22). The percentage of speaking turns coded for explanation and for proof increased considerably after the first quartile (4.55% to 20.13% and 0.65% to 5.03%, respectively). For pictorial virtual manipulatives, the levels of justification also increased as the discussions progressed (see Figure 23), but not to the same extent as the combined virtual manipulatives. The percentage of speaking turns coded for explanation and for proof increased after the first quartile (9.76% to 15.93% and 1.83% to 3.85%, respectively). For tutorial virtual manipulatives, the most frequent occurrence of proof happened in the first quartile of the discussions (2.52%). However, for the rest of the discussion, proof accounted for less than 1% of the speaking turns (see Figure 24). The most frequent occurrence of explanation also happened in the first quartile of the discussions (22.69%). Thereafter, the percentage of explanations dwindled to 10% or less for the remaining portion of the discussions.

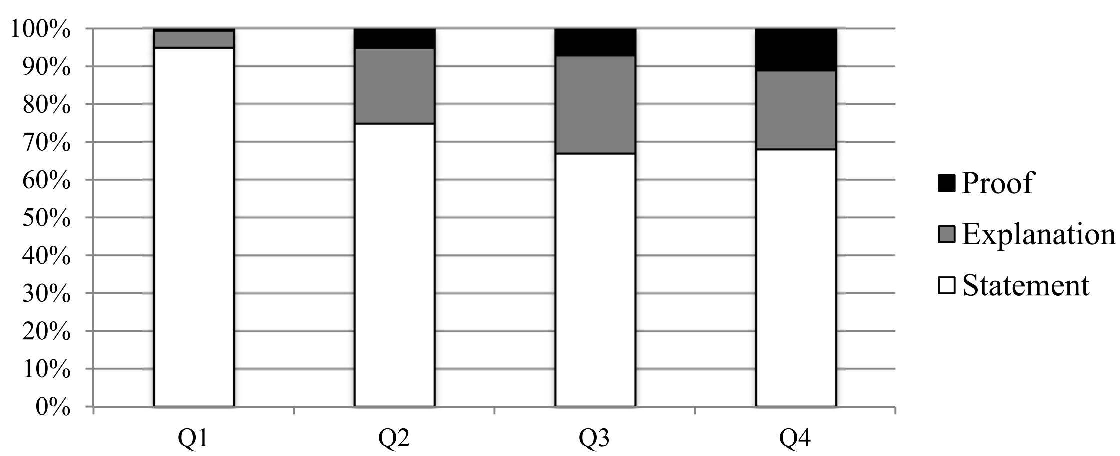


Figure 22. Quartile analysis of justification for combined virtual manipulatives.

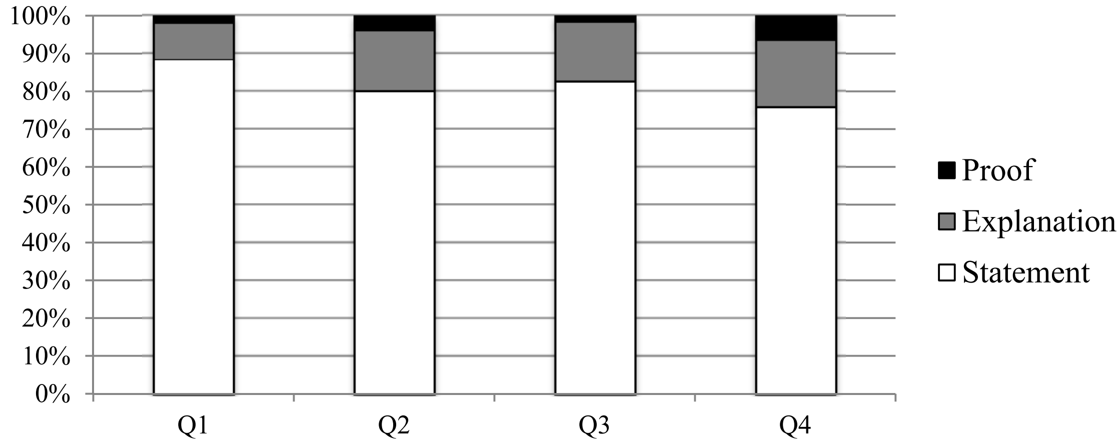


Figure 23. Quartile analysis of justification for pictorial virtual manipulatives.

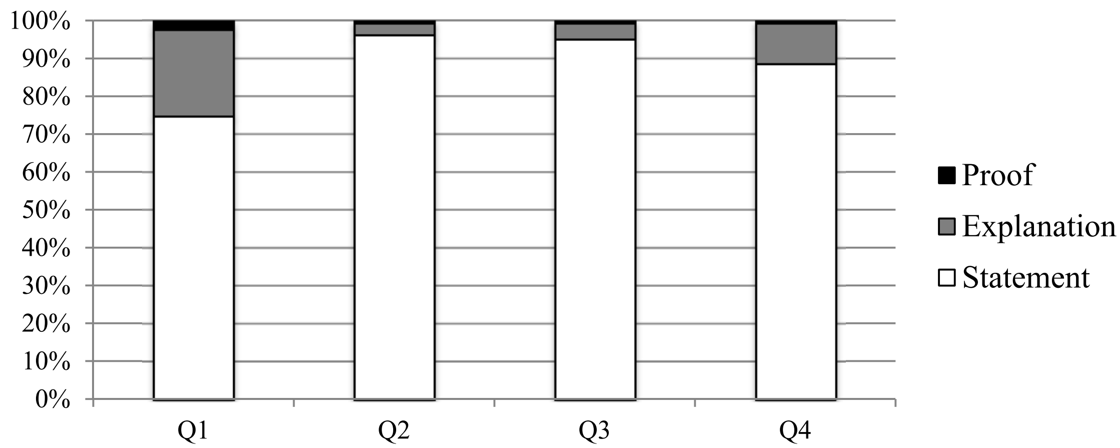


Figure 24. Quartile analysis of justification for tutorial virtual manipulatives.

Collaboration. Figures 25, 26, and 27 compare levels of collaboration across the three virtual manipulative types. For combined virtual manipulatives, the overall proportion of levels of collaboration remained steady throughout the course of the discussions (see Figure 25). For pictorial virtual manipulatives, the percentage of speaking turns coded for build gradually increased until the third quartile (18.65%), and then decreased slightly (see Figure 26). The percentage of speaking turns coded for response remained the same throughout the discussions (approximately 25%). For tutorial

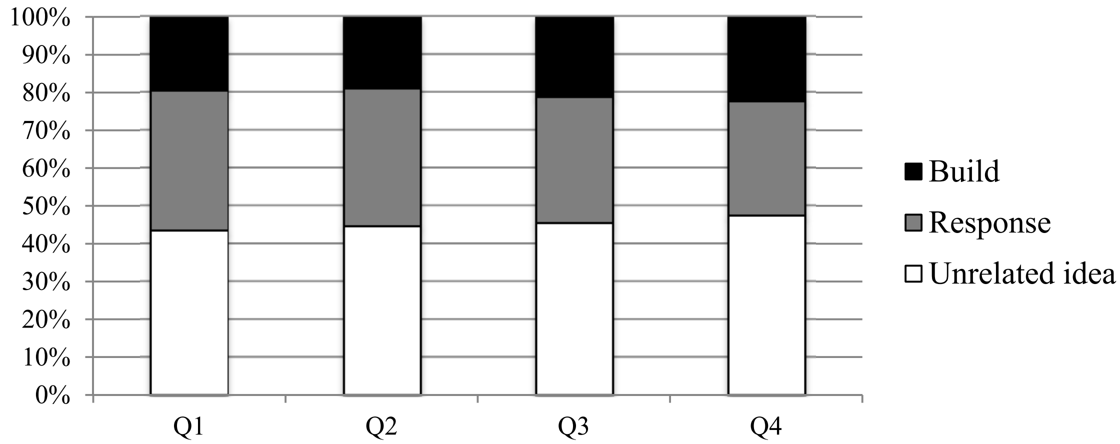


Figure 25. Quartile analysis of collaboration for combined virtual manipulatives.

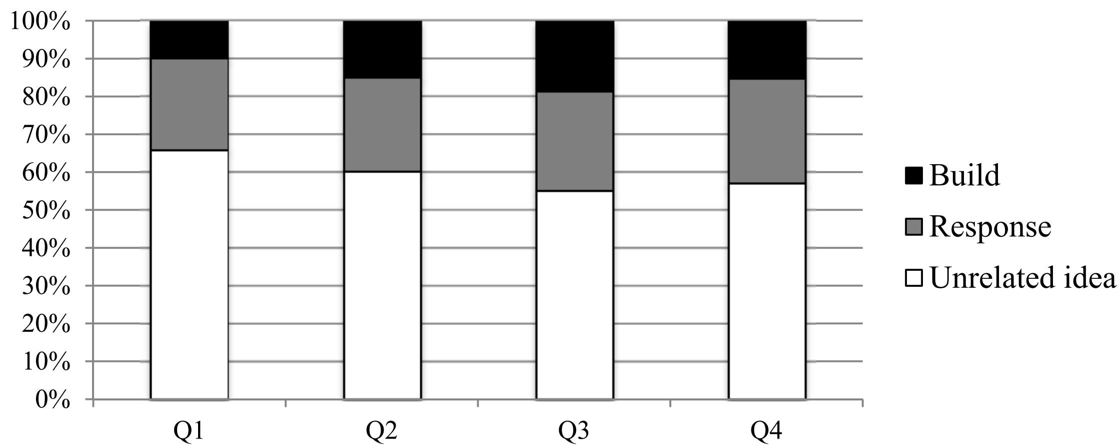


Figure 26. Quartile analysis of collaboration for pictorial virtual manipulatives.

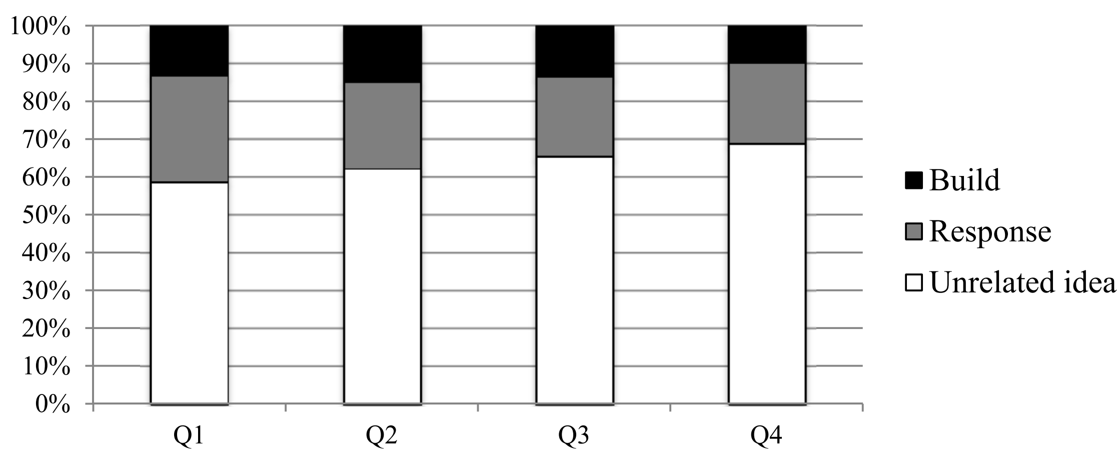


Figure 27. Quartile analysis of collaboration for tutorial virtual manipulatives.

virtual manipulatives, the percentages of speaking turns coded for response and build gradually decreased (27.97% to 21.34% and 13.29% to 9.76%, respectively), while the percentage of speaking turns coded for unrelated ideas rose from 58.74% to 68.90% by the fourth quartile (see Figure 27).

In summary, the quartile analysis confirms that students' discussions when using the tutorial virtual manipulative type consistently reflected lower levels of generalization, justification, and collaboration. In addition, the analysis of discourse from the beginning to the end of each episode shows that when working with the combined and tutorial virtual manipulative types, the level of generalization in students' discussions remained constant. However, when working with the pictorial virtual manipulative type, the level of generalization increased toward the end of students' discussions. Levels of justification in students' discussions remained relatively constant when working with the pictorial and tutorial virtual manipulative types. However, when working with the combined virtual manipulative types, levels of justification in students' discussions increased after the first quarter of the episode. Levels of collaboration remained somewhat constant throughout each episode for all virtual manipulative types.

Overall Analysis of Levels of Discourse

The analysis presented in this section addresses the overarching research question (In what ways do different virtual manipulative types influence the nature of students' mathematical discourse?) in relation to each virtual manipulative type by examining discourse maps.

Analysis of discourse maps. Discourse maps (see Figures 28, 29 and 30) were

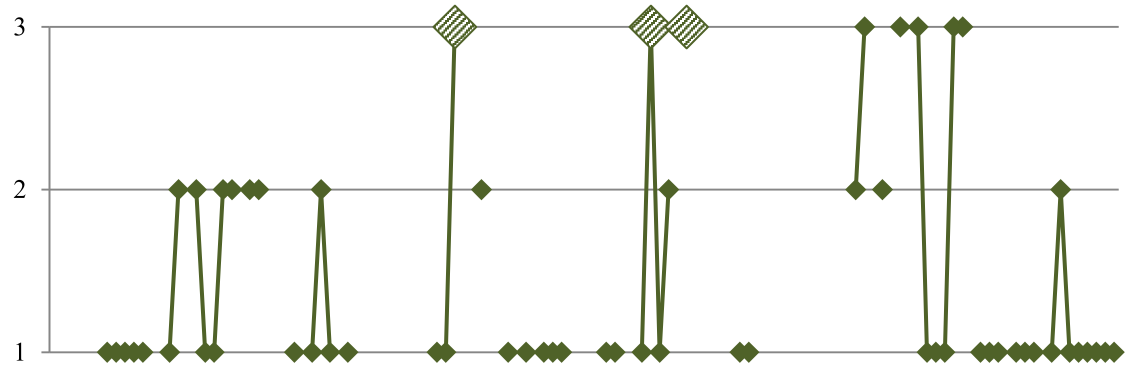


Figure 28. Sample discourse map for generalization using combined virtual manipulative, equivalent fractions, episode C3C.

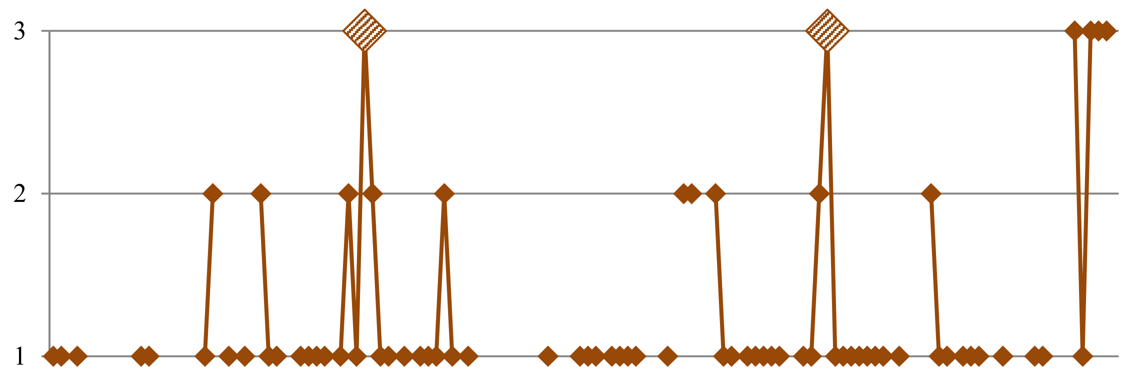


Figure 29. Sample discourse map for justification using pictorial virtual manipulative, base 10 blocks, episode P1B.

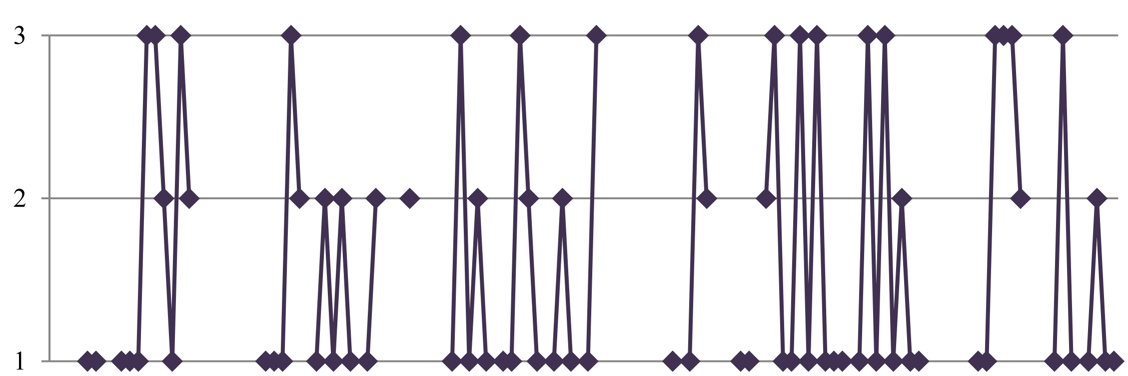


Figure 30. Sample discourse map for collaboration, episode P1C.

created to track the levels of discourse in each episode. Each point on the map represents a coded speaking turn in the order that they occurred in the discussion. An examination of these discourse maps reveals additional patterns and trends in students' interactions with each other when working with virtual manipulatives.

Discourse maps for generalization reveal that the majority of speaking turns coded for the highest level occurred in response to specific questions on the student task sheets. For example, in the lesson using the combined virtual manipulative, Equivalent Fractions, students were asked, "How are the numerators and denominators related to each other in each set of equivalent fractions?" Most of the students' responses to this question were coded as the highest level of generalization (e.g., "The numerators are just counting up one. But the denominators are skipping two"). However, in other occurrences, students made high-level generalizations without the solicitation of the student task sheet (e.g., "Wait, equivalent means the same as that. So you have to fill it in all the way"). In Figure 28, these unsolicited high-level generalizations are indicated on the discourse map by larger, shaded points. Unsolicited high levels of generalization rarely occurred with combined and pictorial virtual manipulatives. They did not occur at all with tutorial virtual manipulatives.

Discourse maps for justification reveal a similar pattern. The majority of speaking turns coded for the highest level of justification occurred in response to specific questions on the student task sheets. For example, part of a question asked, "How do you know?" This question typically elicited explanations and proofs of how the student arrived at his or her answer. However, students also made high-level justifications during the normal course of their discussions (e.g., “Because there are four groups and we need to split these two tens into four groups evenly so one-half. We can get two halves out of this and then two halves again, and that would be four halves.” Episode P1B). In Figure 29, these unsolicited high levels of justification are indicated on the discourse map by larger shaded points. Unsolicited high levels of justification occurred with all virtual manipulative types.

The discourse maps for collaboration (see Figure 30) all followed a similar pattern. For all virtual manipulative types, levels of collaboration fluctuated from the lowest to the highest levels throughout the entire discussion. This indicates that collaboration levels did not change over the course of a discussion and were not impacted by specific questions on the student sheet.

Combined virtual manipulatives. A major affordance of combined virtual manipulatives is their ability to simultaneous link representations of mathematics concepts (Moyer-Packenham & Westenskow, 2013). This dynamic linking assists students in making connections among mathematical ideas and supports their problem-solving strategies. In the following excerpt from episode C3B, Brandon and Bonnie are using a combined virtual manipulative, equivalent fractions, to identify and create models for given equivalent fractions (see Figure 31). This exchange illustrates how the virtual manipulative assisted the students in articulating a rule for making equivalent fractions.

89	Brandon	“How are the numerators and denominators related to each other?” …Umm…How are the numerators and denominators related… So we did one-half, one… two, and then two-fourths, two… four, and then three-sixths…
90	Bonnie	You just add the denominator and add the numerator of how many there is, right?

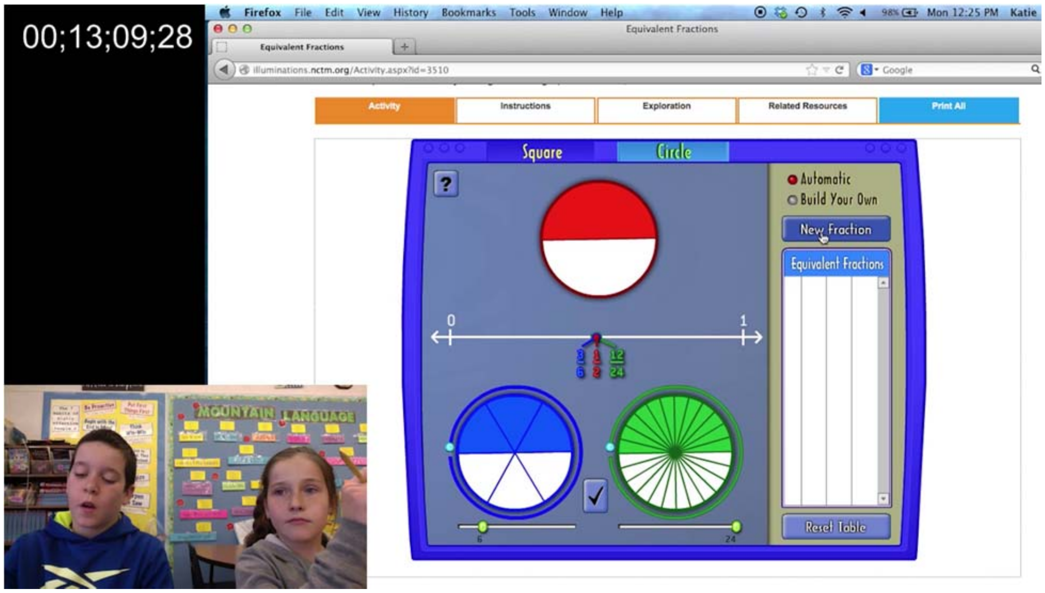


Figure 31. Brandon and Bonnie working on equivalent fractions, episode C3B.

91	Brandon	You just…like if that’s one-half, and that’s two-fourths, the number is one, so you add one to make it two. The number is two, so you add two to make it four.
92	Bonnie	It’s like two, you add two.
93	Brandon	So you just add it to itself.
94	Bonnie	Add itself, yeah.

In referring to his previous experience with making equivalent fractions on the virtual manipulative, Brandon was able to reason through a strategy for finding other equivalent fractions (line 91).

In addition to assisting students in generalizing and making connections among mathematical ideas, the combined virtual manipulatives also supported the students’ justification and collaboration efforts. In the following excerpt from episode C2A, Aaron and Abbie are using a combined virtual manipulative, Cubes, to determine the volume of a given rectangular prism (see Figure 32). This exchange illustrates how students typically used the combined virtual manipulatives to support the explanations of their problem-solving strategies.

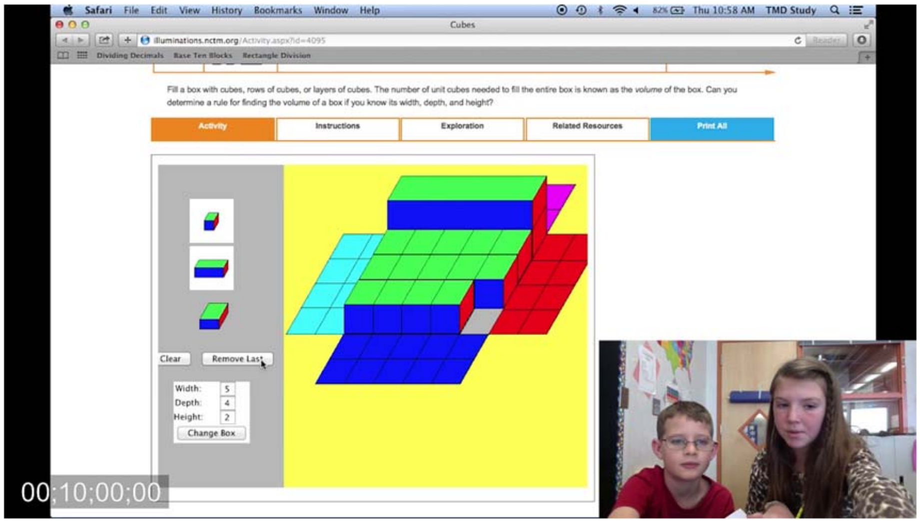


Figure 32. Aaron and Abbie working on cubes, episode C2A.

132	Aaron	"How many unit blocks are in one layer? Write a multiplication equation to represent one layer." 1, 2, 3, 4…by 5. There's 20 blocks. The multiplication is 4 times 5 equals 20.
133	Abbie	Wait, what?
134	Aaron	See 4…5. Read number 4.
135	Abbie	"How many unit blocks are in one layer?"
136	Aaron	"Write a multiplication equation to represent one layer." Four times 5 equals 20. So each layer block is 20 unit blocks. 'Cause look. (Motioning on screen with mouse) 4 by 5. If you were to do it, it would take us 20 of these. See 20.
137	Abbie	Yes, I see. …ok… So what did you write?
138	Aaron	4 times 5 equals 20. It says, "Write a multiplication equation to represent one layer."
139	Abbie	Yeah, I know.
140	Aaron	"How many unit blocks are in the whole box?" …20…40! Yeah, because look. (Points to computer screen) It takes 2 layer blocks to fill this up and each layer block…
141	Abbie	Forty, yeah. You're right.

In this example, the representations served as a communication tool between Aaron and Abbie. Aaron used the pictorial and numeric representations on the virtual manipulative to show how he arrived at his answer of 20 blocks for a layer and 40 blocks for the total volume, and Abbie verified his solution by referring to the pictorial image (lines 136, 140–141).

Pictorial virtual manipulatives. The main difference between the combined and the pictorial virtual manipulative types is the lack of simultaneous linking of representations. With only pictorial images, these virtual manipulatives rely on just one type of representation, yet these virtual manipulatives are more versatile in their use for mathematical problem solving. Because pictorial virtual manipulatives provide fewer supports for mathematical thinking, students must put forth more effort into collaboration and communication in order to successfully solve problems.

Sometimes student pairs were successful in their communications when using pictorial virtual manipulatives. At other times, they were less successful. The following excerpt from episode P1A illustrates a less successful attempt (see Figure 33). Here, Aaron and Abbie are using a pictorial virtual manipulative, Base 10 Blocks, to solve a word problem:

> "Nancy's poster for the school council election covers a space of 6.4 square meters. She wants to divide the poster into 4 equal sections for her slogan. How space will be in each section? Hint: one 10 by 10 square represents 1 square meter"

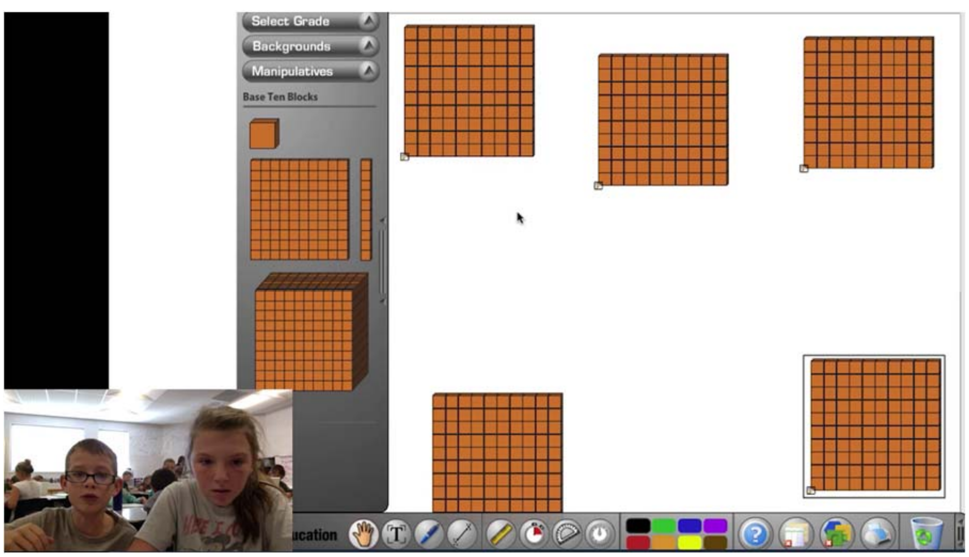


Figure 33. Aaron and Abbie working on base 10 blocks, episode P1A.

83	Aaron	Ok. Let’s use these base 10 block things.
84	Abbie	Well, now 6…
85	Aaron	So 6 point 4.
86	Abbie	So 6 of those. (Aaron places 10x10 blocks)
87	Aaron	And then you can do the 4 tenths if you want. Wait! We have to make this look cool.
88	Abbie	No you don’t.
89	Aaron	I want to. (Arranges blocks in a pattern) Then we have to connect them. Then 4 little teeny… Oh wait, no!
90	Abbie	It’s 6. (Places 2 more 10x10 grids)
91	Aaron	And then put 2… put 1 in between each one and put 2 in there…. Fine you don’t have to make it look cool.
92	Abbie	No…. So that is 6 point 4?
93	Aaron	Yep, that is what it wanted us to do.
94	Abbie	“She wants to divide the poster into 4 equal sections for slogans.” (Aaron starts to move 10x10 grids on top of one another) Just wait a minute…. I don’t get what you’re doing.
95	Aaron	(Arranges blocks) That’s not gonna work. Let’s break up these 2….
96	Abbie	You have to break them up…. Only…
97	Aaron	Whoa!!
98	Abbie	Break up only 2, I think. …You just glued those 2. You need to break it up.
99	Aaron	Break up. That sounds like a love thing.
100	Abbie	Ok, now’s there’s 4….
101	Aaron	Let’s do it 3 at a time.
102	Abbie	Watch. One…. Wait…
103	Aaron	It doesn’t connect.
104	Abbie	Just go like this. (Takes control of mouse)
105	Aaron	If only there was like 2 mouses. (Abbie separates blocks into 4 equal groups)
106	Abbie	This is going to take a while.

Although the students received instruction on the functionality of the virtual manipulative (e.g., dragging one piece on top of another would snap them together, clicking on a hammer icon would break up blocks into smaller pieces), there were no explicit instructions of how to set up and organize a solution for this problem. As illustrated by Aaron and Abbie, students sometimes had difficulties in communicating to their partner what they thought should be done when they did not have control of the mouse (lines 104–106).

At other times, the students were able to successfully communicate using the pictorial virtual manipulatives. In the following excerpt from episode P2C, Colton and Callie are using the pictorial virtual manipulative, shape sorter, to identify triangles that fit one or both of two rules: All angles are acute, and all angles are congruent (see Figure 34).

90	Callie	So all angles are acute. All angles are congruent. Ok. So all 7 triangles…. So this triangle....
91	Colton	…has 2 acute.
92	Callie	All angles are congruent—that's not true. All of the angles are not the same length.
93	Colton	Well it doesn't have all angles acute. So…it wouldn't be…
94	Callie	So it wouldn't go in either. …This one. It definitely has all acute angles. But…
95	Colton	Are all the angles congruent?
96	Callie	No. Wait.
97	Colton	No, they're not. Like the acute angles. Like one of them is like bigger and the other ones are smaller (making larger and smaller angles with hands).

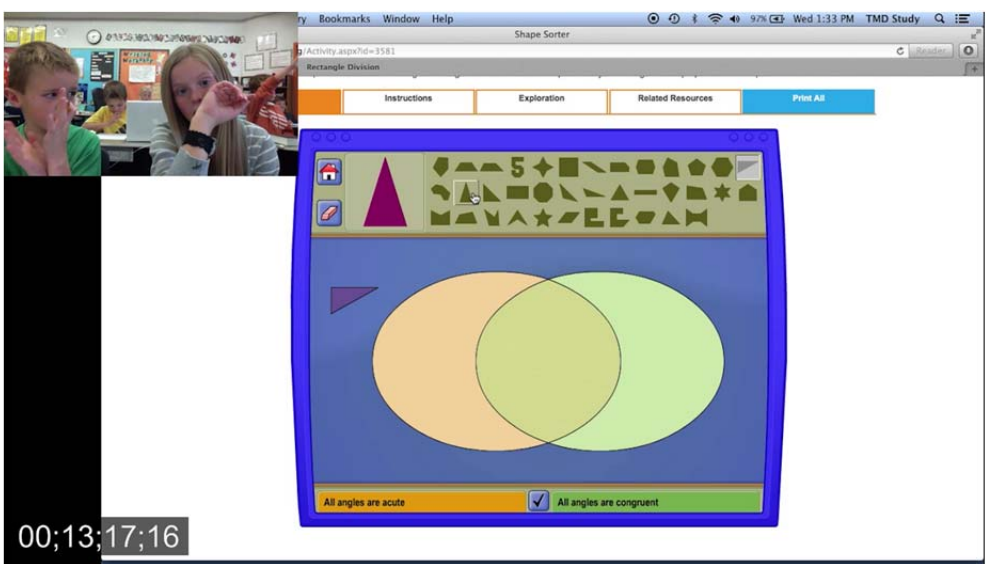


Figure 34. Colton and Callie working on shape sorter, episode P2C.

98	Callie	Oh, yeah. 'Cause on this shape…
99	Colton	That would just be in the red.
100	Callie	This one?
101	Colton	Yeah.
102	Callie	Then next shape. This one…
103	Colton	That's a right angle, so…
104	Callie	Yeah, so it's not all acute. And it's not congruent. So…ok. This one. It definitely does not have all acute angles… and it's a huge line. It doesn't match up with anything else. So it's neither. Next one. That's an obtuse angle, so it's not all acute and it's not all congruent.
105	Colton	This one…
106	Callie	All acute, right? And then it's all…
107	Colton	…congruent.
108	Callie	It's congruent, so…and then… This one. All angles are acute.
109	Colton	And… I don't think they're all congruent. Check. (VM feedback: all correct)
110	Callie	Yep. We got them all right.

As illustrated by Colton and Callie, it was possible for students to use the affordances of the pictorial virtual manipulatives to communicate their mathematical ideas. For example, Colton connected his own knowledge of acute and obtuse angles to the angles of the triangles on the virtual manipulative by gesturing with his hands the relative sizes of the angles (line 97). By combining the images on the pictorial virtual manipulative with physical gestures, these students effectively communicated their mathematical thinking.

Tutorial virtual manipulatives. One feature of tutorial virtual manipulatives is the structured nature of how they present mathematical concepts and how students are expected to interact with the virtual manipulative. As shown in the quantitative analyses, when working with these structured environments, students typically had lower levels of discourse (i.e., generalization, justification, and collaboration) than when working with the other virtual manipulative types. The following excerpt from episode T3B illustrates a typical exchange as students worked with tutorial virtual manipulatives. Here, Brandon and Bonnie are working with the tutorial virtual manipulative, adding fractions, to solve 1/10 + 1/2 (see Figure 35).

89	Brandon	One-tenth plus one-half.
90	Bonnie	So that's 2. That's 10.
91	Brandon	That's 5.
92	Bonnie	Wait! It's 20.
93	Brandon	No it's not. Look, 2 can go into 10.
94	Bonnie	Oh, ok. That's 10 and that's 5.
		(VM feedback: try a different numerator)
95	Brandon	Oh, this should be 1.
		(VM feedback: correct)

In this example, Brandon and Bonnie's discourse consisted of short statements aimed at solving the given addition problem. Neither student supplied any reasoning for their answers. Even when a mistake was made (typing 2/10 instead of 1/10), Brandon simply corrected the error and moved on without commenting on why it was incorrect (line 95). This short exchange illustrates the tendency for students, when working with

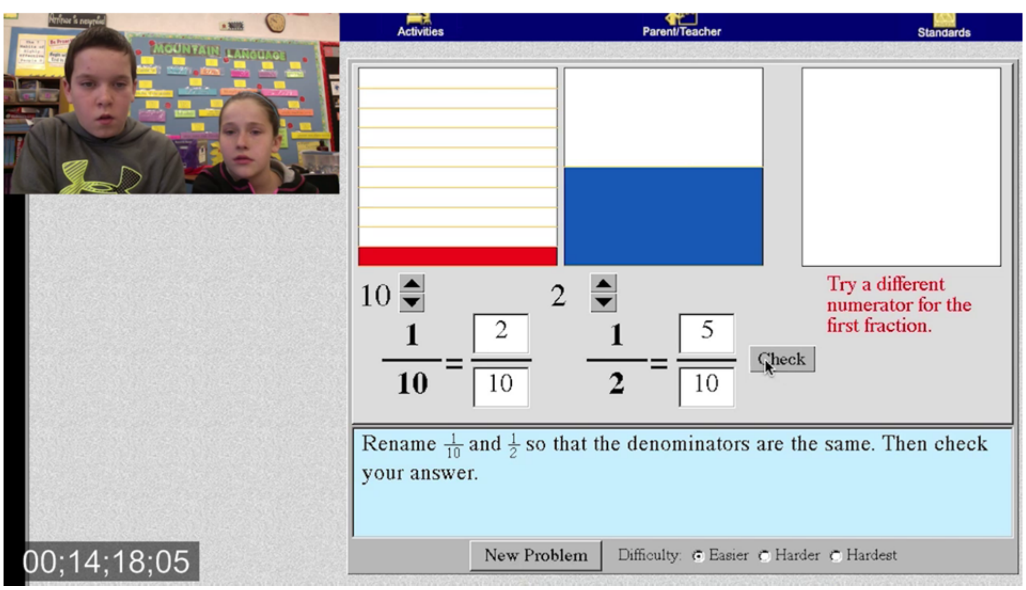


Figure 35. Brandon and Bonnie working with adding fractions, episode T3B.

tutorial virtual manipulatives, to solve the problem presented to them, and then move on to the next problem without providing explanations or justifications for what they are doing or generalizing to other contexts.

Students' discussions when using tutorial virtual manipulatives were marked with declarations of boredom and complaints of how effort much was required of them. The following excerpt from episode T3A of Aaron and Callie working with the Adding Fractions virtual manipulative demonstrates this sentiment (see Figure 36).

131	Aaron	How many are we supposed to do? …20. No, 30.
132	Callie	It says until teacher says it's time.
133	Aaron	That's a bunch! Ok. Let's just quickly do this. (Clicks arrow repeatedly to change denominator) …This should make it easier. This should make it easy because it's really six and this one's five.
134	Callie	Six and five.
135	Aaron	That's nice! So we already know the answer. Like, off the bat, is 11. Eleven-thirtieths (VM feedback: correct)

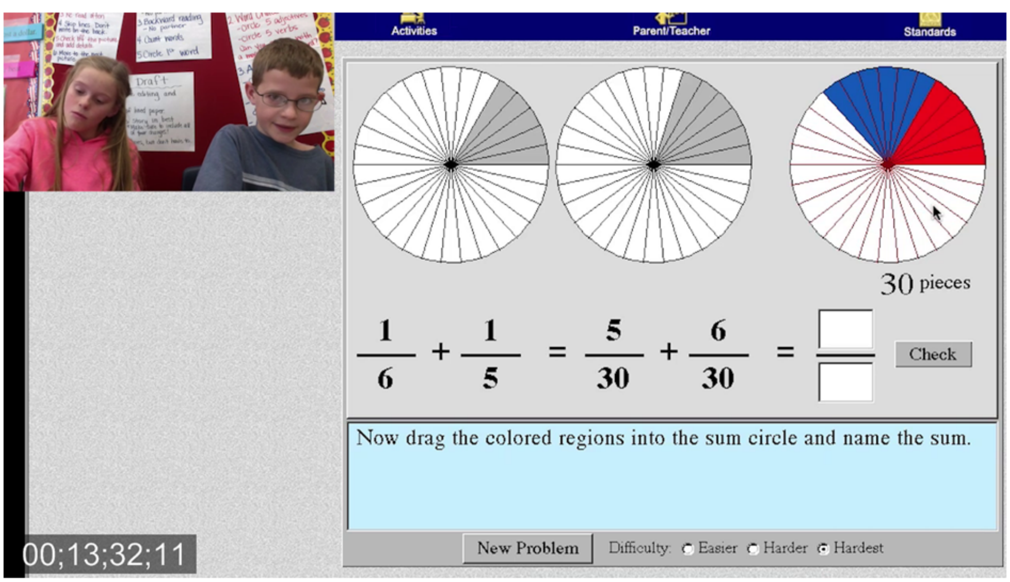


Figure 36. Aaron and Callie working with adding fractions, episode T3A.

136	Aaron	(Clicks 'New Problem') Whoa! We haven't dragged the circle ones. Oh, that's cool. Wait, does it go on the… Ok. So one-sixth plus one-fifth equals… Oh, we need to write… (VM feedback: correct)
137	Aaron	(Waiting for Callie to finish writing)…Next.
. . .		
158	Callie	So…
159	Aaron	I'll start on this real quick. I don't really see the point of us needing to talk through this 'cause it's kind of…simple.

As shown by this example, Aaron did not "see the point" of discussing these problems with his partner (line 159). This attitude was evident in his desire to do the problem on his own. This particular tutorial virtual manipulative gave frequent feedback to students as it guided them through each step of the process. Therefore, most of the students' interactions occurred with the virtual manipulative itself, and not with each other.

In summary, the results of this study identified three distinct partner dispositions: fast-paced, minimal, and executive. These partner dispositions influenced how the students' collaborated with each other, but did not influence differences in students' levels of generalization and justification.

The results of the study showed that the use of different virtual manipulative types influenced the nature of students' mathematical discourse. When working with combined virtual manipulatives, students' discussions reflected higher levels of generalization, justification, and collaboration. This shows that the affordances of combined virtual manipulatives typically encouraged and enhanced student interaction. When working with tutorial virtual manipulatives, students' discussions reflected lower levels of generalization, justification, and collaboration. This shows that the affordances of tutorial virtual manipulatives typically constrained student interaction. Further analysis indicated that collaboration took place consistently throughout all discussions, regardless of virtual manipulative type. However, high levels of generalization and justification occurred less frequently. Unsolicited high levels of justification occurred when using all virtual manipulative types. However, unsolicited high levels of generalization did not occur when using tutorial virtual manipulatives. Virtual manipulative type did not have a statistically significant influence on the amount of mathematical discussion, but it did have an influence on the quality of the discussion.

CHAPTER V

DISCUSSION

The use of technology in current classrooms necessitates a reconsideration of how to leverage technology, such as virtual manipulatives, to most effectively enhance student learning and classroom discourse. This study focused on the influence of different virtual manipulative types on the nature of students' TMD. TMD is defined as discourse in which students use technological representations (e.g., virtual manipulatives) to mediate discussion while engaging in mathematical tasks (see Figure 2). Technology has the potential to enhance and limit communication of mathematical ideas by enabling or restricting access to multiple modalities of mathematical representations (e.g., numeric, pictorial). The overall nature of students' TMD is influenced by the affordances of the technology tools in use, the quality of the mathematical tasks in which the students are engaged, and the broader discourse environment and socio-mathematical norms present in the classroom. The TMD framework provides a means for examining classroom mathematical discourse in light of technology tools available in the classroom.

The purpose of this study was to (a) describe and categorize the nature of students' mathematical discourse as they worked with various virtual manipulative types and (b) to develop theory on the interactions among student-led discourse, virtual manipulatives, and mathematical tasks. This discussion of the results has five sections. The first section describes three distinct partner dispositions identified in this study. The second section describes trends and patterns of variations in students' mathematical discourse when working with each virtual manipulative type. These patterns and trends are used to develop theory on how classroom discourse, technology tools, and mathematical tasks influence TMD. The third section discusses this study's implications for educators. The last two sections identify limitations of the study and suggestions for future research, respectively.

Partner Dispositions

Three distinct partner dispositions were identified in this study that reflect unique characteristics impacting the overall quality of the discourse of each student pair. First, Aaron and Abbie's discussions were characterized by fast-paced and enthusiastic comments. Although they eventually succeeded in completing the assigned tasks, this student pair frequently diverged from the task at hand to talk about topics not related to the assigned tasks. Second, Brandon and Bonnie's discussions were characterized by minimal conversation. They frequently disengaged from what the other was saying and wanted to complete the assigned task as quickly as possible. Third, Colton and Callie's discussions were characterized by task-oriented and executive exchanges that resulted in high levels of collaboration.

The diversity of dispositions identified in these student pairs reflects the complexity of classroom mathematical discourse—even among partners. Many possible explanations exist for the observed differences in these partner dispositions: differences in individuals' personalities, differences in how the students view themselves as learners (Sinclair, 2005), how they view each other (Kotsopoulos, 2010), differences in students' level of academic achievement (Iiskala et al., 2011), or differences in classroom sociomathematical norms (Cobb et al., 1993). Any of these factors could have an impact on the working disposition of a student pair. In theory, various partner dispositions (including those identified in this study) are possible depending on the unique combination of students' characteristics. The presence of multiple partner dispositions during mathematics instruction presents a challenge to teachers as they work to ensure that all students develop understanding of mathematical concepts.

Influence of Virtual Manipulative Types

One overarching research question with four subquestions guided this study. The main research question was: In what ways do different virtual manipulative types influence the nature of students' mathematical discourse? Specific dimensions of discourse—generalization, justification, collaboration, and gestures—were identified by the subquestions. The following sections address the influence of three virtual manipulative types—combined, pictorial, and tutorial—on each dimension of discourse.

Influence of Combined Virtual Manipulatives

Overall, when working with the combined virtual manipulative type, students' discussions reflected statistically significant higher levels of generalization, justification, and collaboration than when working with the other virtual manipulative types. In addition, the students' physical gestures when using the combined virtual manipulative type reflected a balance between beat (i.e., emphasis) and deictic (i.e., pointing). This indicated that affordances of the combined virtual manipulative type enhanced students' mathematical discourse. It has been suggested that the simultaneous linking of representations, as present in the combined virtual manipulative type, supports students in connecting ideas and generalizing concepts (Moyer-Packenham & Westenskow, 2013). For example, when students observe multiple representations of equivalent fractions change in conjunction with each other (e.g., pictorial image, number line model, and numeric symbols) and all representing the same concept, they make comparisons and see patterns more readily.

The multiple representations of mathematics concepts also provide more tools for students to use when justifying and explaining their thinking. For example, while justifying his or her solution for finding the volume of a rectangular prism, a student may point to and comment on the pictorial and numeric representations on a computer screen. With respect to developing theory, the findings of this study indicate that the multiple representations displayed by the virtual manipulatives enabled the students to generalize and justify mathematics concepts effectively in their communications. This is especially evident as levels of justification in students' discussions increased from the beginning to the end of each episode when using combined virtual manipulatives. The linked representations quickly enabled students to justify their solutions in an effective manner. This pattern is similar to findings of Ares and colleagues (2008) who noted that collective representations encouraged students to interact with each other and comment on each other's solutions.

Influence of Pictorial Virtual Manipulatives

Overall, when working with the pictorial virtual manipulative type, students' discussions reflected slightly greater quantities of discourse than when working with the other virtual manipulative types. Students also displayed the highest frequency of gestures (physical and computer based) when working with this virtual manipulative type. This indicated that there was a greater need for the students to communicate the meaning of the representations with each other. The meaning of the representation was not as explicit as with the combined virtual manipulatives. Therefore, the students had to assume responsibility for making connections for themselves. Sometimes, students were successful in making these connections. At other times, they struggled to communicate the mathematical meaning of the manipulative.

Another possible theory for this pattern is that the pictorial virtual manipulative type offered more opportunities for creative variation in how students could interpret the mathematical representations (Moyer-Packenham & Westenskow, 2013). For example, when working with the Base 10 Blocks virtual manipulative, students demonstrated multiple methods for dividing 6.4 into four equal groups. One student pair broke each whole (10 by 10 block) into 10 pieces, and then equally distributed the 64 tenths into four groups. Another student pair first positioned one whole in each corner of the workspace, broke the remaining two wholes into tenths, and then equally distributed the remaining 24 tenths. Both student pairs correctly arrived at the same answer of 1.6. Typically, when engaging in discourse, students' levels of generalization and justification were not as high with the pictorial virtual manipulative type as with the combined virtual manipulative type. The quartile analysis revealed that students' discussions focused on describing their actions with the virtual manipulatives, and that they rarely commented on why they had chosen a certain method or why that method led to a correct answer. However, levels of generalization in students' discussions tended to increase toward the end of the episodes. This indicates that a certain amount of time may be required for students to start to make high-level generalizations when working with pictorial virtual manipulatives. Although the pictorial virtual manipulatives elicited slightly more discussion, the discussion reflected mostly middle to lower levels of discourse.

Influence of Tutorial Virtual Manipulatives

Overall, when working with the tutorial virtual manipulative type, students' discussions reflected the lowest levels of generalization, justification, and collaboration than when working with the other virtual manipulative types. Students also displayed the lowest frequency of gestures (physical and computer based) when working with the tutorial virtual manipulative type. This indicated that features in the tutorial virtual manipulatives discouraged students' mathematical discourse. Tutorial virtual manipulatives are characterized by structured environments that guide students step-by-step toward an understanding of mathematical processes and/or concepts. Oftentimes, they include multiple linked representations of mathematics concepts, and students receive audio or written feedback on their responses throughout their work. As discussed previously, other studies point toward the effectiveness of multiple linked representations in instruction. Prior research shows that the direct feedback provided by tutorial virtual manipulative types is valuable and effective in helping students to learn (Reimer & Moyer, 2005; Steen et al., 2006; Suh & Moyer, 2007). But in these prior studies, the students worked individually and did not work with a partner, so there was no need for discussion.

Despite the positive affordances of tutorial virtual manipulatives when students work alone (i.e., simultaneous linking of representations, direct feedback), the student pairs' discussions in this study reflected lower levels of discourse when using tutorial virtual manipulatives. This pattern remained constant throughout the course of each episode. One possible theory is that the students assumed a more passive role because the tutorial was guiding them through the mathematical processes. Due to the extremely structured nature of the tutorials, students did not feel the need to generalize or justify their answers with each other—nor did they tend to collaborate and build on each other's ideas as much as when working with other virtual manipulative types. Instead, their focus was on responding to the tutorials' direct feedback. Interaction with their partner was secondary to their interaction with the tutorial virtual manipulative. Another possible theory is that the tasks presented by the tutorial virtual manipulatives were not as cognitively demanding as the tasks presented with the other virtual manipulative types (Smith & Stein, 1998). In each tutorial, students were guided through the steps of how to solve the problems. The tasks required little to no need for problem solving—as illustrated by Aaron's comment, "I don't really see the point of us needing to talk through this 'cause it's kind of…simple" (episode T3A, line 159). This means that, theoretically, the type of mathematical task posed by each virtual manipulative may be more influential than the affordances it offers when students work in pairs.

Implications for Educators

A major goal of educational research is to inform classroom practice. Patterns and trends emerging from the results of this study indicate that different virtual manipulative types may be best suited for different stages of learning and instruction or for different instructional arrangements (e.g., partners versus individual learners). Therefore, if further research on TMD continues to find similar patterns and trends, generalizing to multiple populations, educators may use this information in the planning of curriculum and instruction.

Findings from this study suggest that pictorial virtual manipulatives may be more useful as students are starting to develop their understanding of a mathematics concept. This virtual manipulative type offers more flexibility than the other virtual manipulative types and lends itself to an open exploration of mathematical ideas. The lack of simultaneous linking of representations makes it necessary for the teacher to support and scaffold students' learning experiences in such a way to highlight important concepts—an instructional technique appropriate to the early stages of a learning cycle (Hendrickson et al., 2010).

Likewise, findings from this study suggest that combined virtual manipulatives may be best suited for when students are in the process of solidifying and making connections between mathematics concepts and representations, particularly if they are working together in pairs. In this study, students' discussions when using this virtual manipulative type typically reflected higher levels of generalization, justification, and collaboration. Through such robust discussion, students are more likely to learn mathematics in a meaningful way (Imm & Stylianou, 2012; Mueller, 2009; Piccolo et al., 2008; Sfard, 2007). This virtual manipulative type has the affordance of simultaneous linking of representations, which explicitly supports students in making connections among mathematical representations. This linking of representations is critical as students engage in mathematics and as they solidify their understanding (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Hendrickson et al., 2010).

Lastly, findings from this study suggest that tutorial virtual manipulatives are most advantageous for the practice of concepts and skills. Tutorials are designed to walk a student through a concept at his or her own pace. In this study, students' discussions when using this virtual manipulative type typically reflected lower levels of generalization, justification, and collaboration. Although the structured nature of the tutorial virtual manipulative type effectively guided the students through the mathematical concepts, it did not encourage discussion between students. Therefore, if a teacher's goal is to engage students in mathematical discourse, tutorial virtual manipulative types may not be an effective instructional strategy; they would be more effective for individualized instruction or practice.

Limitations

As with all studies, there were limitations that affect the generalizability of these results. The two main limitations were sample characteristics and the broader classroom environment.

This study was designed as an exploratory study of the nature of students' TMD. At the time of this study, the construct of TMD was beginning to emerge and research on it was in initial stages. Therefore, the purpose of this research was to identify and characterize the construct (Lesh & Lovitts, 2000). The sample size was small and limited to six fifth-grade students. With so few participating students, the variations in students' characteristics can have a profound effect on comparison results. These students were accustomed to working with technology on a regular basis. In addition, the participating students were all from the same school that served a white middle-class population with limited diversity. It is possible that findings may differ with other populations such as students of different ages, ethnic groups, or socio-economic status. Other factors that may have influenced the results of this study include students' achievement, students' familiarity with the virtual manipulatives, and students' perceptions of the virtual manipulative types. However, these factors were beyond the scope of this study.

The broader classroom environment may also have been a contributing factor in the results of this study. Because the student pairs came from three different classrooms, it is possible that differences in classroom culture related to the sharing of ideas may have influenced how students interacted with each other in partner situations. The role of the teacher in influencing the nature of classroom discourse was not addressed in this study.

Suggestions for Future Research

In this study, TMD was examined in relation to variations in technology tools. Figure 37 identifies elements of the TMD Framework that were the focus of this study. The study focused on an in-depth analysis of students classroom discourse (i.e., generalization, justification, collaboration, and gestures) as influenced by different virtual manipulative types (i.e., combined, pictorial, and tutorial). In this study there was a

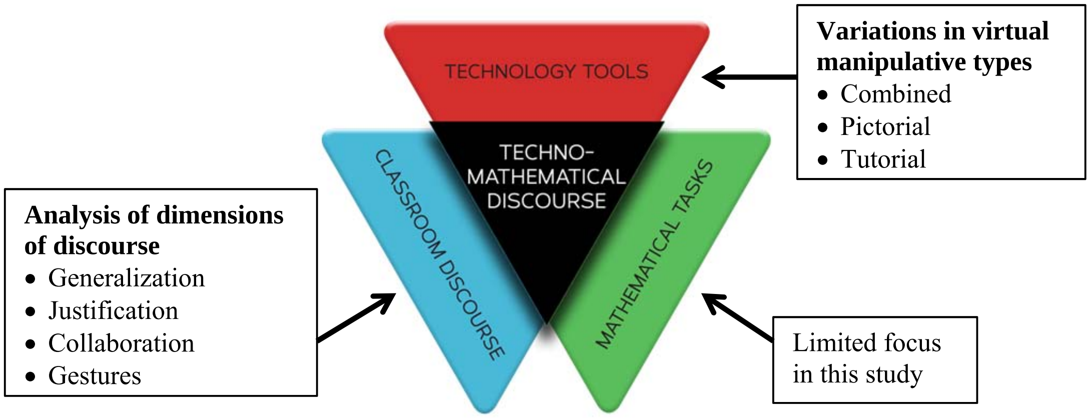


Figure 37. Elements of TMD examined in the current study.

limited focus on the influence of variations of the mathematical tasks in which students engaged.

The present study represents one possible variation of study related to TMD. Other variations need further investigation. For example, future research could focus on other factors related to technology tools, such as students' familiarity with the technology tools, students' perceptions of the technology, or differences in platform (e.g., mouse-controlled versus touch-screen devices). This study did not focus on factors of the broader classroom environment related to classroom discourse. Future research could be conducted on these factors, such as the role of the teacher or varying levels of student achievement. This study examined students' discussions during division, geometry, and fractions units. Future research could examine the influence of variations in mathematical tasks, such as procedural versus conceptual tasks, specific mathematical domains (e.g., fractions, integers, or place value), or lesson formats (e.g., inquiry- versus direct-instruction). Investigation of these factors was beyond the scope of this study. However, their examination could deepen understanding of how students interact with each other when engaging in mathematical tasks through the use of technology.

This was an exploratory study, designed to identify the characteristics of TMD. In order to progress the development of the TMD Framework, more investigations are needed to determine its generalizability and robustness. Similar studies with more diverse populations and larger sample sizes could help to determine if the results of this study were unique to these students or if interactions with these virtual manipulative types are common in the larger population. The comparison of students' discourse with and without technology could also aid in refining the TMD Framework.

Conclusion

This study represents an intersection of two aspects of instruction: the nature of mathematical discourse and the use of virtual manipulatives in the classroom. Extensive research on the former aspect, discourse, reveals that the creation of meaningful discourse in mathematics classrooms is a complex and delicate activity (Walshaw & Anthony, 2008). Yet, few studies have examined the impact of technology on classroom discourse practices. The purpose of the present study was to describe, categorize, and interpret students' discussions as they worked with different virtual manipulative types. This study was built on the premise (a) that mathematics learning occurs when students communicate ideas and discuss mathematics concepts one with another, (b) that virtual manipulatives offer unique affordances that support students' learning of mathematics, and (c) that meaningful discourse takes place when students engage in cognitively demanding tasks.

The results of this study indicated statistically significant differences in levels of student discourse when using different virtual manipulative types. When working with the combined virtual manipulative type, students' discourse reflected considerably higher frequencies of physical and computer-based gestures, and statistically significant higher levels of generalization, justification, and collaboration. When working with the tutorial virtual manipulative type, students' discussions reflected consistently lower frequencies of gestures and lower levels of discourse. When working with the pictorial virtual manipulative type, students' discussions reflected lower levels of discourse as well. However, pictorial virtual manipulatives were associated with the largest amount of discussion among student pairs and the highest quantity of gesture use. One explanation of these variations is that unique affordances of each virtual manipulative type had a direct influence on how students discussed and communicated their mathematical ideas. Most notably, the simultaneous linking of representations present in the combined virtual manipulatives seemed to support students' ability to generalize concepts and justify solutions. However, even though the tutorial virtual manipulative type linked representations simultaneously, its structured manner of presenting learning activities actually discouraged student-student interactions.

The results of this study suggest that in order to encourage meaningful TMD, teachers should choose technology tools (e.g., virtual manipulatives) that combine multiple representations and provide the opportunity to engage in cognitively demanding tasks. Tutorial technology tools have been shown to be effective learning instruments. However, the results of this study indicate that tutorial virtual manipulatives did not encourage meaningful mathematical discourse with these student pairs. This means that the tutorial virtual manipulative type may be better suited for the practice of mathematics concepts or for individual learning.

The patterns and trends identified in this study contribute to the existing literature on the complex issues that surround mathematical discourse and the use of technology in the classroom. While this exploratory study aimed to develop the construct of TMD by examining the interactions among partner discourse, virtual manipulatives, and mathematical tasks, further studies with broader, more diverse populations will contribute to its generalizability.

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APPENDICES

Appendix A

Institutional Review Board (IRB) Certificate

Institutional Review Board

USU Assurance: FWA#00003308

UtahState University
OFFICE OF RESEARCH AND GRADUATE STUDIES

Expedite #6

Letter of Approval

FROM:

Melanie Domenech Rodriguez, IRB Chair

True M. Rubal, IRB Administrator

To:	Patricia Moyer-Packenham, Katie Anderson
Date:	August 12, 2013
Protocol #:	5246
Title:	Examining The Impact Of Different Virtual Manipulative Types On The Nature Of Students' Small-Group Discussions: An Exploratory Mixed Methods Case Study Of Techno-Mathematical Discourse
Risk:	Minimal risk

Your proposal has been reviewed by the Institutional Review Board and is approved under expedite procedure #6 (based on the Department of Health and Human Services (DHHS) regulations for the protection of human research subjects, 45 CFR Part 46, as amended to include provisions of the Federal Policy for the Protection of Human Subjects, November 9, 1998):

Collection of data from voice, video, digital, or image recordings made for research purposes.

This approval applies only to the proposal currently on file for the period of one year. If your study extends beyond this approval period, you must contact this office to request an annual review of this research. Any change affecting human subjects must be approved by the Board prior to implementation. Injuries or any unanticipated problems involving risk to subjects or to others must be reported immediately to the Chair of the Institutional Review Board.

Prior to involving human subjects, properly executed informed consent must be obtained from each subject or from an authorized representative, and documentation of informed consent must be kept on file for at least three years after the project ends. Each subject must be furnished with a copy of the informed consent document for their personal records.

Appendix B

Virtual Manipulatives Used in Study

Table B1

Order of Virtual Manipulatives Used in Study

Order	Cycle 1 Division	Cycle 2 Geometry	Cycle 3 Fractions
First	Combined	Pictorial	Combined
	Rectangle division	Shape sorter	Equivalent fractions
Second	Pictorial	Tutorial	Pictorial
	Base 10 blocks	Coordinate geometry math Interactive	Fraction pieces
Third	Tutorial	Combined	Tutorial
	Dividing decimals	Cubes	Fractions – adding

Table B2

Combined Virtual Manipulatives Used in Study

Name	URL	Domain	CCSSM
Rectangle division	http://nlvm.usu.edu/en/nav/frames_asid_193_g_2_t_1.html?from=category_g_2_t_1.html	Division	5.NBT.6
Cubes	http://illuminations.nctm.org/Activity.aspx?id=4095	Geometry	5.MD.5b
Equivalent fractions	http://illuminations.nctm.org/ActivityDetail.aspx?ID=80	Fractions	5.NF.1

Table B3

Pictorial Virtual Manipulatives Used in Study

Name	URL	Domain	CCSSM
Base 10 blocks	http://www.glencoe.com/sites/common_assets/mathematics/ebook_assets/vmf/VMF-Interface.html	Division	5.NBT.7
Shape sorter	http://illuminations.nctm.org/ActivityDetail.aspx?ID=34	Geometry	5.G.3,4
Fraction pieces	http://nlvm.usu.edu/en/nav/frames_asid_274_g_2_t_1.html	Fractions	5.NF

Table B4

Tutorial Virtual Manipulatives Used in Study

Name	URL	Domain	CCSSM
Dividing decimals	http://www.glencoe.com/sites/common_assets/mathematics/im1/concepts_in_motion/interactive_labs/M1_05/M1_05_dev_100.html	Division	5.NBT.7
Coordinate geometry math interactive	http://edcar-cdn.pbs.org/u/pr/KAET/Coordinate%20Geometry%20Math%20Interactive_f1a70291-8992-4aae-8c53-879a879dcda3/interface.swf	Geometry	5.G.1
Fractions – adding	http://nlvm.usu.edu/en/nav/frames_asid_106_g_2_t_1.html?from=topic_t_1.html	Fractions	5.NF.1,2

Appendix C

Classroom Demographic Questionnaire

Classroom Demographic Questionnaire
Technology Use in the Classroom

Teacher: ______________________________ Number of Years Taught: _________

Number of Students (2013-14): ___________

Part I: Please respond to the following questions.

1. Approximately what percentage of mathematics instructional time do your students spend:
   a. Listening to lecture/teacher demonstrations and taking notes? _________%

   b. Working collectively as a whole class _________%

   c. Working in small groups? _________%

   d. Working individually and independently? _________%
   (total 100%)

Part II: Please circle your best response for each question.

2. How often do your students use computers for instructional activities?

   Every day   A few days a week   Once a week   Once a month

3. How often do your students use computers for mathematics activities?

   Every day   A few days a week   Once a week   Once a month

4. When engaged in mathematics activities, how often do your students:

   a. use the computer to work with virtual manipulatives?

   Every day   A few days a week   Once a week   Once a month

   b. use the computer for fluency practice (drill activities)?

   Every day   A few days a week   Once a week   Once a month

   c. use the computer for assessment purposes?

   Every day   A few days a week   Once a week   Once a month

Appendix D

Lesson Observation Protocol

Lesson Observation Protocol

Date: ____________________________ VM type: ________________________
Grade: ___________________________ Teacher: _________________________
Student 1: ________________________ Student 2: _______________________

Exploration Segment	Gestures Used	Teacher Interactions & Expectations	Affective Factors	Other
Part A				
Part B				

Appendix E

Sample Lesson Plan

Equivalent Fractions

Subject/Strand/Topic	Grade(s)	Common Core Expectations
Fractions	5^{th}	5.NF.1

Key Concepts
The numerators and denominators of equivalent fractions are proportional to each other. Multiplying the numerator and the denominator of any fraction by the same number will result in an equivalent fraction. Equivalent fractions occupy the same location on a number line.

Virtual Manipulative
Equivalent Fractions (combined) http://illuminations.nctm.org/Activity.aspx?id=3510

Materials
Student Task Sheet, Pencils, 1 computer per 2 students

Introduction (10 min)

1. Introduce topic
    a. Activate prior knowledge by having students identify fractions from picture models (square and circle)
    b. Define **numerator** (top: how many pieces you have) and **denominator** (bottom: how many equal-sized pieces are in the whole).
    c. **How do you know if two fractions are equivalent?** Invite a few student responses (full understanding not required at this time).
2. Introduce virtual manipulative.
    a. Provide a brief instruction on features/aspects of the virtual manipulative.
        i. **Goal**: create a blue and a green fraction equivalent to the red fraction.
        ii. **Square/Circle**: switches between different models.
        iii. **Sliders**: change the denominator, change the numerator—can also click on the sections to select them.
        iv. **Checkmark**: if fractions are correct, they will be added to the chart to the right. If they are incorrect, students should fix the error.
        v. **Automatic/Build Your Own**:
3. Briefly go over the structure of the task sheet. Part A orients the students to the features of the virtual manipulative. Part B guides them through an exploration of the mathematical concepts, and contains partner discussion questions.

Exploration (20-30 min)

1. Students will work in pairs—each pair at one computer.
2. Instruct students to open Safari and go to the TMD Study website and navigate to the Equivalent Fractions: Illuminations link.
3. Students should spend about 10 minutes on Part A and 10-20 minutes on Part B.
4. Circulate around the room encouraging students to talk with their partners about the mathematics. Remind students of time constraints and encourage them to finish.

Conclusion (20 min)

1. Have a whole-class discussion/sharing of answers on task sheet & what they learned. Discuss relationships in sets of equivalent fractions. Use questions such as the following to guide the discussion.
    a. **How do you know if two fractions are equivalent?**
    b. **How can you make equivalent fractions?**
    c. **Is there a limit to the number of equivalent fractions for any fraction?**
2. Check for understanding: Write a fraction on the board. Have students identify new equivalent fractions.

Appendix F

Sample Student Task Sheet

Equivalent Fractions

Name: ___________________________ **Date:** ___________

Teacher: _________________________

Link:
http://illuminations.nctm.org/Activity.aspx?id=3510
Instructions: Complete the task sheet as you work with the virtual manipulative. **Remember to talk with your partner** about what you observe happening on the virtual manipulative.

Part A

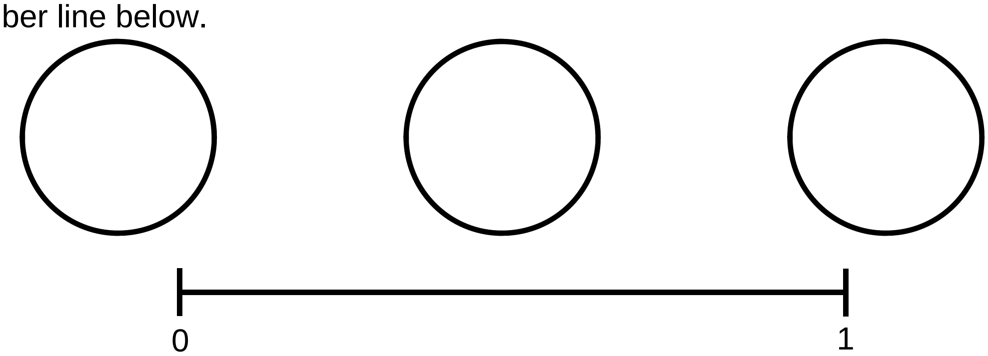
1. Select **Build Your Own**. Select **Circle**.
   a. Set the denominator of the red circle to 2. Highlight 1 red section. What fraction does this represent? Where is it on the number line? Talk to your partner and record your answer.

   b. Set the denominator of the blue circle to 10. Highlight enough blue sections to represent the same amount as the red fraction. What fraction does this represent? Where is it on the number line? Talk to your partner and record your answer.

   c. Set the denominator of the green circle to 6. Highlight enough green sections to represent the same amount as the red fraction. What fraction does this represent? Where is it on the number line? Talk to your partner and record your answer.

   d. Click on the **checkmark**. What happened?

   e. Draw and label the models of each fraction and locate each fraction on the number line below.



Part B

2. Click **Reset Table**. Select **Automatic**. Find equivalent blue and green fractions for 3 different red fractions. Be sure to check your answers each time. Talk to your partner and record your sets of fractions. (Hint: click **New Fraction** to get a new red fraction)

	Blue Fraction	Red Fraction	Green Fraction
1			
2			
3			

   a. Choose one set of equivalent fractions to draw and label.

   b. How are the numerators and denominators related to each other in each set of equivalent fractions? Talk to your partner and record your answer.

3. Click **Reset Table**. Select **Build Your Own**. Use the virtual manipulative to find 2 equivalent fractions for each of the following fractions. Talk to your partner and record your answers.
   a. $\frac{3}{4} =$

   b. $\frac{2}{3} =$

   c. $\frac{5}{7} =$

4. If you have extra time, experiment with the **Square** fraction models. Remember to talk to your partner as you experiment!

CURRICULUM VITAE

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EDUCATION

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Education, Utah State University.
Specialization: Curriculum and Instruction.
Emphasis: Mathematics Education and Leadership.
Dissertation Title: Anderson-Pence, K. L. (2014). *Examining the Impact of Different Virtual Manipulative Types on the Nature of Students' Small-Group Discussions: An Exploratory Mixed Methods Case Study of Techno-Mathematical Discourse*.

M.Ed. May 2007
Master of Education, Southern Utah University.
Emphasis: Elementary Mathematics Education.

B.S. April 2000
Elementary Education, Brigham Young University.
Utah Instructional II Certificate, 1-8 (2004).

Fluent in Portuguese.

EMPLOYMENT HISTORY

UTAH STATE UNIVERSITY

University Supervisor of Student Teachers, Elementary Education Program (2009–10, 2011–12).
School of Teacher Education and Leadership.
College of Education and Human Services, Utah State University, Logan, UT.
Responsibilities included supervising pre-service elementary teachers during their student teaching placements, acting as liaison with cooperating teacher, and planning and conducting weekly seminars.

University Instructor (2009–10, 2011–present).
School of Teacher Education and Leadership.
College of Education and Human Services, Utah State University, Logan, UT.
Responsibilities included teaching graduate and undergraduate courses in the elementary

education program, collaborating with other instructors, developing course syllabi, revising courses based on course evaluation feedback, creating assessments using online course management system (Canvas), and supervising students during practicum placements.

Graduate Research Assistant (2009–10, 2011–present).
School of Teacher Education and Leadership.
College of Education and Human Services, Utah State University, Logan , UT.
Responsibilities included providing research assistance for Dr. Patricia Moyer-Packenham (Mathematics Education) and Dr. Dicky Ng (Mathematics Education) in various research projects.

PUBLIC SCHOOL TEACHING EXPERIENCE – 11 YEARS

Elementary School Teacher, Grade 6, All subjects (2010–11)
Alpine School District, Highland, Utah.

District Math Specialist, Grades 5-6 (2007–09)
Alpine School District, American Fork, Utah.
Designed, organized, and taught professional development for grades 5-6 teachers. Coached and provided model mathematics lessons in select classrooms.

Elementary School Teacher, Grades 3, 4 & 6. All subjects (2000–07).
Alpine School District, Orem, Utah.

AWARDS & PROFESSIONAL RECOGNITION

Graduate Research Assistant of the Year (2013)	Department of Teacher Education and Leadership, Utah State University, Logan, Utah
Presidential Award for Excellence in Mathematics and Science Teaching FINALIST (2007)	Utah State Office of Education
Best of Alpine: In Recognition of Outstanding Service (2007)	Alpine School District, American Fork, Utah
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RESEARCH

RESEARCH INTERESTS

Virtual manipulatives in the mathematics classroom
Mathematics classroom discourse
Mathematics teacher development

PUBLICATIONS

Journal Articles (Refereed)
Anderson-Pence, K. L., Moyer-Packenham, P. S., Westenskow, A., Shumway, J., & Jordan, K. (in press). Visualizing mathematics: Relationships between visual static models and students'

written solutions on fractions. *International Journal for Mathematics Teaching and Learning*.

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Anderson, K. L. (2012). Pattern-block frenzy. *Teaching Children Mathematics, 19*(2), 116-121.

Moyer-Packenham, P. S., Ulmer, L. A. & Anderson, K. L. (2012) Examining pictorial models and virtual manipulatives for third-grade fraction instruction. *Journal of Interactive Online Learning. 11*(3), 103-120.

Conference Proceedings (Refereed)

Moyer-Packenham, P. S., Shumway, J. F., Bullock, E., Tucker, S. I., Anderson-Pence, K., Westenskow, A., Boyer-Thurgood, J., Maahs-Fladung, C., Symanzik, J., Mahamane, S., MacDonald, B., & Jordan, K., The Virtual Manipulatives Research Group at Utah State University. (2014, April). Young children's learning performance and efficiency when using virtual manipulative mathematics iPad apps. Paper presented at the annual National Council of Teachers of Mathematics Research Conference (NCTM), New Orleans, Louisiana.

Anderson-Pence, K. L., & Moyer-Packenham P. S. (2014, January). Techno-mathematical discourse. *Proceedings of the 12th annual Hawaii International Conference on Education* (HICE), (pp. 1466-1483), Honolulu, Hawaii, ISSN# 1541-5880.

Boyer-Thurgood, J., Moyer-Packenham, P., Tucker, S. , Anderson, K. L., Shumway, J., Westenskow, A., & Bullock, E. (2014, January). Kindergarteners' Strategy Development During Combining Tasks on the iPad. *Proceedings of the 12th Annual Hawaii International Conference on Education* (HICE), (pp. 1113-1114), Honolulu, Hawaii, ISSN# 1541-5880.

Moyer-Packenham. P. S., Anderson, K. L., Shumway, J. F., Tucker, S., Westenskow, A., Boyer-Thurgood, J., & Bullock, E., Mahamane, S., Baker, J., Gulkilik, H., Maahs-Fladung, C., Symanzik, J., & Jordan, K., The Virtual Manipulatives Research Group at Utah State University. (2014, January). Developing research tools for young children's interactions with mathematics apps on the iPad. *Proceedings of the 12th Annual Hawaii International Conference on Education* (HICE), (pp. 1685-1694), Honolulu, Hawaii, ISSN# 1541-5880.

Tucker, S. I., Moyer-Packenham, P. S., Boyer-Thurgood, J. M., Anderson, K. L., Shumway, J. F., Westenskow, A., & Bullock, E., The Virtual Manipulatives Research Group at Utah State University. (2014, January). Literature supporting investigations of the nexus of mathematics, strategy, and technology in children's interactions with iPad-based virtual manipulatives. *Proceedings of the 12th annual Hawaii International Conference on Education* (HICE), (pp. 2338-2346), Honolulu, Hawaii, ISSN# 1541-5880.

Ng, D., & Anderson, K. L. (2011). Cognitive empathy and mathematics teaching, In B. Ubuz (Ed.) *Proceedings of the 35th annual conference of the International Group for the Psychology of Mathematics Education* (PME) (Vol. 3 pp. 273-280). Ankara, Turkey: PME.

Articles Submitted

Moyer-Packenham, P. S., Baker, J., Westenskow, A., Anderson, K. L., Shumway, J., Rodzon, K., Ng, D., & Jordan, K. (under review 2013). *Virtual manipulatives versus classroom instruction: Hidden predictors of achievement.*

Westenskow, A., Moyer-Packenham, P. S., Anderson-Pence, K. L., Shumway, J., & Jordan, K. (under review 2013). *Cute drawings? The disconnect between students' pictorial representations and their mathematics responses to fraction questions.*

Articles in Progress

Anderson-Pence, K. L. *Classroom action research as a catalyst for change in mathematics instruction: A teacher's perception* (working title).

Anderson-Pence, K. L. *Techno-mathematical discourse: A framework for examining technology-driven discussions in mathematics classrooms* (working title).

Brown, A. & Anderson-Pence, K. L. *Elementary pre-service teachers' perceptions of a distance mathematics methods course* (working title).

Monroe, E. E., Bahr, D., Wentworth, N., & Anderson-Pence, K. L. *Action research projects in elementary mathematics professional development: Teachers implementing instructional change* (working title).

Shumway, J., Westenskow, A., Moyer-Packenham, P. S., Anderson-Pence, K. L., Tucker, S., & Boyer-Thurgood, J. *Comparing students' errors: The relationship between different representational modalities and students' fractional understanding* (working title).

RESEARCH ACTIVITIES

Captivated! Young Children's Learning Interactions with iPad Mathematics Apps. (2013–present). Obtained IRB approval for the project; pilot tested iPad-based interview protocols; conducted iPad-based interviews with participants; observed interviews; documented methods history in preparation for the writing of forthcoming papers; collected and coded data using qualitative video coding software. Project has resulted in multiple paper presentations at the Hawaii International Conference on Education (2014). Utah State University (with Dr. Patricia Moyer-Packenham and Dr. Cathy Maahs-Fladung).

Pictorial Models and Virtual Manipulatives for Fraction Instruction (2011–2012). Conducted library and document searches for a literature review on virtual and pictorial representations of fraction concepts used by students of different ability levels. Project resulted in paper accepted to Journal of Interactive Online Learning. Utah State University (with Dr. Patricia Moyer-Packenham).

Distance Education Delivery Methods Project (2010–present). Analyzed student response data. Project resulted in a presentation at the AMTE Annual Conference (2010) and a paper submitted for publication. Utah State University (with Dr. Amy Brown).

Grades 3-4 Fractions and Virtual Manipulatives Mathematics Project (2009–present). Taught third- and fourth-grade mathematics fraction units; collected, coded, and analyzed data. Conducted library and document searches for a literature review on the use of technology in mathematics instruction. Project has resulted in multiple papers (in progress) and presentations at the NCTM Research Presession (2013), AERA Conference (2013), and the SSMA Conference (2011). Utah

State University (with Dr. Patricia Moyer-Packenham and Dr. Kerry Jordan).

Utah Elementary Mathematics Endorsement Project (2006–present). Taught in-service lessons, coded, and analyzed participants' final projects, conducted, transcribed, coded, and analyzed interviews of teachers on their experiences with conducting action research in their classrooms. Project has resulted in multiple papers (in progress). Brigham Young University (with Dr. Eula Monroe, Dr. Damon Bahr, and Dr. Nancy Wentworth).

Quadrilaterals Study (2010–2011). Taught fifth-grade mathematics lessons on quadrilaterals, recorded and analyzed video data on classroom interactions and informal assessments, and collected pre- and post-test data on students' understanding of quadrilateral relationships. Project resulted in a presentation at the CMC-South Conference (2011). Utah State University (with Dr. Dicky Ng).

Cognitive Empathy Study (2009–2010). Conducted and transcribed interviews of pre-service teachers' on their experiences in mathematics and perceptions of their ability to reach struggling learners. Project resulted in a paper accepted to the PME Conference (2011). Utah State University (with Dr. Dicky Ng).

RESEARCH SUPERVISION

Committee Member & Mentor – Undergraduate Honors Program Committee
Jessica Billingsly (2012–13); Utah State University

GRANTS FUNDED

Research and Projects Grant. ($1,000). (2014) Utah State University, Graduate Student Senate. Funding awarded for the conducting of dissertation research study.

Graduate Student Travel Award. ($1,000). (2014) Utah State University, Office of Research and Graduate Studies, College of Education. Travel funding awarded for presentation at the Hawaii International Conference on Education Annual Meeting in Honolulu, Hawaii.

Graduate Research Assistant. ($20,000). *Captivated! Young Children's Learning Interactions with iPad Mathematics Apps.* (2013–14). Utah State University, Vice President for Research RC Funding. Lead PI – Patricia Moyer-Packenham, Co-PI – Cathy Maahs-Fladung, and the Virtual Manipulatives Research Group. Project goal: build theory and knowledge about the nature of young children's ways of thinking and interacting with virtual manipulatives using touch-screen mathematics apps on the iPad. My role: obtain IRB approval, pilot test problem-based interview protocols, conduct iPad-based interviews with participants, observe interviews, document methods history in preparation for the writing of forthcoming papers, collect and code video data, collaborate on publications and presentations focusing on young children's interactions with iPad apps.

Graduate Student Travel Award. ($500). (2013) Utah State University, Office of Research and Graduate Studies, College of Education Travel funding awarded for presentation at the NCTM Annual Meeting in Denver, Colorado.

Graduate Student Travel Grant. ($500). (2013) Utah State University, Center for Women and Gender. Travel funding awarded for presentation at the AMTE Annual Conference in Orlando, Florida.

Graduate Research Assistant ($35,000). *Virtual Manipulatives Research Group: Effects of Multiple Visual Modalities of Representation on Rational Number Competence*. (2011–12). Utah State University, Vice President for Research SPARC Funding. Lead PI – Patricia Moyer-Packenham; Collaborating Faculty – Kerry Jordan, Dicky Ng, and Kady Schneiter. My role: design lesson plans for experimental classroom, teach experimental lessons at research sites, conduct data collection and analysis, participate in research team meetings, collaborate on publications and presentations focusing on using virtual manipulatives to teach rational number concepts.

Graduate Student Professional Conference Award. ($600). (2011) Utah State University, Graduate Student Senate and College of Education. Travel funding awarded for presentation at the SSMA Annual Meeting in Colorado Springs, Colorado.

TEACHING

UNIVERSITY TEACHING

Utah State University, Logan, Utah (2009-14)
College of Education and Human Services

TEAL 6522/ TEPD 5522 – Mathematics for Teaching K–8: Rational Numbers and Proportional Reasoning (Spring 2014).
Graduate Course. Elementary Mathematics Endorsement course. Provides practicing teachers a deeper understanding of rational numbers, operations with rational numbers, and proportionality, and instructional strategies to facilitate the instruction of this content for elementary students. Course delivered via interactive broadcast (distance education).

ELED 4060 – Teaching Mathematics & Practicum Level III (Spring 2010, Spring 2012, Fall 2012, Spring 2013, Fall 2013).
Undergraduate Course. Relevant mathematics instruction in the elementary and middle-level curriculum; methods of instruction, evaluation, remediation, and enrichment. Includes supervision of students in a field experience practicum.

ELED 5150 – Student Teaching Seminar & Supervision (Fall 2009, Fall 2011).
Undergraduate Course. Supervision of student in teachers in primary grade (1-3) and upper-elementary (4-6) classrooms. Student teachers are expected to demonstrate competency in designing and implementing a developmentally appropriate learning environment. Students demonstrate professionalism in teaching as they begin their transition from university student to professional teacher. Includes weekly support seminars and monthly observations.

CURRICULUM DEVELOPMENT

Utah State University, Logan, Utah (2013–present)
Elementary Mathematics Teacher Academy – Developed course materials for master's level courses for Utah State University's Elementary Mathematics Teacher Academy (EMTA). Course designed to develop teachers' mathematical knowledge for teaching aligned with the Common Core State Standards for Mathematics. Materials developed included readings, video lectures, application assignments, and assessments for online course delivery. Developed the following 20 sixth-grade curriculum modules:

6.RP.1	Language of Ratios (2013)
6.RP.2	Rates & Ratios (2013)
6.RP.3a	Ratio Tables & Coordinate Planes (2013)
6.RP.3b	Unit Pricing & Constant Speed (2013)
6.RP.3c	Ratios & Percentages (2013)
6.RP.3d	Converting Measurement Units (2013)
6.RP	Big Idea: Proportional Reasoning (2013)
6.NS.5	Meaning of Positive & Negative Numbers (2013)
6.NS.6a,c	Rational Numbers on the Number Line (2013)
6.NS.6b,c	Ordered Pairs on the Coordinate Plane (2013)
6.NS.7a,b	Ordering Rational Numbers (2013)
6.NS.7c,d	Absolute Value (2013)
6.NS.8	Real-World Problems on the Coordinate Plane (2013)
6.NS	Big Idea: Rational Numbers (2013)
6.G.1(part 1)	Finding Area by Rectangle Composition (2013)
6.G.1(part 2)	Finding Area by Triangle Composition (2013)
6.G.2	Volume of Rectangular Prisms (2013)
6.G.3	Polygons on the Coordinate Plane (2013)
6.G.4	Surface Area (2013)
6.G	Big Idea: Van Hiele Levels of Geometric Thought (2013)

SERVICE

PRESENTATIONS

Invited Addresses

California Mathematics Council

Anderson, K. L. & Ng, D. (2011, November). *Square or Rhombus?: Developing Relational Thinking Through Geometric Reasoning Part 2*. Workshop Presentation. California Mathematics Council South (CMC-SOUTH) 52nd Annual Mathematics Conference. Palm Springs, California.

Ng. D. & Anderson, K. L. (2011, November). *Square or Rhombus?: Developing Relational Thinking Through Geometric Reasoning Part 1*. Workshop Presentation. California Mathematics Council South (CMC-SOUTH) 52nd Annual Mathematics Conference. Palm Springs, California.

International Presentations – Scholarship

Hawaii International Conference on Education (HICE)

Anderson, K., & Moyer-Packenham, P. S. (2014, January). *Techno-mathematical discourse*. Research Presentation, 12th Annual Hawaii International Conference on Education (HICE), Honolulu, Hawaii.

Boyer-Thurgood, J., Moyer-Packenham, P. S., Shumway, J., Westenskow, A., Tucker, S., Anderson, K., & Bullock, E. (2014, January). *Kindergarteners' Strategy Development During Combining Tasks on the iPad*. Research Presentation, 12th Annual Hawaii International Conference on Education (HICE), Honolulu, Hawaii.

Moyer-Packenham. P. S., Shumway, J., Westenskow, A., Tucker, S., Anderson, K., Boyer-Thurgood, J., & Bullock, E. (2014, January). *Young Children's Mathematics Interactions with Virtual Manipulatives on iPads.* Research Presentation, 12th Annual Hawaii International Conference on Education (HICE), Honolulu, Hawaii.

Tucker, S. I., Moyer-Packenham, P. S., Boyer-Thurgood, J. M., Anderson, K. L., Shumway, J., Westenskow, A., & Bullock, E. (2014, January). *The Nexus of Mathematics, Strategy, and Technology in Second-Graders' Interactions with an iPad-Based Virtual Manipulative*. Paper Session, 12th Annual Hawaii International Conference on Education (HICE), Honolulu, Hawaii.

National Presentations – Scholarship

American Educational Research Association (AERA)

Moyer-Packenham, P. S., Jordan, K., Anderson, K., Baker, J., Shumway, J., & Westenskow, A. (2013, April). *Hidden Predictors of Achievement: The Equalizing Effect of Virtual Manipulatives for Mathematics Instruction.* Paper Session. Annual Meeting of the American Educational Research Association (AERA). San Francisco, California.

Association of Mathematics Teacher Educators (AMTE)

Shumway, J., Bostwick, A., Anderson, K. L., & Tucker, S. (2013, January). *Building Partnerships: A Collaborative Lesson-Study Experience in a Preservice Mathematics Methods Course.* Presentation, 17th Annual AMTE Conference, Orlando, Florida.

National Council of Teachers of Mathematics (NCTM)

Anderson, K. L. (2013, April). *Pattern Block Frenzy: Proportional Reasoning with Technology.* Presentation, 91st Annual Meeting of the National Council of Teachers of Mathematics (NCTM), Denver, Colorado.

Anderson, K. L., Westenskow, A., & Moyer-Packenham, P. S. (2012, April). *Teachers' Resources for Using Virtual Manipulatives to Teach Fraction Concepts.* Presentation, 90th Annual Meeting of the National Council of Teachers of Mathematics (NCTM), Philadelphia, Pennsylvania.

Ng, D. & Anderson, K. L. (2011, April). *Square or Rhombus? Helping Students Develop Relational Thinking through Geometry.* Presentation, 89th Annual Meeting of the National Council of Teachers of Mathematics (NCTM), Indianapolis, Indiana.

National Council of Teachers of Mathematics Research Presession (NCTM)

Moyer-Packenham, P. S., Baker, J., Anderson, K. L., Shumway, J. F., Westenskow, A., & Jordan, K. (2013, April). *Hidden Achievement Predictors: Equalizing Effects of Virtual Manipulatives.* Paper Session. National Council of Teachers of Mathematics (NCTM) Research Presession, Denver, Colorado.

School Science and Math Association (SSMA)

Moyer-Packenham, P. S., Jordan, K., Ng, D., Anderson, K., Baker, J., Rodzon, K., Shumway, J., & Westenskow, A. (2011, November). *School Mathematics Research on Virtual Manipulatives: A Collaborative Team Approach*. Research Presentation. School Science and Math Association (SSMA) Annual Meeting. Colorado Springs, Colorado.

Brown, A. B. & Anderson, K. L. (2010, November). *Broadcast, Online, Hybrid? Teaching Elementary Math Methods in Distance Education Environments.* Research Presentation, School Science and Math Association (SSMA) Annual Meeting, Ft. Meyers, Florida.

State & Regional Presentations

Consortium for Mathematics Educational Enhancement (CMEE)

Anderson, K. L. & Welch, A. (2007, November). *Motivating Students in Mathematics Through Technology: an Introduction to eNLVM.* Workshop Presentation, Consortium for Mathematics Educational Enhancement (CMEE) Annual State Conference, Salt Lake City, Utah.

T^3: Integrating Math and Technology

Anderson, K. L. & Nielsen, M. (2003, February). *Tumbling Tetronimoes & Broken Calculators.* Workshop Presentation, T^3: Integrating Math and Technology Annual State Conference, Salt Lake City, Utah.

Utah Council of Teachers of Mathematics (UCTM)

Anderson-Pence, K. L. (2013, November). *Techno-Mathematical Discourse: A Framework for Analyzing Math Talk While Working with Virtual Manipulatives.* Annual Conference of the Utah Council of Teachers of Mathematics (UCTM), Salt Lake City, Utah.

Anderson, K. L. (2012, October). *Pattern Block Frenzy: Proportional Reasoning with Technology.* Annual Conference of the Utah Council of Teachers of Mathematics (UCTM), American Fork, Utah.

Anderson, K. L. (2011, November). *Square or Rhombus?: Developing Relational Thinking Through Geometric Reasoning*. Annual Conference of the Utah Council of Teachers of Mathematics (UCTM), Salt Lake City, Utah.

Anderson, K. L. (2011, May). *Minilessons for Multiplying and Dividing Fractions*. Utah Council of Teachers of Mathematics (UCTM) Outreach Conference, Cedar City, Utah.

Anderson, K. L. & Leder, A. (2008, October). *Minilessons for Extending Multiplication.* Annual Conference of the Utah Council of Teachers of Mathematics (UCTM), Sandy, Utah.

Anderson, K. L. (2006, October). *Unraveling Integers.* Annual Conference of the Utah Council of Teachers of Mathematics (UCTM), Salt Lake City, Utah.

Anderson, K. L. (2005, October). *K-6 Mathematics Benchmarking System.* Annual Conference of the Utah Council of Teachers of Mathematics (UCTM), Clearfield, Utah.

Anderson, K. L. (2004, October). *The Ups and Downs of 3rd grade Algebra.* Annual Conference of the Utah Council of Teachers of Mathematics (UCTM), Sandy, Utah.

NATIONAL LEADERSHIP & SERVICE

Reviewer (2013–present)	*School Science and Mathematics*. School Science and Mathematics Association.
Reviewer (2012–present)	*Teaching Children Mathematics*. National Council of Teachers of Mathematics

STATE LEADERSHIP & SERVICE

Board of Directors (2009–11)	Utah Council of Teachers of Mathematics. Webmaster.
Board of Directors (2009–11)	Utah Council of Teachers of Mathematics. Monthly Newsletter Editor.
Board of Directors (2007–09)	Utah Council of Teachers of Mathematics. K-3 Representative. Wrote problems for state math competition.
Committee Member (2007–09)	State Mathematics Education Coordinating Committee (Utah)
Committee Member (2006–08)	State of Utah Instructional Materials Committee
District Math Coordinator (2001–04)	Grade 3: Alpine School District, American Fork, Utah

STATE SERVICE – OUTREACH FOR PUBLIC SCHOOLS

Utah

Edith Bowen Lab School, Logan, Utah. *Professional Development School Partnership* (September 2013). Taught a 4th grade model minilesson on relational thinking with multiplication.

Alpine School District, Orem, Utah. *Utah Elementary Mathematics Endorsement Project* (March 2012). Taught a 6th grade model lesson during a ratios and proportions unit.

Utah State Office of Education. *Utah's Common Core State Standards (CCSS) 6th Grade Committee.* (December 2010). Worked with a team of 25 sixth-grade teachers to interpret core standards, suggest instructional strategies, identify useful resources, and construct sample formative assessment tasks.

Emery County School District, Castledale, Utah. *K-6 Mathematics Content and Selected Topics from Algebra and Geometry.* (2008-09). Lead Instructor, two workshops for 25 grades 4-6 teachers.

Alpine School District, American Fork, Utah. *Utah Elementary Mathematics Endorsement.* (2006-08). Co-instructor, two cohorts of 20 grades 2-6 teachers.

Alpine School District, American Fork, Utah. *Making Meaning for Operations.* (2008). Lead Instructor, eight-week workshop for 6 participants.

Alpine School District, American Fork, Utah. *Introduction to eNLVM.* (October-November 2006). Lead Instructor, four-week workshop for 12 participants.

Alpine School District, American Fork, Utah. *Motivating Students to Engage in Mathematics Using Technology.* (March 2006). Lead Instructor, one-day workshop for 50 participants.

Alpine School District, American Fork, Utah. *Literacy and Technology in the Content Areas.* (January 2006). Lead Instructor, one-day workshop for 50 participants.

Alpine School District, American Fork, Utah. *Making Meaning for Operations.* (2003). Lead Instructor, eight-week workshop for 20 participants.

Alpine School District, American Fork, Utah. *Building a System of Tens.* (2002). Lead Instructor, eight-week workshop for 20 participants.

Alpine School District, American Fork, Utah. *Math Support Group for Investigations: Grade 3.* (2001-2004). Lead Facilitator. Year-long meetings for 6 participants.

INTERNATIONAL MATHEMATICS EXPERIENCES

October 2011	Mathematics Education Delegation, People to People Ambassadors. São Paulo, Brazil. Rio de Janeiro, Brazil. Visited primary/secondary schools and universities; met with Brazilian mathematics education researchers, classroom teachers, and students. Topics discussed: Use of technology in mathematics education, ethnomathematics, culturally relevant pedagogy, and teacher professional development.
November 2007	Mathematics Education Delegation: Education Forum. People to People Ambassadors. Cairo, Egypt. Visited primary/secondary schools and universities. Topics discussed: Use of technology in education, teacher professional development.

PROFESSIONAL DEVELOPMENT ATTENDED

April 2012	Grant Proposal Writing Workshop, Utah State University, Logan, Utah.
July 2008	Developing Mathematical Ideas Institute. Mt. Holyoke College, South Hadley, Massachusetts.
July 2008	Math Coaching Institute. Mt. Holyoke College, South Hadley, Massachusetts.
April 2008	Professional Development with Catherine T. Fosnot, American Fork, Utah.
June 2007	Professional Development with Robert Marzano. American Fork, Utah.
July 2003	Investigations Summer Institute: Building Computational Fluency from Kindergarten through Grade 5. Cedar Hills, Utah.