5-2015

Classification of Five-Dimensional Lie Algebras with One-Dimensional Subalgebras Acting as Subalgebras of the Lorentz Algebra

Jordan Rozum

Utah State University

CLASSIFICATION OF FIVE-DIMENSIONAL LIE ALGEBRAS WITH ONE-DIMENSIONAL SUBALGEBRAS ACTING AS SUBALGEBRAS OF THE LORENTZ ALGEBRA

by

Jordan Rozum

A thesis submitted in partial fulfillment
of the requirements for the degree

of

MASTER OF SCIENCE

in

Mathematics

Approved:

Ian Anderson Major Professor	Charles Torre Committee Member
Mark Fels Committee Member	Mark McLellan Dean of Graduate Studies

UTAH STATE UNIVERSITY
Logan, Utah

2015

ABSTRACT

Classification of Five-Dimensional Lie Algebras with

One-Dimensional Subalgebras Acting as

Subalgebras of the Lorentz Algebra

by

Jordan Rozum, Master of Science

Utah State University, 2015

Major Professor: Dr. Ian Anderson
Department: Mathematics and Statistics

Motivated by A. Z. Petrov's classification of four-dimensional Lorentzian metrics, we provide an algebraic classification of the isometry-isotropy pairs of four-dimensional pseudo-Riemannian metrics admitting local slices with five-dimensional isometries contained in the Lorentz algebra. A purely Lie algebraic approach is applied with emphasis on the use of Lie theoretic invariants to distinguish invariant algebra-subalgebra pairs. This method yields an algorithm for identifying isometry-isotropy pairs subject to the aforementioned constraints.

(186 pages)

DEDICATION

This thesis is dedicated to the one who has given me the most enthusiastic encouragement, always stayed by my side, and given frequent and unsolicited dictation assistance.

This thesis is dedicated to my dog, Hendrix.

ACKNOWLEDGMENTS

First, I would like to thank my advisors, Professors Ian Anderson and Charles Torre for their advice and guidance on this project. I am especially grateful for the long hours Ian Anderson has spent with me reviewing this thesis. In addition, I would like to thank the final member of my thesis committee, Dr. Mark Fels, for the time and effort he has put forth in fulfilling this role.

I also acknowledge Jesse Hicks' contributions to this project, including providing Maple databases, helping to find several difficult changes of basis, and having many fruitful discussions about the "big picture".

As this thesis is the final product of my time at Utah State University, I would also like to take this opportunity to thank those who have helped me throughout my time here. Dr. Shane Larson was my first research advisor and taught me what it means to be a good scientist, helping me to form my commitment to outreach. Dr. David Peak has always given me excellent advice and driven me to apply myself in new endeavors. Finally, Karalee Ransom has fixed far too many problems for me to enumerate here and I am forever grateful to her.

Of course, I must also thank my friends and family who have encouraged me throughout my career thus far (and I strongly suspect they will continue to encourage me). In particular, I would like to thank my wife, Rachel Rozum, whose influence has greatly improved my diligence and character.

Jordan Rozum

Contents

List of Tables

List of Figures

CHAPTER 1

INTRODUCTION AND SUMMARY OF RESULTS

1.1. Introduction

A pseudo-Riemannian manifold $(\mathcal{M}, g)$ is an n-dimensional manifold equipped with a metric of signature (p, q). The isometry group G of $(\mathcal{M}, g)$ is the group of diffeomorphisms on $\mathcal{M}$ that preserve g with functional composition as the group operation. In the book Einstein Spaces [**10**], A. Z. Petrov gives a local classification of four-dimensional Lorentzian metrics according to the algebraic structure of the isometry algebra and the signature of the metric on the orbits. As such, Petrov's results provide a systematic approach to finding exact solutions in general relativity as well as to the equivalence problem of four-dimensional Lorentzian metrics with symmetry. However, there is reason to believe that small gaps exist in Petrov's classification (see for example [**4**]) and therefore, an independent verification of these results is desirable. This thesis provides that verification for a significant portion (to be made precise shortly) of the metrics classified in [**10**]. Whereas Petrov's approach is a combination of geometric, algebraic, and inductive arguments, the approach taken here is purely algebraic.

From the perspective of the study of group actions on manifolds, the local classification of isometries and metrics can be subdivided into two branches according to whether or not the group action admits a local slice. The notion of a slice characterizes in a precise sense when the group orbits at each point are equivalent as homogeneous spaces (see Chapter 2, Definition 21). If the group action admits a local slice, the problem of classifying isometries and metrics can be reduced to the case of transitive isometry, i.e., the study of homogeneous spaces admitting pseudo-Riemannian metrics in dimensions two, three, and four. The homogeneous case further splits into the cases of reductive and non-reductive isotropy (see Chapter 2, Definition 40). See Figure 1.1 We pause here to remark that in the Riemannian case, local slices and reductive complements always exist, so the complexity of the classification problem is greatly reduced (for a more complete treatment of slices, see [**8**]).

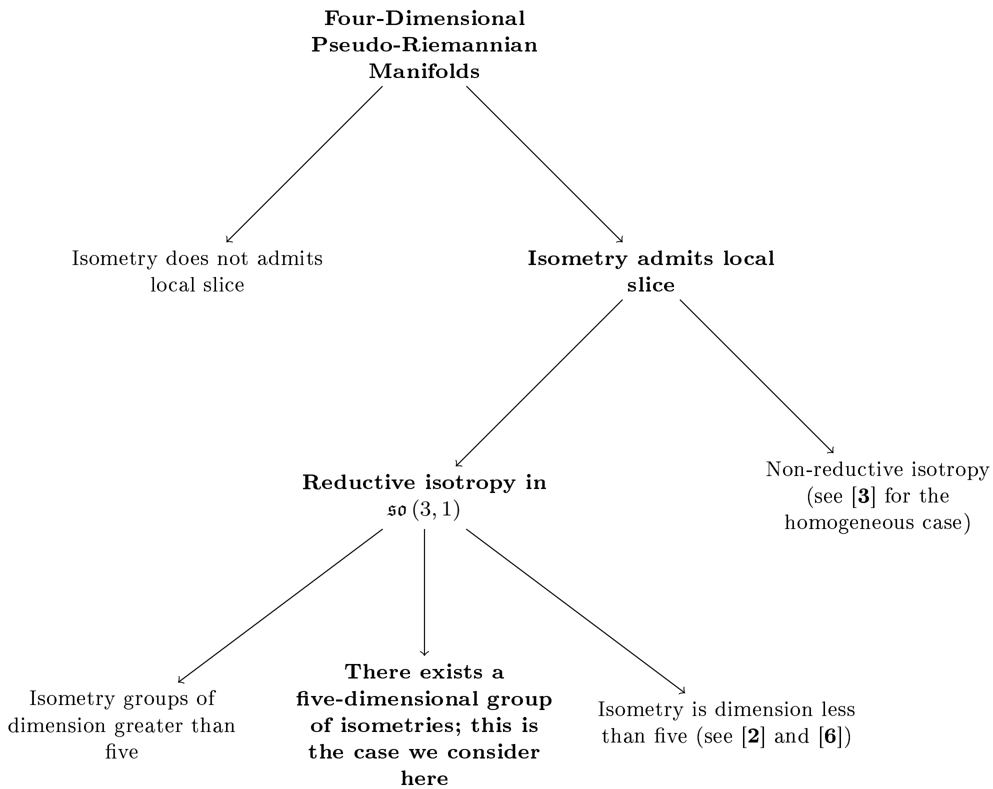

FIGURE 1.1. Summary of pseudo-Riemannian manifolds considered in this thesis.

The case of non-reductive isotropy has been studied in [**3**], which gives an algebraic classification of non-reductive homogeneous pseudo-Riemannian spaces of dimension four. The reductive case has been done in dimensions two and three in [**2**] and [**6**]. We extend this latter study by considering four-dimensional homogeneous space-times admitting five-dimensional isometry groups. For completeness, we also examine the case of five-dimensional isometry on three-dimensional homogeneous spaces and find one case that seems to have been overlooked in [**2**]. Also of note is that a similar algebraic approach to the one undertaken here was applied to the classification of homogeneous Einstein-Maxwell spaces in [**5**].

In summary, this thesis examines those space-times for which there is a five-dimensional group of isometries admitting a local slice and having reductive isotropy. In these cases, there is a direct correspondence between metrics on the orbit manifold and metrics on the reductive complement to the isotropy subalgebra that are isotropy invariant (see Theorem 67). In this way, our problem reduces to an algebraic classification of Lie algebra-subalgebra pairs $(\mathfrak{g}, \mathfrak{h})$ where

(1) $\mathfrak{g}$ is five-dimensional
(2) $\mathfrak{h} \subset \mathfrak{g}$ is reductive
(3) the adjoint representation of $\mathfrak{h}$ on a reductive complement $\mathfrak{m}$ is a subalgebra of $\mathfrak{so}(3,1)$.

Two pairs $(\mathfrak{g}_1, \mathfrak{h}_1)$ and $(\mathfrak{g}_2, \mathfrak{h}_2)$ are considered equivalent if there is a Lie algebra isomorphism $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\phi(\mathfrak{h}_1) = \mathfrak{h}_2$.

This algebraic classification is achieved by applying the "Schmidt method" outlined in [**11**]. The key idea behind this method is to fix the adjoint action of the isotropy to act as a subalgebra of $\mathfrak{so}(3,1)$. The next step of the Schmidt method is to enforce the Jacobi identities and normalize the structure constants to identify all Lie algebras of this form up to real change of basis using a standard classification, e.g. [**12**], as is used here. Finally, the isotropy is placed in some convenient form via automorphism.

After the imposition of the Jacobi identities, the structure constants may still contain several parameters and the Lie algebraic classification may depend on these parameters non-trivially. Thus the straightforward approach of simply trying to find appropriate changes of basis by inspection becomes unmanageable and cases are easily missed. Therefore, at each stage in the classification, we determine a Lie theoretic invariant with which to split cases. Not only does this help ensure the integrity of the classification by providing a robust organizational structure, it also yields an algorithmic approach for determining to which standard pair an algebra-subalgebra pair belongs. We believe that the use of Lie theoretic invariants to enhance the Schmidt method is the primary technical contribution of this thesis.

After performing the Schmidt method to generate the algebra-subalgebra pairs corresponding to space-times for which there is a group of isometries that admits a local slice, is five-dimensional, and has reductive isotropy, we compare our results to those obtained by Petrov. All of the reductive, five-dimensional algebras of Killing fields given by Petrov for Lorentzian metrics are found among the list we generate. Special care must be taken in determining which isometry-isotropy pairs can be realized as the *complete* isometry-isotropy pair of some Lorentzian metric. While we find many algebra-subalgebra pairs that are not among the isometry-isotropy pairs in [**10**] of the appropriate dimension, all such "missing" isometry algebras in fact correspond to metrics that admit more than five isometries (see Chapter 8).

Together with [**2**], the algebraic results presented here make significant progress toward a classification of isometry-isotropy pairs on homogeneous Lorentzian manifolds of dimension four or less. To complete the classification of homogeneous space-times of dimension four or less, isometries of dimension greater than five must be considered. The methods used in this thesis are easily applied to such cases The more difficult problem lies in the case of space-times which do not admit local slices as these do not lend themselves to purely algebraic considerations. Examples of such spaces are known to exist in Petrov's classification ([**10**]), see for instance Example 34.

1.2. Summary of Results

When the isotropy is of dimension two or greater, the classification is straightforward and these cases are briefly discussed in Chapter 3. The bulk of this thesis is the classification of five-dimensional Lie algebras with reductive one-dimensional subalgebras, as pairs. Each isotropy subalgebra is chosen with a particular adjoint action as a starting point. These are labeled "$F8$" for the two-dimensional isotropy or "$F10$" through "$F14$" for the one-dimensional isotropy; this is reflected in the names chosen for the pair designations. The algebra-subalgebra pairs found are summarized in Tables 1.1 through 1.4. The invariants distinguishing each pair are summarized in the diagrams in Figures 1.2 through 1.6, which provide a complete algorithm for determining the pair designation of a given algebra-subalgebra pair of the type considered in this thesis. The algebras given in the tables below are Lie algebras from the classification given by Snobl and Winternitz in [**12**], from which the algebra naming conventions also derive. The structure equations for these algebras can be found in Appendix A.

TABLE 1.1. Summary of classified algebra-subalgebra pairs in the $F11$ (loxodrome) case and the cases of two-dimensional isotropy (spanned by a null rotation and a boost).

Pair Designation	Algebra	Parameters	Isotropy
$(F8,0)$	$\mathfrak{s}_{5,35}$	$a=-1$	e_4,e_5
$(F8,1)$	$\mathfrak{sl}(2,\mathbb{R})\oplus 2\mathfrak{n}_{1,1}$		e_3+e_4, e_2-2e_5
$(F11,0)$	$\mathfrak{s}_{5,11}$	$\alpha=\tan\theta$, $\gamma=0$, $\beta=-\tan\theta$	e_5

TABLE 1.2. Summary of $F12$ (rotation) algebra-subalgebra pairs.

Pair Designation	Algebra	Parameters	Isotropy
$(F12,0)$	$\mathfrak{s}_{3,3}\oplus\mathfrak{s}_{2,1}$	$a=0$	e_3
$(F12,1)$	$\mathfrak{sl}(2,\mathbb{F})\oplus\mathfrak{s}_{2,1}$		e_1-e_3
$(F12,2)$	$\mathfrak{so}(3,\mathbb{R})\oplus\mathfrak{s}_{2,1}$		e_1
$(F12,3)$	$\mathfrak{sl}(2,\mathbb{F})\oplus 2\mathfrak{n}_{1,1}$		e_1-e_3
$(F12,4)$	$\mathfrak{sl}(2,\mathbb{F})\oplus 2\mathfrak{n}_{1,1}$		$e_1-e_3-2e_4$
$(F12,5)$	$\mathfrak{so}(3,\mathbb{R})\oplus 2\mathfrak{n}_{1,1}$		e_1
$(F12,6)$	$\mathfrak{so}(3,\mathbb{R})\oplus 2\mathfrak{n}_{1,1}$		e_1-e_4
$(F12,7)$	$\mathfrak{s}_{3,3}\oplus 2\mathfrak{n}_{1,1}$	$a=0$	e_3
$(F12,8)$	$\mathfrak{s}_{4,7}\oplus\mathfrak{n}_{1,1}$		e_4
$(F12,9)$	$\mathfrak{s}_{5,45}$		e_5
$(F12,10)$	$\mathfrak{s}_{4,12}\oplus\mathfrak{n}_{1,1}$		e_4
$(F12,11)$	$\mathfrak{s}_{5,43}$	$\alpha=0$	e_5

TABLE 1.3. Summary of $F13$ (boost) algebra-subalgebra pairs.

Pair Designation	Algebra	Parameters	Isotropy
$(F13,0)$	$\mathfrak{s}_{3,1}\oplus\mathfrak{s}_{2,1}$	$a=-1$	e_3
$(F13,1)$	$\mathfrak{sl}(2,\mathbb{F})\oplus\mathfrak{s}_{2,1}$		e_2
$(F13,2)$	$\mathfrak{s}_{3,1}\oplus 2\mathfrak{n}_{1,1}$	$a=-1$	e_3
$(F13,3)$	$\mathfrak{s}_{4,6}\oplus\mathfrak{n}_{1,1}$		e_4
$(F13,4)$	$\mathfrak{sl}(2,\mathbb{F})\oplus 2\mathfrak{n}_{1,1}$		e_2
$(F13,5)$	$\mathfrak{sl}(2,\mathbb{F})\oplus 2\mathfrak{n}_{1,1}$		e_2-2e_4
$(F13,6)$	$\mathfrak{s}_{5,44}$		e_5
$(F13,7)$	$2\mathfrak{s}_{2,1}\oplus\mathfrak{n}_{1,1}$		e_2-e_4
$(F13,8)$	$\mathfrak{s}_{5,41}$	$a=b$	e_4-e_5

TABLE 1.4. Summary of $F14$ (null rotation) algebra-subalgebra pairs.

Pair Designation	Algebra	Parameters	Isotropy
$(F14,0)$	$\mathfrak{s}_{5,37}$		e_4

Pair Designation	Algebra	Parameters	Isotropy
$(F14,1)$	$\mathfrak{s}_{5,38}$		e_4
$(F14,2)$	$\mathfrak{sl}(2,\mathbb{F})\oplus 2\mathfrak{n}_{1,1}$		e_3+e_4
$(F14,3)$	$\mathfrak{s}_{4,11}\oplus\mathfrak{n}_{1,1}$		e_5
$(F14,4)$	$\mathfrak{n}_{5,4}$		e_2+e_3
$(F14,5)$	$\mathfrak{s}_{4,1}\oplus\mathfrak{n}_{1,1}$		e_5
$(F14,6)$	$\mathfrak{n}_{5,2}$		e_5
$(F14,7)$	$\mathfrak{s}_{5,20}$		$e_1-e_2-e_3$
$(F14,8)$	$\mathfrak{n}_{5,6}$		e_4
$(F14,9)$	$\mathfrak{s}_{5,14}$		$e_2+e_3+e_4$
$(F14,10)$	$\mathfrak{s}_{5,14}$		e_1-e_3
$(F14,11)$	$\mathfrak{s}_{5,30}$	$a\neq 1$	$e_2+e_3+e_4$
$(F14,12)$	$\mathfrak{s}_{5,30}$	$a\neq 1$	e_2+e_3
$(F14,13)$	$s_{5,32}$		e_2+e_3
$(F14,14)$	$\mathfrak{s}_{5,31}$		e_2+e_3
$(F14,15)$	$\mathfrak{s}_{5,29}$		e_2+e_3
$(F14,16)$	$\mathfrak{s}_{5,30}$	$a=1$	e_2+e_3
$(F14,17)$	$\mathfrak{s}_{4,6}\oplus\mathfrak{n}_{1,1}$		e_2-2e_3
$(F14,18)$	$\mathfrak{s}_{4,6}\oplus\mathfrak{n}_{1,1}$		$e_2-2e_3+2e_5$
$(F14,19)$	$\mathfrak{s}_{4,7}\oplus\mathfrak{n}_{1,1}$		e_3
$(F14,20)$	$\mathfrak{s}_{4,7}\oplus\mathfrak{n}_{1,1}$		e_3-e_5
$(F14,21)$	$\mathfrak{s}_{5,16}$		e_3
$(F14,22)$	$\mathfrak{s}_{5,16}$		e_3+e_4
$(F14,23)$	$\mathfrak{s}_{5,15}$		e_2-e_3
$(F14,24)$	$\mathfrak{s}_{5,15}$		$e_2-e_3+e_4$
$(F14,25)$	$\mathfrak{s}_{4,10}\oplus\mathfrak{n}_{1,1}$		e_3
$(F14,26)$	$\mathfrak{s}_{4,10}\oplus\mathfrak{n}_{1,1}$		e_3+e_5
$(F14,27)$	$\mathfrak{s}_{4,8}\oplus\mathfrak{n}_{1,1}$		$e_2-e_3+e_5$
$(F14,28)$	$\mathfrak{s}_{4,8}\oplus\mathfrak{n}_{1,1}$		e_2-e_3
$(F14,29)$	$\mathfrak{s}_{4,9}\oplus\mathfrak{n}_{1,1}$		e_2
$(F14,30)$	$\mathfrak{s}_{4,9}\oplus\mathfrak{n}_{1,1}$		e_2+e_5
$(F14,31)$	$\mathfrak{s}_{5,19}$	$\alpha\neq 1$	e_3
$(F14,32)$	$\mathfrak{s}_{5,19}$	$\alpha\neq 1$	e_3-e_4
$(F14,33)$	$\mathfrak{s}_{5,17}$	$a\neq 1$	e_2-e_3
$(F14,34)$	$\mathfrak{s}_{5,17}$	$a\neq 1$	$e_2-e_3-e_4$
$(F14,35)$	$\mathfrak{s}_{5,17}$	$a=1$	e_2-e_3
$(F14,36)$	$\mathfrak{s}_{5,18}$		$e_2-e_3-\frac{1}{2}e_4$
$(F14,37)$	$\mathfrak{s}_{5,25}$	$\beta\neq 2\alpha$	e_2+e_4
$(F14,38)$	$\mathfrak{s}_{5,25}$	$\beta\neq 2\alpha$	e_2
$(F14,39)$	$\mathfrak{s}_{5,22}$	$b\neq 1,\ b\neq a+1$	$e_2+e_3+e_4$
$(F14,40)$	$\mathfrak{s}_{5,22}$	$b\neq 1,\ b\neq a+1$	e_2+e_3
$(F14,41)$	$\mathfrak{s}_{5,24}$	$a\neq 1,\ a\neq 2$	e_2+e_4
$(F14,42)$	$\mathfrak{s}_{5,24}$	$a\neq 1,\ a\neq 2$	e_2
$(F14,43)$	$\mathfrak{s}_{5,23}$		e_2+e_3
$(F14,44)$	$\mathfrak{s}_{5,22}$	$b=1$	e_2+e_3
$(F14,45)$	$\mathfrak{s}_{5,21}$		$e_2-e_3+e_4$
$(F14,46)$	$\mathfrak{s}_{5,24}$	$a=1$	e_2-e_3
$(F14,47)$	$\mathfrak{s}_{5,26}$		$e_2+e_3+e_4$
$(F14,48)$	$\mathfrak{s}_{5,26}$		e_2+e_3
$(F14,49)$	$\mathfrak{s}_{5,28}$		e_3-e_4
$(F14,50)$	$\mathfrak{s}_{5,28}$		e_3

Pair Designation	Algebra	Parameters	Isotropy
$(F14, 51)$	$\mathfrak{s}_{5,27}$		$e_2 + e_3 + e_4$
$(F14, 52)$	$\mathfrak{s}_{5,27}$		$e_2 + e_3$
$(F14, 53)$	$\mathfrak{s}_{5,22}$	$b = a + 1$	$e_2 + e_3 + e_4$
$(F14, 54)$	$\mathfrak{s}_{5,22}$	$b = a + 1$	$e_2 + e_3$
$(F14, 55)$	$\mathfrak{s}_{5,25}$	$\beta = 2\alpha$	$e_3 - e_4$
$(F14, 56)$	$\mathfrak{s}_{5,25}$	$\beta = 2\alpha$	e_3
$(F14, 57)$	$\mathfrak{s}_{5,24}$	$a = 2$	$e_2 + e_3 + e_4$
$(F14, 58)$	$\mathfrak{s}_{5,24}$	$a = 2$	$e_2 + e_3$

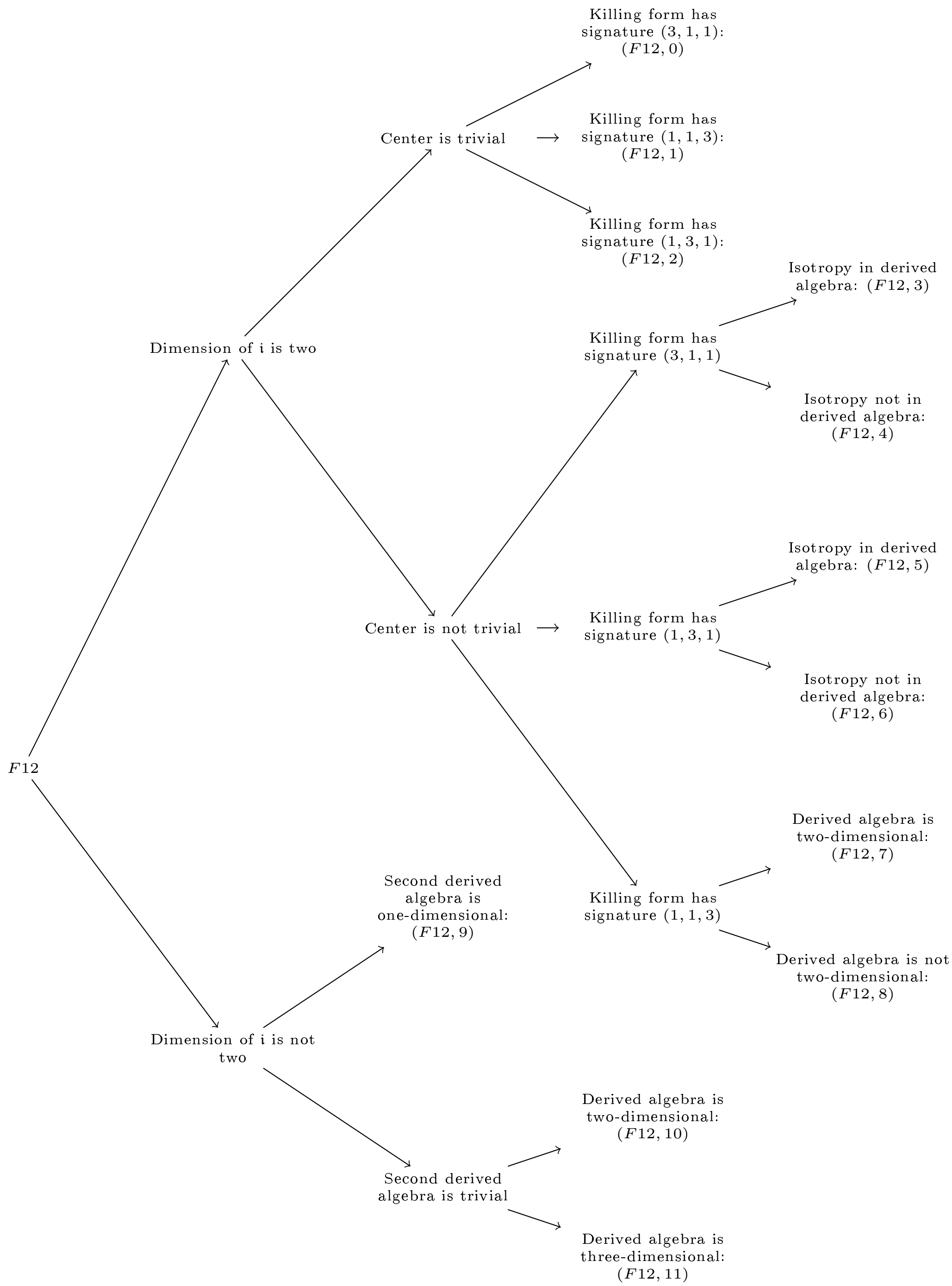

FIGURE 1.2. Summary of F12 (rotation) invariants and case-splitting.

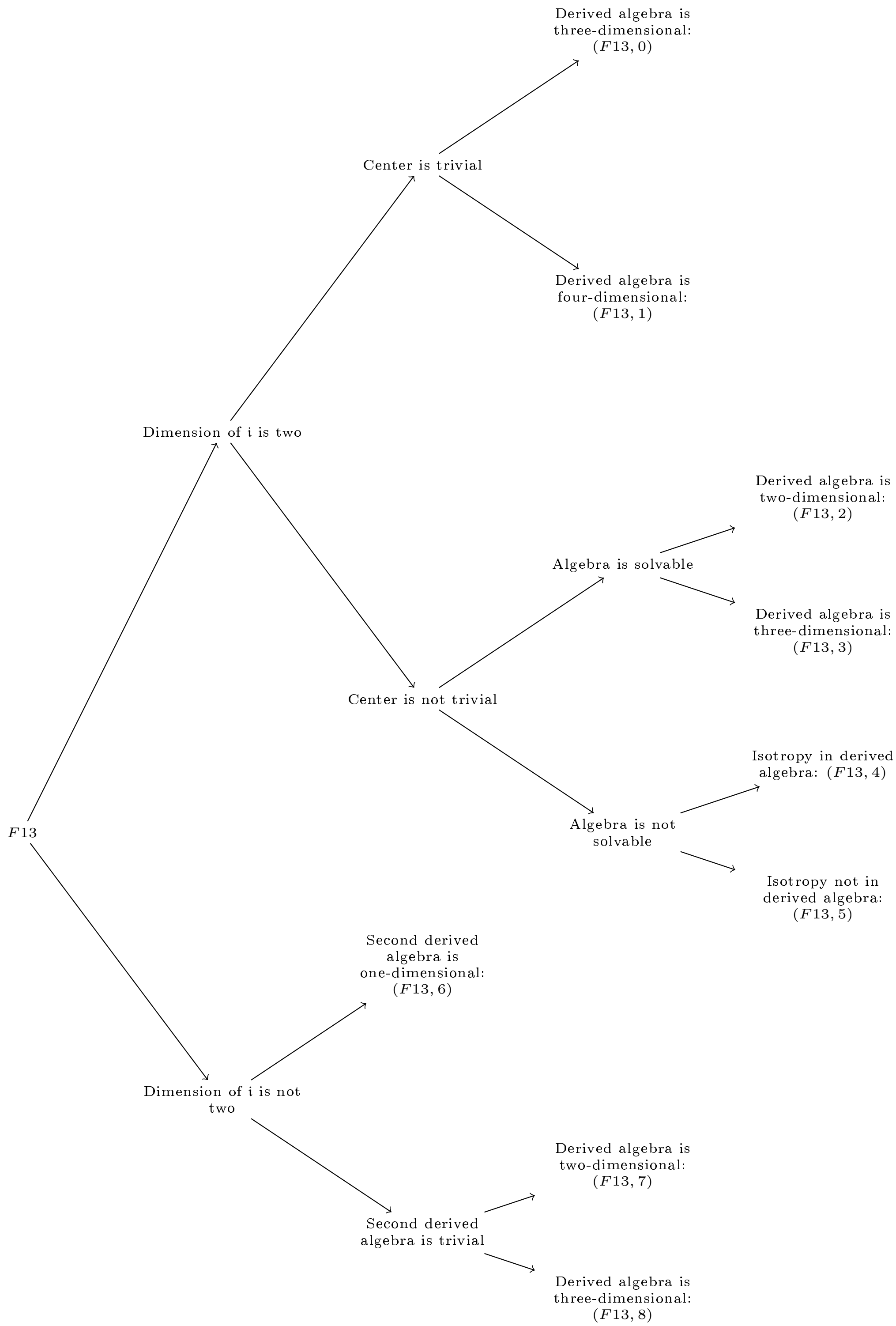

FIGURE 1.3. Summary of F13 (boost) invariants and case-splitting.

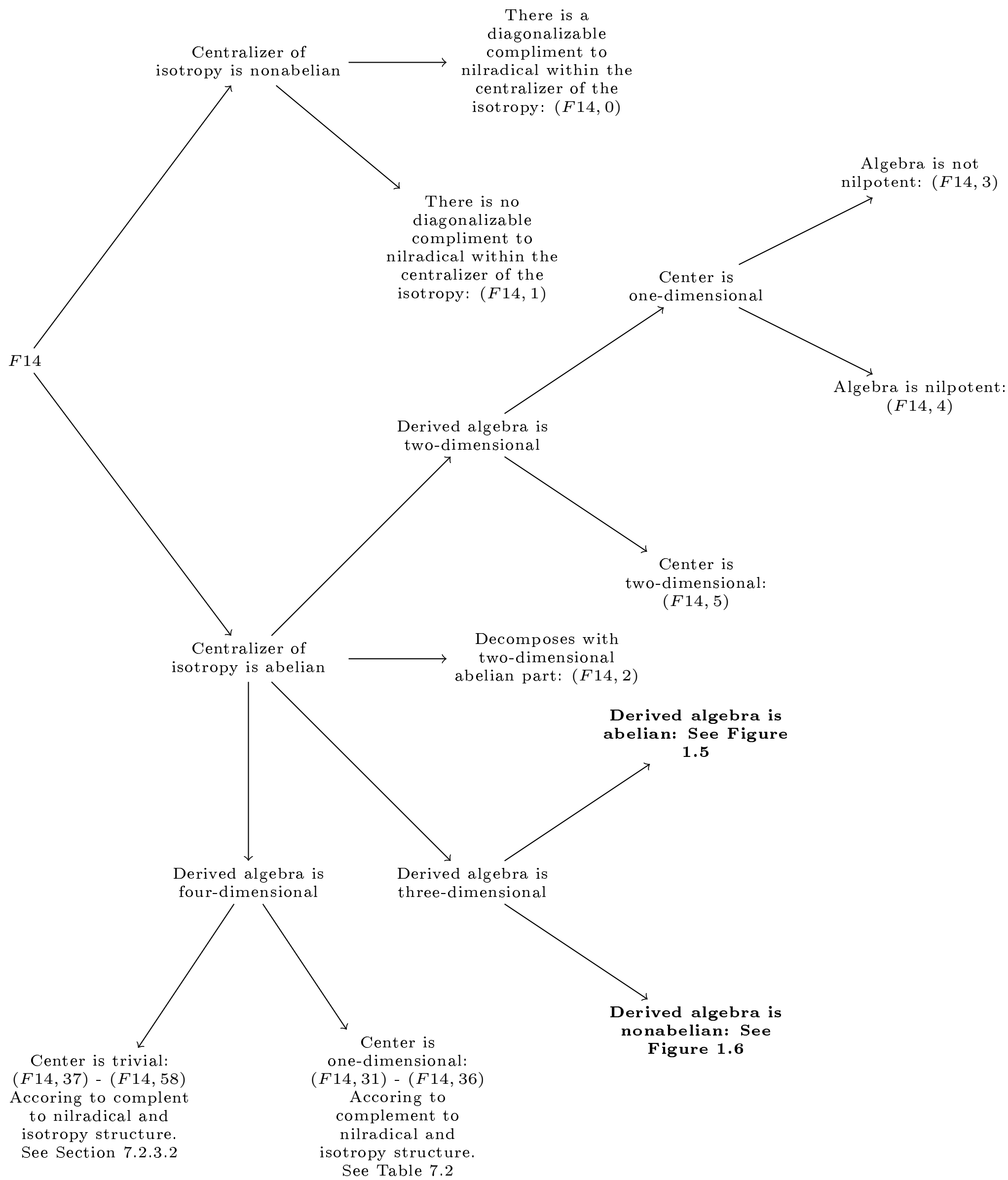

Figure 1.4. Summary of F14 (null rotation) invariants and case-splitting.

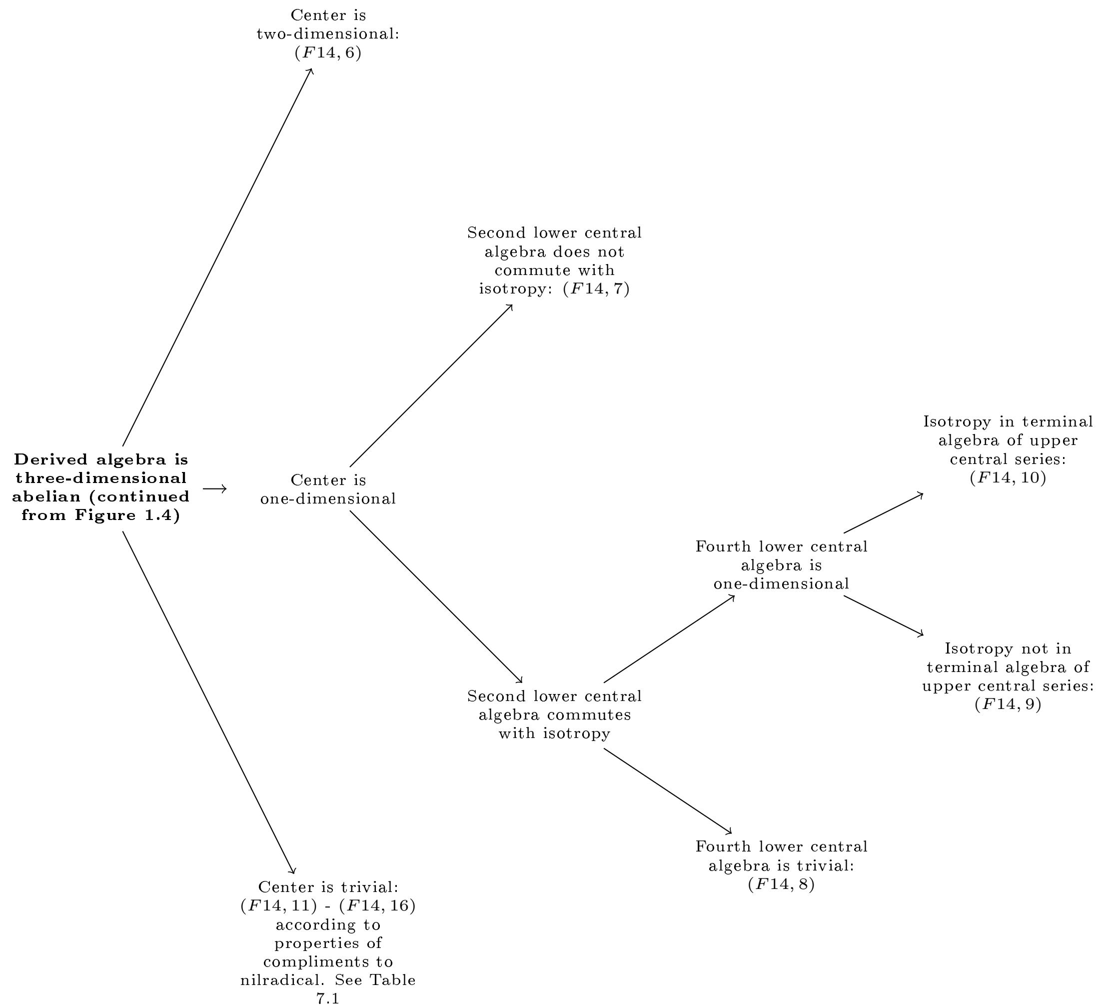

FIGURE 1.5. Summary of F14 (null rotation) invariants and case-splitting. Continued from Figure 1.4.

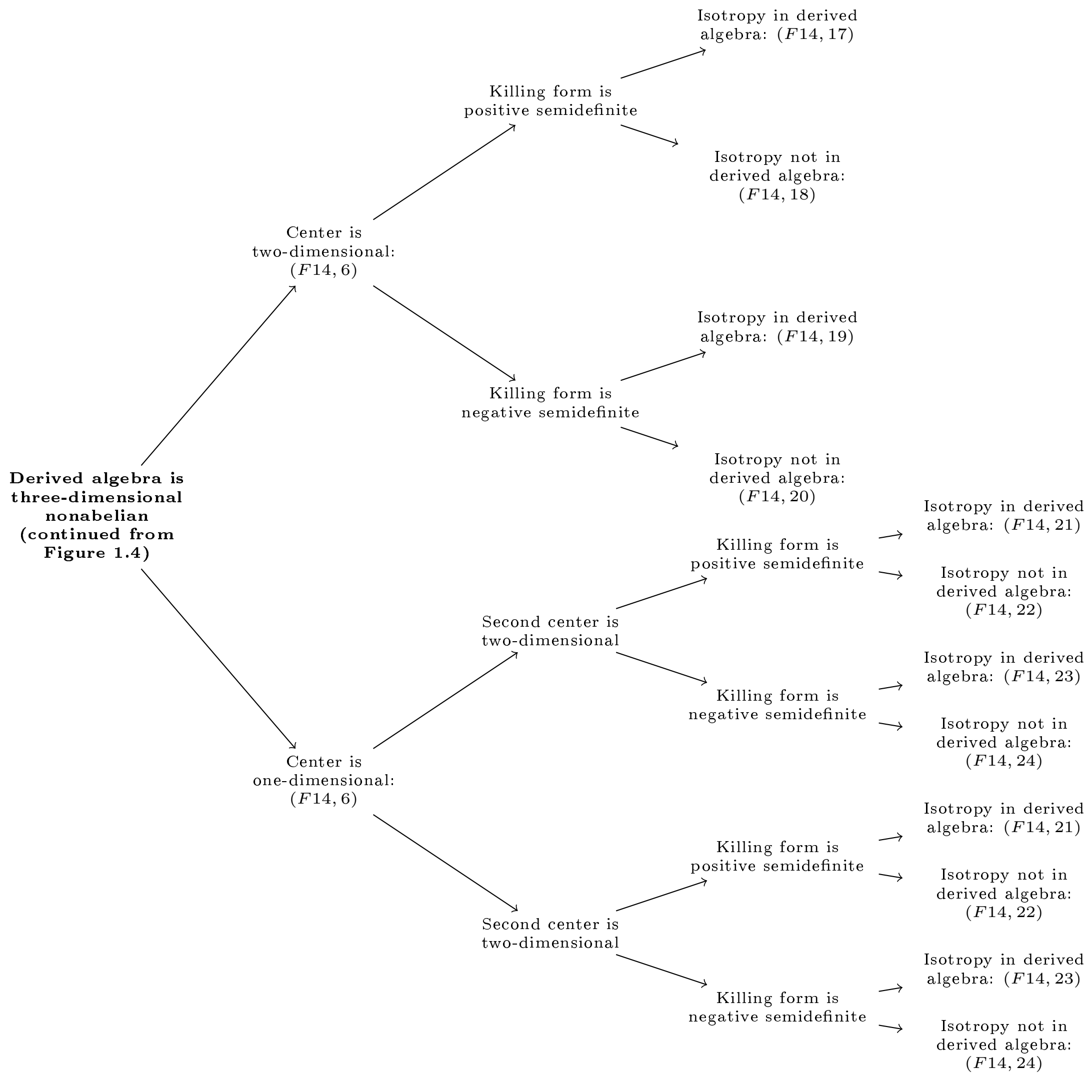

FIGURE 1.6. Summary of F14 (null rotation) invariants and case-splitting. Continued from Figure 1.4.

1.3. Organization Overview

Chapter two outlines the fundamental principles of pertinence to the work with an emphasis on group actions and isometry in Lorentzian space-times. In addition to the introductory principles, it contains an overview of the applicability of the results and a summary of the so-called Schmidt method used to generate them. The Lie algebra classification system used throughout is also discussed.

The classification begins in the third chapter with the case of five-dimensional isometry with isotropy that is not one-dimensional; with the exception of trivial isotropy, only two such cases exist. The next four chapters are organized according to which subalgebra of the Lorentz algebra the isotropy belongs. Chapter four gives the classification of type $F11$ isotropies, or loxodromes. Chapter five classifies rotational isotropy, type $F12$. In chapter six, type $F13$ isotropies, or boosts are classified. Finally, in chapter seven, type $F14$ isotropies, or null rotations, are classified.

Following the classification, the application this work to the study of homogeneous space-times is explored. Specifically, the relationship between this work and the classification of homogeneous Lorentzian space-times in [**10**] is shown explicitly. At the algebraic level, we find exact agreement between Petrov's approach and the approach used here.

The appendices include Lie multiplication tables for the algebras generated, Maple worksheets that follow the classification and basis alignment given in this work, and Maple source for a database of the algebra-subalgebra pairs generated.

CHAPTER 2

PRELIMINARIES

In this preliminary chapter, we give definitions and theorems that are of pertinence to the work. This chapter gives an introduction to the foundational concepts of manifolds and group actions, Lie algebras, and pseudo-Riemannian manifolds. These topics are covered at an introductory level and a more detailed exposition can be found in any introductory differential geometry text, such as [**1**]. Snobl and Winternitz also provide introduction to many Lie theoretic concepts in [**12**]. The Schmidt method, first outlined in [**11**], and is introduced in this chapter and is used later in this thesis to classify algebra-subalgebra pairs.

2.1. Manifolds

We begin with a brief overview of manifolds such as can be found in any introductory text on differential geometry. Of particular importance to this thesis are vector fields and their flows, so we define these and some related concepts now.

Definition 1. A *vector* at a point p in an n-dimensional manifold $\mathcal{M}$ is a derivation on smooth real-valued functions on $\mathcal{M}$. The set of all such vectors at p forms T_p, the *tangent space* at p. A *vector field* is a smooth section of the tangent bundle, denoted $T\mathcal{M}$. Let $\mathfrak{X}(\mathcal{M})$ denote the space of all vector fields on $\mathcal{M}$. The *Lie bracket* of two vector fields X and Y is written $[X, Y]$ and given by $[X, Y](f) = X(Y(f)) - Y(X(f))$ for all $f \in C^{\infty}(\mathcal{M})$.

Definition 2. Let $\phi : \mathcal{M} \to \mathcal{N}$ be a smooth map. The *pushforward of* ϕ *at* x_0 is the map $\phi_* : T_{x_0}\mathcal{M} \to T_{\phi(x_0)}\mathcal{N}$ given by $\phi_*(X)(f) = X(f \circ \phi)$ for all $f \in C^{\infty}(\mathcal{N})$.

Definition 3. An *integral curve* of the vector field X on a manifold $\mathcal{M}$ is a smooth map $\alpha : J \to \mathcal{M}$, where J is an open interval of $\mathbb{R}$, such that $\alpha'(t) = X_{\alpha(t)}$ for all $t \in J$. The integral curve may be specified uniquely via the initial condition $\alpha(0) = p$.

DEFINITION 4. The *flow* of a vector field X on a manifold $\mathcal{M}$ is the one-parameter family of diffeomorphisms $\phi_t : \mathcal{M} \to \mathcal{M}$ with $t \in (-\epsilon, \epsilon)$ such that $\phi_t'(p) = X_{\phi_t(p)}$ for any $p \in \mathcal{M}$ and $\phi_t \circ \phi_s = \phi_{t+s}$ for $t, s, t+s \in (-\epsilon, \epsilon)$.

DEFINITION 5. The *Lie derivative* of a tensor field T along a vector field X (denoted $\mathcal{L}_X T$) is defined via $(\mathcal{L}_X T)_p = \frac{d}{dt}\big|_{t=0} (\phi_t^* T)_p$ where ϕ_t is the flow of X. The Lie derivative measures the rate of change of T along the integral curve of X. It follows from the definition that the Lie derivative $\mathcal{L}_X$ . . .

(1) of a real scalar function f is $X(f)$.
(2) commutes with the exterior derivative (i.e., $\mathcal{L}_X dT = d(\mathcal{L}_X T)$).
(3) is Leibniz with respect to contraction and tensor product.
(4) of a vector field Y is $[X, Y]$.

2.2. Group Actions on Manifolds

Since the focus of this thesis is on isometry and isotropy, an overview of group actions is appropriate. We begin with the definition of a group action.

DEFINITION 6. A (left) *group action* of a group G on a manifold $\mathcal{M}$ is a map $\mu : G \times \mathcal{M} \to \mathcal{M}$ such that $\mu(g, \mu(h, x)) = \mu(gh, x)$ and $\mu(e, x) = x$ where e is the identity element of G. The map $\mu_g : \mathcal{M} \to \mathcal{M}$ is given by $\mu_g(x) = \mu(g, x)$. Given a group action $\mu : G \times \mathcal{M} \to \mathcal{M}$, we say G acts on $\mathcal{M}$ by μ, written $G \circlearrowleft \mathcal{M}$.

Orbits, isotropy, and transitive group actions are of particular interest in this work.

DEFINITION 7. Let G act on $\mathcal{M}$ by μ. The *orbit* of a point $x \in \mathcal{M}$ is the image of μ restricted to x, i.e., $O_G(x) = \{\mu(g, x) : g \in G\}$.

DEFINITION 8. Let G act on $\mathcal{M}$ by μ. The *isotropy* G_x of a point $x \in \mathcal{M}$ is the subgroup of G that fixes x under μ, i.e., $G_x = \{g \in G : \mu(g, x) = x\}$.

DEFINITION 9. Let G act on $\mathcal{M}$ by μ. The *linear isotropy representation* of G_x at $x \in \mathcal{M}$ is the group homomorphism $Is_x : G_x \to GL(T_x\mathcal{M})$ given by $Is_x(g)(X) = \mu_{g*}(X)$. The representation is call *faithful* if $Is_x(g)(X) = X$ implies g is the identity in G.

EXAMPLE 10. Consider the orthogonal group $SO(n+1)$ acting on the n-sphere in $\mathbb{R}^{n+1}$. The isotropy at a point p is the set of all rotations about the ray from the origin through p and thus is diffeomorphic to $SO(n)$. If $n = 2$ and p is on the z-axis, then this isotropy at p is $SO(3)_p = \left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}\right\}$. The linear isotropy representation is thus given by $Is_p\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, which acts on vectors in the plane tangent to the north pole of the sphere. Since the linear isotropy representation takes the same form as the group transformation, it is easy to see that the representation is faithful.

DEFINITION 11. Let G act on $\mathcal{M}$ by μ. If for any $x, y \in \mathcal{M}$, there is $g \in G$ such that $\mu(g, x) = y$ (or, equivalently, $O_G(x) = \mathcal{M}$), the μ is called *transitive* and $\mathcal{M}$ is *homogeneous* under the action of G by μ.

EXAMPLE 12. A manifold $\mathcal{M}$ may be homogeneous under the action of more than one group. Consider the 3-sphere, $\mathcal{S}^3$. If $\mathcal{S}^3$ is though of as a subset of $\mathbb{R}^4$, then under the usual action of the orthogonal group $O(4)$, $\mathcal{S}^3$ is homogeneous. If $\mathcal{S}^3$ is thought of as a subset of $\mathbb{C}^2$, then under the usual action of the unitary group, $U(2)$, $\mathcal{S}^3$ is homogeneous.

We now discuss Lie groups and their actions on manifolds. We first define Lie groups, then cite a well-known and important theorem regarding the geometric structure of Lie subgroups.

DEFINITION 13. A *Lie group* G is a group that is also a differentiable manifold on which group multiplication and multiplication composed with inversion are smooth functions from the product manifold $G \times G$ to G. The *left invariant vector fields* on G are the vector fields $X \in \mathfrak{X}(G)$ such that for any $g, h \in G$, $l_{g*}X_h = X_{gh}$ where l_g is the map given by left multiplication by g. Note that the left-invariant vector fields are uniquely specified by their value at the identity element e of G: $X_g = l_{g*}X_e$.

DEFINITION 14. A *Quotient* G/H of a Lie group G by a Lie subgroup H is the set of cosets $G/H = \{gH : g \in G\}$ where two cosets g_1H and g_2H are equal if there is $h \in H$ such that $g_1 h = g_2$.

THEOREM 15. *(Closed Subgroup Theorem): Let G be a Lie group and H be subgroup of G closed under the subspace topology. Then H is an embedded Lie subgroup of G.*

COROLLARY 16. *Let G be a Lie group with H a topologically closed subgroup of G. Then the natural projection map $\pi : G \to G/H$ induces a manifold structure on G/H and is smooth. For each $gH \in G/H$ there is a neighborhood $\mathcal{U}$ of gH and a smooth local cross section $\sigma : \mathcal{U} \to G$ exists such that $\pi \circ \sigma$ is the identity on G/H. Furthermore, G/H is homogeneous under the natural action of G.*

We now give a simple application of this corollary as an example.

EXAMPLE 17. The Grassmannian manifold $Gr\,(n, r)$ is by definition the manifold of r-subspaces of an n-dimensional vector space. Since any r-plane can be rotated into any other, $O\,(n)$ acts transitively on $Gr\,(n, r)$. The isotropy of the plane spanned by the first r coordinate vectors is given by $O\,(r)$ acting in the plane and $O\,(n - r)$ acting in the complement space. Therefore the Grassmannian can be written $Gr\,(n, r) = O\,(n)\,/\,(O\,(r) \times O\,(n - r))$. Since $O\,(r)$ and $O\,(n - r)$ are topologically closed, Corollary 16 ensures that $Gr\,(n, r)$ is indeed a homogeneous space under the action of $O\,(n)$.

Of particular pertinence to this thesis is the structure of G/H when G is a Lie group acting on a manifold and H is the isotropy at a point. To aid in the study of G/H, we give the following theorem.

THEOREM 18. *If G is a Lie group acting smoothly on a manifold $\mathcal{M}$ via μ, then the isotropy G_{x_0} at an arbitrary and fixed $x_0 \in \mathcal{M}$ is topologically closed in G.*

PROOF. For arbitrary and fixed $x_0 \in \mathcal{M}$, consider the smooth (in particular, continuous) map $\mu_{x_0} : G \to \mathcal{M}$ given by $\mu_{x_0}(g) = \mu(g, x_0)$. The isotropy group G_{x_0} is given by $\mu_{x_0}^{-1}(x_0)$. Since G_{x_0} is the inverse image of a point, it is topologically closed. $\square$

We now define equivariance then cite the Fundamental Theorem of Homogeneous Spaces.

DEFINITION 19. If G acts on manifolds $\mathcal{M}$ and $\mathcal{N}$ by μ and ν respectively, a map $\phi : \mathcal{M} \to \mathcal{N}$ is *G-equivariant* if $\phi(\mu(g, x)) = \nu(g, \phi(x))$.

THEOREM 20. *(The Fundamental Theorem of Homogeneous Spaces): Let the Lie group G act smoothly on $\mathcal{M}$ by a transitive group action μ and let G act on G/G_x by group multiplication where $x \in \mathcal{M}$. Then there exists a G-equivariant diffeomorphism between $\mathcal{M}$ and G/G_x.*

The Fundamental Theorem of Homogeneous Spaces is given with proof in introductory texts (e.g., Theorem 9.3 of Section IV in [**1**]), but may be extended when the isotropy structure is in some sense independent of which point in the manifold is taken as reference. To make this precise, we define a slice of the manifold at a point.

DEFINITION 21. Let G be a group acting smoothly on a manifold $\mathcal{M}$ via μ. A *local cross-section*, $\mathcal{S}$, is a sub-manifold of $\mathcal{M}$ such that for all $x \in \mathcal{S}$ the equality $T_x O_G(x) \oplus T_x\mathcal{S} = T_x\mathcal{M}$ holds. If for any fixed $x_0 \in \mathcal{S}$ there is a smooth function $\gamma : \mathcal{S} \to G$ such that $\mu(\gamma(x_0), x_0) = x_0$ and $G_{x_0} = \gamma(y) G_y (\gamma(y))^{-1}$ for all $y \in \mathcal{S}$, then $\mathcal{S}$ is called a *local slice* and $\mathcal{M}$ is called a *simple G space*. If the image of γ is a subset of G_{x_0} (i.e., $G_{x_0} = G_y$), then $\mathcal{S}$ is called *isotropy preserving*.

REMARK 22. If a group G acting smoothly on a manifold $\mathcal{M}$ via μ admits a local slice, $\mathcal{S}$, then $\mathcal{M}$ admits a local isotropy preserving slice $\mathcal{S}'$ through an arbitrary point $x_0 \in \mathcal{S}$.

PROOF. Fix $x_0 \in \mathcal{S}$ and let $\gamma : \mathcal{S} \to G$ be as given in Definition 21. The isotropy at any $y \in \mathcal{S}$ is of the form $G_y = (\gamma(y))^{-1} G_{x_0} \gamma(y)$. Since μ and γ are smooth, $\mathcal{S}' = \{\mu(\gamma(s), s) : s \in \mathcal{S}\}$ is a local slice with isotropy at each point given by G_{x_0}. □

We now give an example of a slice in the familiar case of rotations in $\mathbb{R}^3$.

EXAMPLE 23. Let $G = SO(3)$ act on $\mathcal{M} = \mathbb{R}^3 \setminus \{0\}$ in the standard way via matrix multiplication. The isotropy at a point x_0 is given by $G_{x_0} = \{A \in G : Ax_0 = x_0\}$. Multiplying by a scalar λ, we see that the isotropy at λx_0 is also G_{x_0}. Furthermore, at any point x_0, the orbit under G is the sphere of radius $||x_0||$ which has its normal direction aligned with the ray given by λx_0. Therefore $\mathcal{S} = \{\lambda x_0 : \lambda \in (0,\infty)\}$ is an isotropy-preserving slice through x_0.

The following example demonstrates that not all group actions yield local slices.

EXAMPLE 24. Let G be the matrix group given by $\left\{\begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : a,b \in \mathbb{R}\right\}$ and let it act on the $z = 1$ plane by matrix multiplication. At a point $(x, y, 1)$, the isotropy is given by $G_{(x,y)} = \left\{\begin{pmatrix} 1 & a & -ay \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in G\right\}$. Note that $G_{(x_1,y_1)} \neq G_{(x_2,y_2)}$ whenever $y_1 \neq y_2$ (i.e., when the orbits are not equal). Thus there is no isotropy preserving local slice at any point in the $z = 1$ plane and by Remark 22, there is no local slice anywhere. This can also be shown by explicitly calculating $gG_{(x,y)}g^{-1}$ for arbitrary g.

THEOREM 25. *(The Fundamental Theorem of Simple G Spaces): Let the Lie group G act smoothly on a manifold $\mathcal{M}$. Suppose there is a local slice $\mathcal{S}$ such that the isotropy at each point in $\mathcal{S}$ is given by the subgroup H. For any fixed $x_0 \in \mathcal{S}$, there exists a local G-equivariant diffeomorphism between a neighborhood $\mathcal{U}$ of $\mathcal{M}$ containing x_0 and $(S \cap \mathcal{U}) \times G/H$.*

PROOF. Since $x_0 \in \mathcal{S}$, the isotropy group at x_0 is $G_{x_0} = H$. Define $\phi : \mathcal{S} \times G/H \to \mathcal{M}$ by $\phi(s, gH) = gs$. Since the isotropies at all points $s \in \mathcal{S}$ are equal to H, the map ϕ is well-defined. By Corollary 16, for each g there is a neighborhood $\mathcal{V}$ of g and smooth local cross section$\sigma : \mathcal{V} \to G$ such that $\phi(s, gH) = \sigma(gH)s$. Thus, because the group action is assumed smooth, ϕ is smooth. Note that for any $h \in G$, $h\phi(s, gH) = hgs = \phi(s, hgH)$, so ϕ is G-equivariant when G acts on $\mathcal{S} \times \mathcal{V}$ in the natural way (by group multiplication in $\mathcal{V} \subset G/H$). It suffices to use the fact that $T_{x_0}O_G(x) \cap T_{x_0}\mathcal{S}$ is trivial for $x_0 \in \mathcal{S}$ and apply the inverse function theorem to show that the restriction of ϕ to$(\mathcal{S} \cap \mathcal{U}) \times G/H$ is a local G-equivariant diffeomorphism. □

This theorem leads to an important result regarding the point-independence of isotropy in manifolds with group actions admitting slices.

COROLLARY 26. *If a manifold $\mathcal{M}$ admit a local slice $\mathcal{S}$ through $x_0 \in \mathcal{M}$ under the smooth action of the group G, and the isotropy at each point in $\mathcal{S}$ is given by the subgroup H. Then there is a neighborhood $\mathcal{U}$ of x_0 such that the isotropy at any point in $\mathcal{U}$ is conjugate to H.*

PROOF. Choose $u \in \mathcal{U}$ and recall $\phi : (\mathcal{S} \cap \mathcal{U}) \times G/H \to \mathcal{U}$ from the previous theorem. Since u is in the image of ϕ and ϕ is invertible, there is $(s, gH) \in (\mathcal{S} \cap \mathcal{U}) \times G/H$ such that $u = gs$ and thus $G_u = gHg^{-1}$. □

2.3. Infinitesimal Group Actions

Following the preceding section, we now treat infinitesimal group actions, beginning with a few fundamental definitions.

DEFINITION 27. An *infinitesimal group action* Γ on a manifold $\mathcal{M}$ is a real finite-dimensional vector space of vector fields on $\mathcal{M}$ that is closed under the Lie bracket.

DEFINITION 28. Let Γ be an infinitesimal group action on $\mathcal{M}$. The *isotropy* of Γ at a point $x \in \mathcal{M}$ is the subalgebra Γ_x of Γ that vanishes at x.

DEFINITION 29. Let Γ be an infinitesimal group action on $\mathcal{M}$. The *linear isotropy representation* of Γ_x at $x \in \mathcal{M}$ is the map $Is_x : \Gamma_x \to \mathfrak{gl}(T_x\mathcal{M})$ given by $Is_x(X)(Y) = [X, Y]_x$. Note that because X vanishes at x, the derivatives of Y at x need not be known. The representation is called *faithful* if $Is_x(X)(Y) = 0$ implies X is the zero vector field.

EXAMPLE 30. The conformal algebra for $\mathbb{R}^3$ with the Minkowski metric is spanned by the following [**7**]:

$$\partial_x, \qquad \partial_y, \qquad \partial_t,$$

$$\boldsymbol{r}_{xy} = -y\partial_x + x\partial_y, \qquad \boldsymbol{r}_{xt} = t\partial_x + x\partial_t \qquad \boldsymbol{r}_{yt} = t\partial_y + y\partial_t$$

$$\partial_u, \quad \boldsymbol{d} = x\partial_x + y\partial_y + t\partial_t, \quad \boldsymbol{v}_\alpha = \alpha(x, y, t)\,\partial_u$$

$$\begin{aligned} \boldsymbol{i}_x &= \left(x^2 - y^2 + t^2\right)\partial_x + 2xy\partial_y + 2xt\partial_t - xu\partial_u \\ \boldsymbol{i}_y &= 2xy\partial_x + \left(y^2 - x^2 + t^2\right)\partial_y + 2yt\partial_t - yu\partial_u \\ \boldsymbol{i}_t &= 2xt\partial_x + 2yt\partial_y + \left(x^2 + y^2 + t^2\right)\partial_t - tu\partial_u \end{aligned}$$

where $\alpha(x, y, t)$ is an arbitrary solution to the wave equation in two spatial and one time dimension. Since the algebra includes all four translations, it is transitive with isotropy spanned by

$\{r_{xy}, r_{xt}, r_{yt}, \boldsymbol{i}_x, \boldsymbol{i}_y, \boldsymbol{i}_t, \boldsymbol{d}\}$. These inversions $\boldsymbol{i}_\mu$ for $\mu \in \{x, y, t\}$ are quadratic in all components. Therefore, at the origin $\mathscr{O}$, their Lie brackets with any other vectors evaluate to zero and thus $Is_{\mathscr{O}}(\boldsymbol{i}_\mu) = \mathbf{0}$ is identically zero. Therefore, the linear isotropy representation is not faithful for this infinitesimal group action.

Definition 31. Let Γ be an infinitesimal group action on $\mathcal{M}$. If at each point $x \in \mathcal{M}$, Γ evaluated at x spans $T_x\mathcal{M}$, then the infinitesimal group action is *transitive* and $\mathcal{M}$ is called homogeneous under the infinitesimal action of Γ.

There is close correspondence between group actions and their infinitesimal counterparts. The following result summarizes this correspondence.

Remark 32. Every smooth group action of a Lie group G by μ on a manifold $\mathcal{M}$ generates an infinitesimal group action as follows. Let X be a left-invariant vector field on G. Then X has an integral curve $\phi : (-\epsilon, \epsilon) \to G$ with $\phi(0)$ equal to the identity in G. The curve $\mu(\phi, x_0)$ has a tangent vector at x_0, Y_{x_0}. Varying x_0 now produces a vector field Y on $\mathcal{M}$. This procedure generates a map $\nu : T_eG \to \mathfrak{X}(\mathcal{M})$ whose image is an infinitesimal group action Γ. If G_{x_0} is the isotropy subgroup of G at x_0 and $X_e \in T_eG_{x_0}$, then the image of ϕ is a subset of G_{x_0}. Thus $\mu(\phi, x_0) = x_0$ and the tangent vector here is the zero vector. Therefore $\nu(T_eG_{x_0}) = \Gamma_{x_0}$.

Example 33. Consider the Special Euclidean group $SE(2)$ acting on the plane with coordinates $(x, y, 1)$. The group action is given by matrix multiplication by $\mu_{(a,b,\theta)} = \begin{pmatrix} \cos\theta & \sin\theta & a \\ -\sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{pmatrix}$. Then the pushforward of left multiplication by (a, b, θ) is given by $(a, b, \theta)_* = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Thus the left-invariant vector fields are spanned by $X_1 = \cos\theta\partial_a - \sin\theta\partial_b$, $X_2 = \sin\theta\partial_a + \cos\theta\partial_b$, and $X_3 = \partial_\theta$. We now find the integral curves with initial position at the identity. For X_1, we have the initial value problem

$$\begin{aligned} \cos\theta(t) &= a'(t) \\ -\sin\theta(t) &= b'(t) \\ 0 &= \theta'(t) \\ (a, b, \theta)(0) &= (0, 0, 0), \end{aligned}$$

which has solution $(a,b,\theta)(t)=(t,0,0)$. For X_2, we have the initial value problem

$$\begin{aligned} \sin\theta(t) &= a'(t) \\ \cos\theta(t) &= b'(t) \\ 0 &= \theta'(t) \\ (a,b,\theta)(0) &= (0,0,0), \end{aligned}$$

which has solution $(a,b,\theta)(t)=(0,t,0)$. For X_3, we have the initial value problem

$$\begin{aligned} 0 &= a'(t) \\ 0 &= b'(t) \\ 1 &= \theta'(t) \\ (a,b,\theta)(0) &= (0,0,0), \end{aligned}$$

which has solution$(a,b,\theta)(t)=(0,0,t)$. Applying these three curves to a point (x_0,y_0) in the plane via the group action gives curves $(x,y)_1(t)=(t,0)$, $(x,y)_2(t)=(0,t)$, and $(x,y)_3(t)=(x_0\cos t+y_0\sin t,-x_0\sin t+y_0\cos t)$. The tangent vectors at (x_0,y_0) are given by $(1,0)$, $(0,1)$, and $(y_0,-x_0)$. Thus the translations are infinitesimally generated by the vector fields ∂_x and ∂_y, while the rotation is generated by $y\partial_x-x\partial_y$.

In light of this correspondence, local isotropy preserving slices may be studied in infinitesimal terms, as in the following example.

Example 34. Consider the special case of the metric in (32.26) in **[10]** , which has Killing vectors (i.e., infinitesimal group action; see Definition 56) given by

$$X_i=\partial_{x^i}\ (i=1,2,3)\quad X_4=x^2\partial_{x^1}+\omega\left(x^4\right)\partial_{x^2}+\lambda\left(x^4\right)\partial_{x^3}$$

with ω and λ not identically zero. The only non-zero Lie bracket is $[X_2,X_4]=X_1$, and the isotropy at (a_0,b_0,c_0,d_0) is spanned by $h=b_0X_1+\omega(d_0)X_2+\lambda(d_0)X_3-X_4$. The adjoint of a generic vector χ with X_2 component α and X_4 component β is

$$ad(\chi)=\begin{pmatrix} 0 & -\beta & 0 & \alpha \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and its exponential is

$$Ad\,(\chi)=\begin{pmatrix}1 & -\beta & 0 & a\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

Thus $Ad\,(\chi)\,(h)$ differs from h only in the X_1 component. Therefore, there is no change of basis such that h is independent of x^4. This implies that the action of isometries does not admit a local slice. [1]

2.4. Lie Algebras

In this section, we give a review of Lie algebras. For a more in-depth exposition, see [**12**]. We begin with the definitions of Lie groups and Lie algebras and the relationship between them.

Definition 35. A (real) *Lie algebra* $\mathfrak{g}$ is a (real) vector space endowed with an anti-symmetric bilinear product $[\cdot,\cdot]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ that obeys the Jacobi identity: $[x,[y,z]]+[z,[x,y]]+[y,[z,x]]=0$ for all $x,y,z\in\mathfrak{g}$. The product $[\cdot,\cdot]$ is the Lie bracket and the structure constants ${C_{ij}}^k$ for a basis $\{e_i\}$ are given by $[e_i,e_j]={C_{ij}}^k e_k$.

Remark 36. As a vector space, a Lie algebra admits a canonical dual space. If, in a given basis $\{e_i\}$, the Lie algebra has structure constants ${C_{ij}}^k$, then the dual basis, $\{\omega^i\}$ (subject to $\omega^i\,(e_j)={\delta^i}_j$), obeys $d\omega^k=-\frac{1}{2}{C_{ij}}^k\omega^i\wedge\omega^j$ and the Jacobi identity becomes $d^2\equiv 0$, where d is the exterior derivative.

Remark 37. Every finite-dimensional Lie group G has a corresponding Lie algebra $\mathfrak{g}$ given by the left-invariant vector fields on G together with vector field commutation as the Lie bracket. Similarly, every finite dimensional Lie algebra $\mathfrak{g}$ has a corresponding simply-connected Lie group G whose left-invariant vector fields give $\mathfrak{g}$.

Now we give some elementary definitions from the study of Lie algebras. Many, but not all, will find application in the algebra-subalgebra classification given in this thesis. Those that are not used are included here for the sake of completeness.

[1]Other such examples of isometry groups in [**10**] that do not admit slices include equations (30.8), (33.1), and (33.54).

DEFINITION 38. The *adjoint* $ad(x)$ of a vector x in a Lie algebra $\mathfrak{g}$ is the linear map $ad(x) : \mathfrak{g} \to \mathfrak{g}$ given by $ad(x)(y) = [x, y]$.

DEFINITION 39. The *Killing form* K on a Lie algebra $\mathfrak{g}$ is the symmetric bilinear form given by $K(x, y) = \mathrm{tr}(ad(x)\, ad(y))$.

DEFINITION 40. A *Lie subalgebra* $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a vector subspace of $\mathfrak{g}$ such that the Lie bracket on $\mathfrak{h}$ is the restriction of the Lie bracket on $\mathfrak{g}$ to $\mathfrak{h}$. A Lie algebra-subalgebra pair is an ordered pair $(\mathfrak{g}, \mathfrak{h})$ such that $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$. The subalgebra $\mathfrak{h}$ is called . . .

(1) *reductive* if there is a vector space complement $\mathfrak{m}$ to $\mathfrak{h}$ such that $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$, where $\oplus$ is the vector space direct sum, and $[\mathfrak{m}, \mathfrak{h}] \subset \mathfrak{m}$. In this case, $\mathfrak{m}$ is called a *reductive complement.*
(2) *symmetric* if there is a reductive complement $\mathfrak{m}$ to $\mathfrak{h}$ such that $[\mathfrak{m}, \mathfrak{m}] \in \mathfrak{h}$. In this case, $\mathfrak{m}$ is called a *symmetric complement.*

DEFINITION 41. An *ideal* $\mathfrak{i}$ in a Lie algebra $\mathfrak{g}$ is a subalgebra such that $[\mathfrak{i}, \mathfrak{g}] \subseteq \mathfrak{i}$. A Lie algebra with only the trivial ideals $\{0\}$ and the algebra itself is called simple.

EXAMPLE 42. Consider the example of the two-dimensional nonabelian Lie algebra with $[e_1, e_2] = e_2$. Let $\mathfrak{h}$ be spanned by e_1. Then let $\mathfrak{m}_1$ be spanned by e_2 and $\mathfrak{m}_2$ be spanned by $e_1 + e_2$. Both $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are vector space complements to $\mathfrak{h}$, but only $\mathfrak{m}_1$ is a reductive complement. Furthermore, since $\mathfrak{m}_1$ is abelian, it is also a symmetric complement. Therefore $\mathfrak{h}$ is a symmetric subalgebra (this implies also that $\mathfrak{h}$ is a reductive subalgebra). Note also that $\mathfrak{m}_1$ is an ideal, though in general, the complement to $\mathfrak{h}$ need not even be a subalgebra.

DEFINITION 43. The *centralizer* $\mathrm{cent}_{\mathfrak{g}}(\mathfrak{h})$ of a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is given by

$$\mathrm{cent}_{\mathfrak{g}}(\mathfrak{h}) \equiv \{x \in \mathfrak{g} : \forall y \in \mathfrak{h}, [x, y] = 0\}.$$

DEFINITION 44. The *normalizer* $\mathrm{norm}_{\mathfrak{g}}(\mathfrak{h})$ of a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is given by

$$\mathrm{norm}_{\mathfrak{g}}(\mathfrak{h}) \equiv \{x \in \mathfrak{g} : \forall y \in \mathfrak{h}, [x, y] \in \mathfrak{h}\}.$$

DEFINITION 45. The *generalized center* $GC_{\mathfrak{g}}(\mathfrak{h})$ of a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is given by

$$GC_{\mathfrak{g}}(\mathfrak{h}) \equiv \{x \in \mathfrak{g} : \forall y \in \mathfrak{g}, [x, y] \in \mathfrak{h}\}.$$

Definition 46. The *upper central series* of a Lie algebra is the series of ideals $\mathfrak{z}_1(\mathfrak{g}) \subseteq \mathfrak{z}_2(\mathfrak{g}) \subseteq \ldots \subseteq \mathfrak{z}_k(\mathfrak{g}) \subseteq \ldots \subseteq \mathfrak{g}$ where

$$\mathfrak{z}_1(\mathfrak{g}) \equiv C(\mathfrak{g}) \equiv GC_{\mathfrak{g}}(0)$$

is the center of $\mathfrak{g}$ and for $k \geq 1$, the ideal $\mathfrak{z}_{k+1}(\mathfrak{g})$ is defined via $\mathfrak{z}_{k+1}(\mathfrak{g}) = GC_{\mathfrak{g}}(\mathfrak{z}_k(\mathfrak{g}))$. A Lie algebra $\mathfrak{g}$ is called abelian if $C(\mathfrak{g}) = \mathfrak{g}$.

Definition 47. The *derived series* of a Lie algebra is the series of ideals $\mathfrak{g} = \mathfrak{g}^{(0)} \supseteq \mathfrak{g}^{(1)} \supseteq \ldots \supseteq \mathfrak{g}^{(k)} \supseteq \ldots$ defined recursively via $\mathfrak{g}^{(k+1)} = \left[\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}\right]$. The Lie algebra $\mathfrak{g}^{(1)}$ is called the derived algebra of $\mathfrak{g}$. A Lie algebra $\mathfrak{g}$ is called solvable if there is $k \in \mathbb{N}$ such that $\mathfrak{g}^{(k)} = 0$.

Definition 48. The *lower central series* of a Lie algebra is the series of ideals $\mathfrak{g} = \mathfrak{g}^0 \supseteq \mathfrak{g}^1 \supseteq \ldots \supseteq \mathfrak{g}^k \supseteq \ldots$ defined recursively via $\mathfrak{g}^{k+1} = \left[\mathfrak{g}^k, \mathfrak{g}\right]$. A Lie algebra $\mathfrak{g}$ is called nilpotent if there is $k \in \mathbb{N}$ such that $\mathfrak{g}^k = 0$.

Example 49. Consider again the simple example of the two-dimensional nonabelian Lie algebra with $[e_1, e_2] = e_2$. The upper central series is given by $\mathfrak{z}_k(\mathfrak{g}) = \operatorname{span}\{e_1\}$ for all k. The derived series is given by $\mathfrak{g}^{(1)} = \operatorname{span}\{e_2\}$ and $\mathfrak{g}^{(k)}$ trivial for all $k > 1$. The lower central series is given by $\mathfrak{g}^k = \operatorname{span}\{e_2\}$ for all $k \geq 1$.

Definition 50. The *radical* $R(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is the maximal solvable ideal in $\mathfrak{g}$.

Remark 51. The radical for a given Lie algebra is unique because the sum of solvable ideals is a solvable ideal.

Definition 52. The *nilradical* $NR(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is the maximal nilpotent ideal in $\mathfrak{g}$.

Remark 53. The nilradical for a given Lie algebra is unique because the sum of nilpotent ideals is a nilpotent ideal.

2.5. Space-Times, Isometry, and Killing Vectors

The following definitions are fundamental to the study of manifolds with metrics and serve to connect the algebraic and group nature of this work to larger geometric concerns. In particular, we consider the action of an isometry group (defined below) on a manifold together with its isotropy at a point.

DEFINITION 54. A *metric* g on an n-dimensional manifold $\mathcal{M}$ is a real, non-degenerate, symmetric, type $(0,2)$ tensor. A metric has *signature* (p,q) if at each point $x \in \mathcal{M}$, there are vector subspaces P and Q of T_xM, of dimension p and q respectively, such that $T_xM = P \oplus Q$, the metric is positive-definite on P, and the metric is negative-definite on Q. *A* metric is *Riemannian* if the metric is positive-definite. A metric is *Lorentzian* if the metric has signature $(1, n-1)$ or $(n-1, 1)$. A *pseudo-Riemannian manifold* $(\mathcal{M}, g)$ is an n-dimensional manifold equipped with a metric of signature (p,q). A *space-time* $(\mathcal{M}, g)$ is a manifold $\mathcal{M}$ equipped with a Lorentzian metric g. (Typically, the dimension of $\mathcal{M}$ is four, but this need not always be the case.)

DEFINITION 55. The *isometry group* G of a space-time $(\mathcal{M}, g)$ is the set of all diffeomorphisms on $\mathcal{M}$ which preserve g, i.e., $G = \{\phi : \mathcal{M} \to \mathcal{M} : g(X,Y) = g(\phi_* X, \phi_* Y)\}$.

DEFINITION 56. The *isometry algebra* Γ of a pseudo-Riemannian manifold $(\mathcal{M}, g)$ is the set of all vector fields on $\mathcal{M}$ such that the Lie derivative of g vanishes along the vector field, i.e., $\Gamma = \{X \in TM : \mathcal{L}_X g = 0\}$. If $X \in \Gamma$, then X is called a *Killing vector.*

THEOREM 57. *The isometry group of any n-dimensional space-time is a Lie group of dimension at most $\frac{n(n+1)}{2}$ and the corresponding Lie algebra is isomorphic to the isometry algebra.*

EXAMPLE 58. In two dimensions, the maximal dimension of the isometry group is three. In the plane, this is realized as two translations and a rotation. On the n-sphere, the isometry group is $O(n+1)$, which is also of maximal dimension. In four-dimensions, the maximal dimension is realized (not uniquely) by the Minkowski metric and its isometry group, the Poincaré group, which consists of three rotations, three boosts, and four translations.

DEFINITION 59. The *isotropy algebra* Γ_{x_0} of at a point x_0 in pseudo-Riemannian manifold$(\mathcal{M}, g)$ is the subalgebra of the isometry algebra formed by the vector fields that vanish at x_0.

THEOREM 60. *If G is the isometry group of a pseudo-Riemannian manifold, then the isotropy algebra is the Lie algebra of the isotropy group G_{x_0} at x_0.*

THEOREM 61. *The linear isotropy representations for the isotropy subgroup of the isometry group and the isotropy subalgebra of the isometry algebra for a pseudo-Riemannian manifold are faithful.*

THEOREM 62. *The isotropy group at x_0 for a pseudo-Riemannian manifold $(\mathcal{M}, g)$ with metric of signature (p,q) is isomorphic to a subgroup of $SO(p,q)$.*

PROOF. Let Γ_{x_0} be the isotropy algebra at x_0 and let G_{x_0} be the isotropy group. Then any $\phi \in G_{x_0}$ has the property that $g_{\phi(x_0)}(\phi_* X_{x_0}, \phi_* Y_{x_0}) = g_{x_0}(X_{x_0}, Y_{x_0})$ for any pair $X_{x_0}, Y_{x_0} \in T_{x_0}\mathcal{M}$, i.e., the metric is preserved by ϕ. Since $g_{\phi(x_0)} = g_{x_0}$ for any $\phi \in G_{x_0}$, the isotropy group preserves g_{x_0}, a quadratic form of signature (p,q). Therefore, G_{x_0} is a subgroup of $SO(p,q)$. □

In a space-time, the orbits through a point under the action of the isometry group can be placed in three broad types according to the signature of the metric on the orbit. This finds application in, for example, Petrov's classification of space-times [**10**]. The following definition describes these three types.

DEFINITION 63. Let V be a p-dimensional subspace of an n-dimensional space-time such that the metric on V has constant signature. The *subspace type of* V is given by the following:

(1) The subspace type is *space-like* if the signature of the metric on the subspace is $(p,0)$.
(2) The subspace type is *time-like* if the signature of the metric on the subspace is $(p-1,1)$.
(3) The subspace type is *null* if the metric on the subspace is degenerate.

Given a group acting on a manifold, the *orbit type* of an orbit through a point is the subspace type of the orbit.

EXAMPLE 64. In Minkowski space with Cartesian coordinates (x,y,z,t), the surface defined by $t=0$ is $\mathbb{R}^3$ and is space-like. The surface defined by $z=0$ is the Minkowski plane and is time-like. The light cone at the origin, defined by $t^2 = x^2+y^2+z^2$ is null.

We now consider sufficient conditions under which a pseudo-Riemannian manifold $(\mathcal{M}, g)$ admits a local slice.

THEOREM 65. *If g is a Riemannian metric on $\mathcal{M}$ with isometry group G, then at any point $x_0 \in \mathcal{M}$, the action of G admits a local slice.*

THEOREM 66. *If g is a Riemannian metric on $\mathcal{M}$ the isotropy algebra is a reductive subalgebra of the isometry algebra.*

If the isometry group acts transitively on a manifold, the Fundamental Theorem of Homogeneous Spaces (Theorem 20) gives a diffeomorphism between the manifold and the quotient of the isometry group by the isotropy group. We now demonstrate that if the isotropy is reductive, this diffeomorphism guarantees a correspondence between metrics on the manifold and metrics on the reductive complement to the isotropy algebra.

THEOREM 67. *Let $(\mathcal{M}, g)$ be a pseudo-Riemannian manifold with transitive isometry group G (with identity e) and isotropy group $H = G_p$ at p. Let the Lie algebra of G be $\mathfrak{g}$ and the Lie algebra of H be $\mathfrak{h}$ with $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Then the metric g on $T_p\mathcal{M}$ induces a metric $\tilde{g}$ of the same signature on the vector space $\mathfrak{m}$. If $\mathfrak{h}$ is reductive and $\mathfrak{m}$ is a reductive complement, $ad\,(\mathfrak{h})$ preserves $\tilde{g}$ infinitesimally.*

PROOF. Let $\phi : G/H \to \mathcal{M}$ be the G-equivariant diffeomorphism between G/H and $\mathcal{M}$ such that $\phi\,(eH) = p$. Further let $\pi : G \to G/H$ be the natural projection map and $\tilde{\pi} : G \to \mathcal{M}$ be given by $\tilde{\pi} = \phi \circ \pi$. Since the kernel of π is H, the pushforward of $\tilde{\pi}$ at the identity, $\tilde{\pi}_* : \mathfrak{g} \to T_p\mathcal{M}$, has kernel $\mathfrak{h}$ and thus $\tilde{\pi}_*$ is bijective when restricted to $\mathfrak{m}$. This generates a metric $\tilde{g} : \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$ given by $\tilde{g} = g \circ \left(\tilde{\pi}_*\big|_{\mathfrak{m}} \times \tilde{\pi}_*\big|_{\mathfrak{m}}\right)$.

Consider $x \in \mathfrak{h}$ and the Killing vector $X \in TM$ with $X_p = \tilde{\pi}_*\,(x)$. If $y, z \in \mathfrak{m}$ with $Y_p = \tilde{\pi}_*\,(y)$ and $Z_p = \tilde{\pi}_*\,(z)$, then the Lie bracket at p of X with Y or Z can be calculated in coordinates ξ^i as $[X, Y]_p = \frac{\partial X^i}{\partial \xi^j}\big|_p Y_p^j \partial_{\xi_p^j} = [X_p, Y_p]$; in particular, Y need only be given at p. Thus,

$$\mathcal{L}_X g\,(Y, Z)\,\big|_p = X\,(g\,(Y, Z))\,\big|_p - g\,([X_p, Y_p]\,, Z_p) - g\,(Y_p, [X_p, Z_p])\,.$$

Since X is a Killing vector, this implies $g\,([X_p, Y_p]\,, Z_p) + g\,(Y_p, [X_p, Z_p]) = 0$. If $\mathfrak{m}$ is a reductive complement to $\mathfrak{h}$, then $\tilde{\pi}_* \circ ad\,(\mathfrak{h})$ acting on $\mathfrak{m}$ induces a family of linear transformations on $T_p\mathcal{M}$ with $\tilde{\pi}_*\,([\mathfrak{h}, \mathfrak{m}]) = [\tilde{\pi}_*\,(\mathfrak{h})\,, \tilde{\pi}_*\,(\mathfrak{m})]$. Therefore $g\,([X_p, Y_p]\,, Z_p) + g\,(Y_p, [X_p, Z_p]) = 0$ is equivalent to $\tilde{g}\,(ad\,(x)\,(y)\,, z) + \tilde{g}\,(y, ad\,(x)\,(z)) = 0$ and thus $ad\,(\mathfrak{h})$ preserves $\tilde{g}$ infinitesimally. $\square$

2.6. Overview of the Schmidt Method

Given a Lie subalgebra, $\mathfrak{h}$, of the Lorentz algebra, $\mathfrak{so}(p,1)$, it is possible to construct a Lie algebra $\mathfrak{g}$ such that $\mathfrak{g}$ has a realization as a Lie algebra of Killing vectors on a pseudo-Riemannian manifold such that those vector fields corresponding to $\mathfrak{h}$ vanish at a point. Note that this Lie algebra of Killing vectors is not necessarily the full isometry algebra (see Section 2.7 for more detail). This procedure and the relevant theorems are derived in [**11**] and summarized here.

Fix $\mathfrak{h} \subset \mathfrak{so}(p,1)$. Then $\mathfrak{h}$ consists of linear maps σ_i on Minkowski space with matrix elements ${\sigma_{i\alpha}}^{\beta}$ and commutators $[\sigma_i, \sigma_j] = {c_{ij}}^k \sigma_k$. Choose $F_k \in \mathfrak{h}$ and $X_\alpha \in \mathfrak{m}$ to be a basis for $\mathfrak{g}$ with commutators

$$
\begin{aligned}
[F_i, F_j] &= {c_{ij}}^k F_k \\
[F_i, X_\alpha] &= {\sigma_{i\alpha}}^{\beta} X_\beta \\
[X_\alpha, X_\beta] &= {\mu_{\alpha\beta}}^{\gamma} X_\gamma + {\lambda_{\alpha\beta}}^k F_k
\end{aligned}
$$

where $\mu_{\alpha\beta}^{\gamma}$ and $\lambda_{\alpha\beta}^{k}$ are arbitrary insofar as the Jacobi identities are satisfied. The relevant Jacobi identities are

$$
\begin{aligned}
{c_{[ab}}^k {c_{c]k}}^r &= 0 \\
{\sigma_{b\alpha}}^{\beta} {\sigma_{a\beta}}^{\gamma} - {\sigma_{a\alpha}}^{\beta} {\sigma_{b\beta}}^{\gamma} - {c_{ab}}^k {\sigma_{k\alpha}}^{\gamma} &= 0 \\
2{\sigma_{a[\alpha}}^{\rho} \mu_{\beta]\rho}^{\gamma} + 2{\sigma_{a[\alpha}}^{\rho} \lambda_{\beta]\rho}^{k} + \mu_{\alpha\beta}^{\rho} {\sigma_{a\rho}}^{\gamma} + \lambda_{\alpha\beta}^{r} {c_{ar}}^k &= 0 \\
\mu_{[\beta\gamma}^{\rho} {\mu_{\alpha]\rho}}^{\kappa} + \lambda_{[\beta\gamma}^{k} {\sigma_{\alpha]k}}^{\kappa} &= 0
\end{aligned}
$$

where square brackets around indicies indicate anti-symmetrization. Thus $\mathfrak{h}$ has basis given by $\{F_k\}$ and is reductive in $\mathfrak{g}$. It is shown in [**11**] that $\mathfrak{g}$ will have a realization as a Lie algebra of Killing vectors on a homogeneous space of dimension $\dim\mathfrak{g} - \dim\mathfrak{h}$. This process allows for the classification of all (abstract) Lie algebra-subalgebra pairs $(\mathfrak{g}, \mathfrak{h})$ arising from reductive isometry-isotropy pairs on homogeneous spaces. In particular, we perform this classification for the case in which $\mathfrak{g}$ is dimension five and $\mathfrak{h}$ is dimension one (for the non-reductive case, see [**3**]).

Let R denote the set of nondegenerate linear transformations $r : \mathfrak{g} \to \mathfrak{g}$ such that $[F_i, r(X_\alpha)] = [F_i, X_\alpha]$ and $r(F_i) = F_i$. Then R is given by computing the matrix centralizers for the adjoint of each $F_i \in \mathfrak{h}$ and taking the intersection. of The crux of the Schmidt method, therefore, is to impose the Jacobi identities on $\mu_{\alpha\beta}^{\gamma}$ and $\lambda_{\alpha\beta}^{k}$, giving a set of quadratic equations, then to select a representation for R-orbits. As a practical matter, however, it is better to follow the procedure below:

(1) Eliminate parameters by imposing the Jacobi identities.
(2) Introduce Lie theoretic branching.
(3) Use R transformations to eliminate further parameters.
(4) Apply transformations to align the algebras with a standard reference, e.g. [**12**], in order to ensure all inequivalent algebra pairs are found.
(5) Repeat the above steps until no more parameters can be eliminated and all inequivalent algebra pairs are found.

Since we classify $(\mathfrak{g}, \mathfrak{h})$ without respect to the reductive complement chosen, we may compose the transformations in steps 3 and 4 without loss of generality.

We find that with the exception of two cases, five-dimensional isometry implies either trivial isotropy or one-dimensional isotropy. Thus, the procedure followed here need not be as general as the one outlined in [**11**]. We first fix a one-dimensional subgroup of the Lorentz group and identify the infinitesimal generator in the standard representation, yielding a linear transformation from $\mathbb{R}^4$ to itself. This transformation, in some convenient basis, is taken to be the adjoint of a particular vector in the isotropy (designated e_5) in a five-dimensional Lie algebra, thus fixing the isotropy and four of the ten Lie brackets. The Jacobi identity is then imposed. Generally, a relatively large family of Lie algebras with many parameters remains. By considering algebraic invariants and choosing appropriate bases, this family is reduced to a list of those Lie algebras that are unique up to real change of basis, as classified by [**12**]. Throughout any changes of basis, the isotropy, as a vector subspace, is noted. The algebra and the isotropy are given in the basis recorded in [**12**]. These results are tabulated in Tables 1.1 through 1.4, while the distinguishing invariants are summarized in Figures 1.2 through 1.6.

2.7. Applicability of the Schmidt Method

Given a pseudo-Riemannian manifold $(\mathcal{M}, g)$, it is of interest to know whether or not its isometry-isotropy algebra-subalgebra pair is discoverable via the Schmidt method. If $\mathcal{M}$ is homogeneous under the action of its isometry group G and the isotropy subalgebra is reductive, then the Schmidt method applies since the isotropy subalgebra is everywhere the same. The difficulty occurs for spaces that are not homogeneous and therefore may or may not have point-dependent isotropy. Corollary 26 shows that isotropy groups conjugate in a neighborhood $\mathcal{U}$ of an isotropy preserving slice $\mathcal{S}$. By Remark 22, if a local slice exists through a point $x_0 \in \mathcal{M}$, then an isotropy preserving local slice through x_0 can be found. In this neighborhood of conjugate isotropies, the isometry-isotropy algebra-subalgebra pair is the same. Furthermore, the manifold is diffeomorphic to the Cartesian product of the slice with a homogeneous space, i.e., $\mathcal{M} \cong (\mathcal{S} \cap \mathcal{U}) \times G/G_{x_0}$. The Schmidt method then applies to the homogeneous space G/G_{x_0} if the isotropy is reductive. Since by Theorem 25 G/G_{x_0} is diffeomorphic to the orbit $O_G(x_0)$, the isometry-isotropy algebra-subalgebra pair on G/G_{x_0} is the same as that on $\mathcal{M}$. The problem then is to determine under what conditions a local slice can be found. Theorems 65-**??** give some sufficient conditions for the existence of a slice and reductive isotropy.

An important limitation of the Schmidt method is that it only guarantees that the algebra-subalgebra pair $(\mathfrak{g}, \mathfrak{h})$ is realizable as Killing vectors on a pseudo-Riemannian manifold $(\mathcal{M}, g)$. It does *not* guarantee that there exists a pseudo-Riemannian manifold for which the isometry-isotropy algebra is identically $(\mathfrak{g}, \mathfrak{h})$, nor does it fix the signature of g, though this may be done "by hand" when generating invariant metrics on the reductive complement to $\mathfrak{h}$. The following examples illustrate this limitation.

EXAMPLE 68. Among the Lie algebras of Killing vectors given by Petrov in [**10**], none has isotropy acting (in any basis) as $B(\theta) = \operatorname{span}\{\cos\theta\,(y\partial_x - x\partial_y) - \sin\theta\,(t\partial_z + z\partial_t)\}$ on a four-dimensional space-time (see Section 2.8 for a discussion of possible isotropy types). The following algebra, however, can be obtained via application of the Schmidt method using the isotropy $B(\theta)$ (in fact, as is shown in Chapter 4, this is the only such algebra).

$$[e_5, e_1] = \cos\theta e_2 \quad [e_5, e_2] = -\cos\theta e_1 \quad [e_5, e_3] = \sin\theta e_4 \quad [e_5, e_4] = \sin\theta e_3$$

The structure equations indicate the existence of four independent and commuting Killing fields. Therefore, the corresponding four-dimensional pseudo-Riemannian manifold must be flat, implying that there are in fact ten Killing vectors. That is, $B(\theta)$ manifests as a Killing vector in the isotropy algebra only when there are additional isotropy Killing vectors. The Schmidt method makes no attempt to account for these additional symmetries.

2.8. Subalgebras of the Lorentz Algebra

Except for two cases, all algebra-subalgebra pairs with isometry dimension five have one-dimensional isotropy. Such algebra-subalgebra pairs are associated with four-dimensional space-times, and the isometry algebra of any such space-time must be a subalgebra of $\mathfrak{so}(3,1)$. Therefore it becomes useful to classify the inequivalent subalgebras of $\mathfrak{so}(3,1)$, as has been done in [**9**]. Three basis elements can be thought of as infinitesimal generators for rotations about a spatial axis, R_x, R_y, and R_z, and three basis elements can be thought of as infinitesimal generators for boosts in each spatial direction, K_x, K_y, and K_z and are referenced as such throughout this thesis. The inequivalent subalgebras of $\mathfrak{so}(3,1)$ are given in Table 2.1 alongside the matrix representation used in this thesis.

The one-dimensional subalgebras of $\mathfrak{so}(3,1)$, $F11 - F14$, are of particular pertinence to this thesis; the only case of isotropy not among these is the two-dimensional isotropy case, which belongs to $F8$ (as a subgroup of $\mathfrak{so}(2,1)$). In the standard action of $\mathfrak{so}(3,1)$, each one-dimensional subalgebra has a geometric interpretation: the $F11$ family of subalgebras gives, for each value of θ, an infinitesimal generator for a loxodromic transformation; the $F12$ and $F13$ algebras are infinitesimal generators for rotations and boosts, respectively; and the $F14$ algebra generates a null-rotation. For the case of one-dimensional isotropy, the classification begins with the isotropy type, and the chapters of this thesis are likewise arranged.

2.9. Classification of Lie Algebras and Subalgebras

Much of this work relies on the complete classification of real Lie algebras up to dimension six presented in [**12**]. The classification is organized to facilitate identification of a given Lie algebra by

$F1$:	$\{B_1, B_2, B_3, B_4, B_5, B_6\}$	$F9$:	$\{B_1, B_2\}$
$F2$:	$\{B_1, B_2, B_3, B_4\}$	$F10$:	$\{B_3, B_4\}$
$F3$:	$\{R_x, R_y, R_z\}$	$F11$:	$\{B(\theta)\}$
$F4$:	$\{R_z, K_x, K_y\}$	$F12$:	$\{R_z\}$
$F5$:	$\{B(\theta), B_3, B_4\}$	$F13$:	$\{K_z\}$
$F6$:	$\{B_1, B_3, B_4\}$	$F14$:	$\{R_y + K_z\}$
$F7$:	$\{B_2, B_3, B_4\}$	$F15$:	$\{0\}$
$F8$:	$\{B_2, B_3\}$		

$$B_1 = 2R_z \qquad B_2 = -2K_z \qquad B_3 = -R_y - K_z$$
$$B_4 = R_x - K_y \qquad B_5 = R_y - K_x \qquad B_6 = R_x + K_y \qquad B(\theta) = \cos(\theta) R_z - \sin(\theta) K_z$$

$$R_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad R_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad R_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K_y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K_z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

TABLE 2.1. Inequivalent subalgebras of $\mathfrak{so}(3,1)$, labeled $F1 - F15$.

computation of its nilradical and three central series: the upper central series, lower central series, and derived series. It is derived, however, through consideration of the Jordan normal form of the nonnilpotent elements on the nilradical and indeed, this is often a necessary computation for the proper identification of a Lie algebra. Where possible, this work aims to carry out the classification through consideration of the three central series as this is computationally simpler. Typically, only the dimensions of the algebras in these series are needed. Another useful invariant is the signature of the Killing form, which is invariant under real change of basis. Unfortunately, there are distinct real algebras which cannot be distinguished by these invariants and we must return to the consideration of the Jordan normal form of the nonnilpotent elements' adjoint action on the nilradical.

In addition to distinguishing real Lie algebras, we must also distinguish isotropy subalgebras of a given Lie algebra. Most frequently this is accomplished by considering whether or not the isotropy falls within an invariant, easily identified subalgebra such as the derived algebra. In some cases, however, this is insufficient and we calculate special subalgebras containing the isotropy. For example, the largest ideal in the nilradical containing the isotropy is used to distinguish certain isotropies of types $F12$ and $F13$, while the existence or nonexistence of a four-dimensional algebra other than the nilradical that contains the isotropy distinguishes several isotropies of type $F14$.

CHAPTER 3

ISOTROPY OF DIMENSION OTHER THAN ONE

The problem of classifying, as pairs, five-dimensional isometry algebras with zero dimensional isotropies is equivalent to classifying five-dimensional Lie algebras, which has been done by [**12**]. If the isotropy is three-dimensional or four dimensional, the isometry must act on a (homogeneous) space of dimension two or one, but these spaces admit maximal isometries of dimension three and one, respectively. In either case, the isometry must be of dimension strictly less than five and thus these cases do not yield algebra-subalgebra pairs of appropriate dimensions.

Consider the case of a five-dimensional isometry algebra with reductive two-dimensional isotropy subalgebra. We follow the general procedure in [**2**] deviating only in notation and the Lie algebra classification used. An algebra-subalgebra pair not found in [**2**] is presented for this case, as the Jacobi identities are not as restrictive as claimed there.

Not all two-dimensional subalgebras of $\mathfrak{so}\,(2,1)$ can generate algebras of the form required for execution of the Schmidt method. That is, given a subalgebra of $\mathfrak{so}\,(2,1)$ spanned by $\{F_1, F_2\}$ with commutator $[F_i, F_j] = c_{ij}{}^k F_k$, there is not necessarily choice of structure constants $\sigma_{i\alpha}{}^{\beta}$, $\mu_{\alpha\beta}{}^{\gamma}$, and $\lambda_{\alpha\beta}{}^{k}$ such that

$$
\begin{aligned}
[F_i, F_j] &= c_{ij}{}^k F_k \\
[F_i, X_\alpha] &= \sigma_{i\alpha}{}^{\beta} X_\beta \\
[X_\alpha, X_\beta] &= \mu_{\alpha\beta}{}^{\gamma} X_\gamma + \lambda_{\alpha\beta}{}^{k} F_k
\end{aligned}
$$

is a Lie algebra (in particular, the Jacobi identities might not be satisfied). We therefore begin by determining which two-dimensional subalgebras of $\mathfrak{so}\,(2,1)$ are possible choices for isotropy algebras. Choose a basis $\{e_i\}$ and let the isotropy be spanned by basis vectors e_4 and e_5. The only two-dimensional subalgebras of $\mathfrak{so}\,(2,1)$ are non-abelian, and so we may take $[e_4, e_5] = e_4$. If e_4 represents a boost or rotation, then we may take $[e_4, e_1] = \pm e_2$ and $[e_4, e_2] = e_1$ with all other brackets with e_4 giving zero (except $[e_4, e_5] = e_4$ from earlier). In this case, the Jacobi identities on e_4, e_5, and each of e_1 and e_2 give the following:

$$0 = [e_4,[e_1,e_5]] + [e_5,\pm e_2] \pm e_2 \tag{3.1}$$

$$0 = [e_4,[e_2,e_5]] + [e_5,e_1] - e_1 \tag{3.2}$$

The e_2 component of Equation 3.2 $[e_1,e_5]_{e_1} + [e_5,e_2]_{e_2} + 1 = 0$ and the $e_1$component of Equation 3.2 gives $[e_2,e_5]_{e_2} + [e_5,e_1]_{e_1} - 1 = 0$. These are mutually exclusive, and so the Jacobi Identities cannot be satisfied if e_4 represents a rotation or a boost. Therefore, e_4 must represent a null rotation and we may take $[e_4,e_5] = e_4$,$[e_4,e_2] = -e_3$ and $[e_4,e_1] = e_2$. Since the adjoint of e_5 restricted to the span of the first three basis vectors is an element of $\mathfrak{so}(2,1)$, it is traceless. This requirement, together with the Jacobi identities, forces the structure equations to take the following form:

$$\begin{aligned}
[e_1,e_2] &= a_1e_1 - a_1d_3e_3 \\
[e_1,e_3] &= -a_1e_2 \\
[e_1,e_4] &= -e_2 \\
[e_1,e_5] &= -e_1 + d_2e_2 + d_3e_3 \\
[e_2,e_3] &= a_1e_3 \\
[e_2,e_4] &= e_3 \\
[e_2,e_5] &= -d_2e_3 \\
[e_3,e_5] &= e_3 \\
[e_4,e_5] &= e_4
\end{aligned}$$

The eigenvalues of the adjoint of e_5 restricted to the span of the first three basis vectors has real, distinct eigenvalues and therefore acts as a boost. Therefore, we execute the Schmidt method using as isotropy the subalgebra of $\mathfrak{so}(2,1)$ spanned by a boost and a null rotation. For consistency, we denote this isotropy type $F8$, though here the isotropy is to be thought of as a subgroup of $\mathfrak{so}(2,1)$ rather than of $\mathfrak{so}(3,1)$. We have already imposed the Jacobi identities, and so we now consider invariant characteristics in order to eliminate parameters and identify unique algebra-subalgebra pairs. We find two distinct pairs.

The derived algebra is spanned by $\{e_1, e_2, e_3, e_4\}$, and so the second derived algebra is spanned by $\{a_1e_1, e_2, e_3\}$. Thus the second derived algebra is two-dimensional if and only if $a_1 = 0$, and in this case, the change of basis $\left(e_3, e_2, -e_1 + \frac{d_3}{2}e_3, e_4, -d_2e_4 - e_5\right)$ gives the algebra and its isotropy in standard form with identification $(F8, 0)$; this is the case found in [**2**]. If $a_1 \neq 0$, the second derived algebra is three-dimensional and the change of basis $\left(2e_1 - d_3e_3, \frac{2}{a_1}e_2, \frac{1}{a_1^2}e_3, -\frac{1}{a_1^2}e_3 + \frac{1}{a_1}e_4, \frac{1}{a_1}e_2 + d_2e_4 + e_5\right)$ gives the isotropy in standard form with identification $(F8, 1)$.

CHAPTER 4

F11: LOXODROMES

The F11 family of subalgebras of $\mathfrak{so}(3,1)$ in standard coordinates and basis is given by the loxodrome $\{B(\theta)\}$ where

$$B(\theta) \equiv \begin{pmatrix} 0 & \cos\theta & 0 & 0 \\ -\cos\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta \\ 0 & 0 & -\sin\theta & 0 \end{pmatrix}$$

and $\theta \in \left(0, \frac{\pi}{2}\right)$. Let $\mathfrak{g}_{F11}$ be a generic five-dimensional Lie algebra with basis $\{e_i\}$ and $ad(e_5) = B(\theta) \oplus (0)$ so that $\{e_5\}$ is a subalgebra of isotropy type $F11$. This determines all Lie brackets involving the isotropy, e_5. The Lie brackets are thus of the form

$$\begin{aligned} [e_5, e_1] &= -\cos\theta e_2 \\ [e_5, e_2] &= \cos\theta e_1 \\ [e_5, e_3] &= -\sin\theta e_4 \\ [e_5, e_4] &= -\sin\theta e_3 \\ [e_\alpha, e_\beta] &= \mu_{\alpha\beta}{}^\gamma e_\gamma + \lambda_{\alpha\beta} e_5 \end{aligned}$$

where Greek indices run from 1 to 4. The structure constants $\mu_{\alpha\beta}{}^\gamma$ and $\lambda_{\alpha\beta}$ are subject to the Jacobi identities, which require that $\mu_{\alpha\beta}{}^\gamma$ and $\lambda_{\alpha\beta}$ are identically zero (see Appendix B.2 for details).

The change of basis $(e_3 + e_4, e_3 - e_4, e_1, e_2, \sec\theta e_5)$ gives the algebra pair $(F11, 0)$ in standard form.

CHAPTER 5

F12: ROTATIONS

The F12 subalgebra of $\mathfrak{so}(3,1)$ in standard coordinates and basis is given by the rotation $\{R_z\}$ where

$$R_z \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $\mathfrak{g}_{F12}$ be a generic five-dimensional Lie algebra with basis $\{e_i\}$ and $ad(e_5) = R_z \oplus (0)$ so that $\{e_5\}$ is a subalgebra of isotropy type $F12$. Immediately, the Jacobi identity with basis vectors e_1, e_2, and e_5 gives $0 = [e_5, [e_1, e_2]]$ so $[e_1, e_2]$ has no e_1 or e_2 component. Thus, let the structure constants, ${C_{ij}}^k$ be given by the following:

$$\begin{aligned}
[e_1, e_2] &= a_3e_3 + a_4e_4 + a_5e_5 \\
[e_1, e_3] &= b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4 + b_5e_5 \\
[e_2, e_3] &= c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4 + c_5e_5 \\
[e_1, e_5] &= e_2 \\
[e_1, e_4] &= d_1e_1 + d_2e_2 + d_3e_3 + d_4e_4 + d_5e_5 \\
[e_2, e_4] &= g_1e_1 + g_2e_2 + g_3e_3 + g_4e_4 + g_5e_5 \\
[e_2, e_5] &= -e_1 \\
[e_3, e_4] &= h_1e_1 + h_2e_2 + h_3e_3 + h_4e_4 + h_5e_5 \qquad (5.1)
\end{aligned}$$

To enforce the remainder of the Jacobi identities, consider the Maurer Cartan forms, $\{\omega^i\}$ (where $\langle \omega^i, e_j \rangle = \delta^i_j$). The exterior derivative on the Maurer Cartan forms is given by $d\omega^k = -\frac{1}{2}{C_{ij}}^k \omega^i \wedge \omega^j$ and the Jacobi identities on the Lie algebra are equivalent to the integrability condition $d^2 \equiv 0$. Immediately, this condition requires that the structure equations take the following form:

$$\begin{aligned}
[e_1, e_2] &= a_3e_3 + a_4e_4 + a_5e_5 \\
[e_1, e_3] &= b_1e_1 + b_2e_2 \\
[e_2, e_3] &= -b_2e_1 + b_1e_2 \\
[e_1, e_5] &= e_2 \\
[e_1, e_4] &= d_1e_1 + d_2e_2 \\
[e_2, e_4] &= -d_2e_1 + d_1e_2 \\
[e_2, e_5] &= -e_1 \\
[e_3, e_4] &= h_3e_3 + h_4e_4 + h_5e_5
\end{aligned} \tag{5.2}$$

Applying the change of basis $(e_1, e_2, e_3 - b_2e_5, e_4 - d_2e_5, e_5)$, we can take $b_2 = d_2 = 0$ without loss of generality. Then $d^2\omega^1 = 0$ forces $h_5 = 0$ as well, yielding the following Lie brackets:

$$\begin{aligned}
[e_1, e_2] &= a_3e_3 + a_4e_4 + a_5e_5 \\
[e_1, e_3] &= b_1e_1 \\
[e_2, e_3] &= b_1e_2 \\
[e_1, e_5] &= e_2 \\
[e_1, e_4] &= d_1e_1 \\
[e_2, e_4] &= d_1e_2 \\
[e_2, e_5] &= -e_1 \\
[e_3, e_4] &= h_3e_3 + h_4e_4
\end{aligned} \tag{5.3}$$

Let $\mathfrak{n}$ be the centralizer of the isotropy subalgebra (note from the structure constants that $\mathfrak{n}$ always forms a subalgebra). We seek the largest subalgebra $\mathfrak{i}$ of $\mathfrak{n}$ that is also an ideal in $\mathfrak{g}_{F12}$. Let $x = \alpha e_1 + \beta e_2$ be an arbitrary linear combination of e_1 and e_2 and let $y = \mu e_3 + \lambda e_4 + \nu e_5$ be a fixed vector in $\mathfrak{n}$. The Lie bracket is given by

$$[x, y] = \left(\mu b_1 + \lambda d_1 - \frac{\beta}{\alpha}\nu\right)\alpha e_1 + \left(\mu b_1 + \lambda d_1 + \frac{\alpha}{\beta}\nu\right)\beta e_2. \tag{5.4}$$

If $[x, y] \in \mathfrak{n}$, then $\beta\nu + \alpha\nu = 0$. Since α and β are arbitrary, we require $\nu = 0$. Thus, $\mathfrak{i}$ must take the form $\mathfrak{i} = \{\mu e_3 + \lambda e_4 : \mu b_1 = -\lambda d_1\}$. If $b_1 = d_1 = 0$, then $\mathfrak{i}$ is a two-dimensional subalgebra of $\mathfrak{n}$, and furthermore is an ideal in $\mathfrak{g}_{F12}$. If either b_1 or d_1 is non-zero, then $\mathfrak{i}$ is at most one-dimensional and $\mathfrak{n}$ has no two-dimensional subalgebra that is also an ideal in $\mathfrak{g}_{F12}$. Therefore, $b_1 = d_1 = 0$ if and only if $\dim \mathfrak{i} = 2$.

5.1. The Dimension of $\mathfrak{i}$ is Two

If $\dim \mathfrak{i} = 2$, then $b_1 = d_1 = 0$ the Jacobi identity requires that either $a_3 = a_4 = 0$ or $h_3 = h_4 = 0$; this requirement ensures that the Jacobi identity is completely satisfied. These can be distinguished by the dimension of the center.

5.1.1. The Center is Trivial. If the center is trivial, then either h_3 or h_4 is non-zero and $a_3 = a_4 = 0$. Since the span of e_3 and e_4 completely decomposes from the rest of the algebra as the two-dimensional non-abelian algebra, there is a basis in which the structure constants take the following form, with the Jacobi identity completely satisfied (see B.3 for details):

$$\begin{aligned} [e_1, e_2] &= a_5 e_5 \\ [e_1, e_5] &= e_2 \\ [e_2, e_5] &= -e_1 \\ [e_3, e_4] &= e_3 \end{aligned} \tag{5.5}$$

By scaling e_1 and e_2 by $|a_5|^{-1/2}$, we may take $a_5 \in \{-1, 0, 1\}$. The Killing form is given by

$$\begin{pmatrix} 2a_5 & 0 & 0 & 0 & 0 \\ 0 & 2a_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

and thus the sign of a_5 determines its signature, giving three distinct real Lie algebras. For $a_5 = 0$, the change of basis $(-e_1, -e_2, e_5, e_3, -e_4)$ gives the algebra pair in standard form with identification $(F12, 0)$. For $a_5 = 1$, the change of basis $(-e_2 + e_5, 2e_1, -e_2 - e_5, e_3, e_4)$ gives the algebra pair in standard form with identification $(F12, 1)$. For $a_5 = -1$, the change of basis $(e_5, e_2, e_1, e_3, e_4)$ gives the algebra pair $(F12, 2)$ in standard form.

5.1.2. The Center is not Trivial.

If the center is not trivial, then it is two dimensional and $h_3 = h_4 = 0$. The Killing form is given by

$$\begin{pmatrix} 2a_5 & 0 & 0 & 0 & 0 \\ 0 & 2a_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix},$$

and thus the sign of a_5 determines its signature. If $a_5 = 0$, the derived algebra is two-dimensional if and only if $a_3 = a_4 = 0$. If $a_5 \neq 0$, then the isotropy is in the derived algebra if and only if $a_3 = a_4 = 0$. If either a_3 or a_4 is non-zero then with either the change of basis $(e_1, e_2, a_3e_3 + a_4e_4, e_3, e_5)$ or $(e_1, e_2, a_3e_3 + a_4e_4, e_4, e_5)$, we can take $a_3 = 1$ and $a_4 = 0$ without loss of generality. Thus, there are six inequivalent algebra-subalgebra pairs in this case. The changes of basis are summarized in Table 5.1.

a_5	a_3	Change of Basis	Pair Designation
1	0	$(-e_2 + e_5, 2e_1, -e_2 - e_5, e_3, e_4)$	$(F12, 3)$
1	1	$(-e_2 + e_3 + e_5, 2e_1, -e_2 - e_3 - e_5, e_3, e_4)$	$(F12, 4)$
−1	0	$(e_5, e_2, e_1, e_3, e_4)$	$(F12, 5)$
−1	1	$(e_3 - e_5, -e_2, e_1, e_3, e_4)$	$(F12, 6)$
0	0	$(-e_1, -e_2, e_5, e_3, e_4)$	$(F12, 7)$
0	1	$(-e_3, -e_1, e_2, -e_5, e_4)$	$(F12, 8)$

TABLE 5.1. Summary of changes of basis to standard form for $\mathfrak{g}_{F12}$ when $\dim \mathfrak{i} \geq 2$ and the center is non-empty.

5.2. The Dimension of $\mathfrak{i}$ is Less than Two

If the dimension of $\mathfrak{i}$ is less than two, than at least one of b_1 and d_1 is non-zero. We apply the change of basis $\left(e_1, e_2, \frac{1}{b_1}e_3, e_4 - \frac{d_1}{b_1}e_3, e_5\right)$ if $b_1 \neq 0$, and the change of basis $\left(e_1, e_2, \frac{1}{d_1}e_4, e_3 - \frac{b_1}{d_1}e_4, e_5\right)$ if $b_1 = 0$ (and $d_1 \neq 0$). The Jacobi identities then require that the structure equations take on the following form, with appropriate relabeling of arbitrary constants (see B.3 for details):

$$\begin{aligned} [e_1, e_2] &= a_4 e_4 \\ [e_1, e_3] &= e_1 \\ [e_2, e_3] &= e_2 \\ [e_1, e_5] &= e_2 \\ [e_2, e_5] &= -e_1 \\ [e_3, e_4] &= h_4 e_4 \end{aligned} \tag{5.6}$$

The Jacobi identities are fully satisfied if either $a_4 = 0$ or $h_4 = -2$. The derived series is given by $\mathfrak{g}_{F12}^{(1)} = \operatorname{span}\{e_1, e_2, a_4 e_4, h_4 e_4\}$, then $\mathfrak{g}_{F12}^{(2)} = \operatorname{span}\{a_4 e_4\}$, and finally, $\mathfrak{g}_{F12}^{(3)} = \{0\}$. Thus $a_4 = 0$ if and only if $\dim \mathfrak{g}_{F12}^{(2)} = 0$ and in that case, $h_4 = 0$ if and only if $\dim \mathfrak{g}_{F12}^{(1)} = 2$. The changes of basis are summarized in Table 5.2.

a_4	h_4	Change of Basis	Pair Designation
$a_4 \neq 0$	-2	$(-a_4 e_4, e_1, -e_2, -e_3 - e_4, e_5)$	$(F12, 9)$
0	0	$(e_1, e_2, -e_3, -e_5, e_4)$	$(F12, 10)$
0	$h_4 \neq 0$	$(e_4, -e_1 - e_2, -e_1 + e_2, -e_3, -e_5)$	$(F12, 11)$, $\beta = h_4$

TABLE 5.2. Summary of changes of basis to standard form for $\mathfrak{g}_{F12}$ when $\dim \mathfrak{i} < 2$.

CHAPTER 6

F13: BOOSTS

The F13 subalgebra of $\mathfrak{so}(3,1)$ in standard coordinates and basis is given by the rotation $\{K_z\}$ where

$$K_z \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $\mathfrak{g}_{F13}$ be a generic five-dimensional Lie algebra with basis $\{e_i\}$ and $ad(e_5) = K_z \oplus (0)$ so that $\{e_5\}$ is a subalgebra of isotropy type $F13$. Immediately, the Jacobi identity with basis vectors e_1, e_2, and e_5 gives $0 = [e_5, [e_1, e_2]]$ so $[e_1, e_2]$ has no e_1 or e_2 component. Thus, let the structure constants, ${C_{ij}}^k$ be given by the following:

$$\begin{aligned}
[e_1, e_2] &= a_3e_3 + a_4e_4 + a_5e_5 \\
[e_1, e_3] &= b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4 + b_5e_5 \\
[e_2, e_3] &= c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4 + c_5e_5 \\
[e_1, e_5] &= -e_2 \\
[e_1, e_4] &= d_1e_1 + d_2e_2 + d_3e_3 + d_4e_4 + d_5e_5 \\
[e_2, e_4] &= g_1e_1 + g_2e_2 + g_3e_3 + g_4e_4 + g_5e_5 \\
[e_2, e_5] &= -e_1 \\
[e_3, e_4] &= h_1e_1 + h_2e_2 + h_3e_3 + h_4e_4 + h_5e_5
\end{aligned} \tag{6.1}$$

To enforce the remainder of the Jacobi identities, consider the Maurer Cartan forms, $\{\omega^i\}$ (where $\langle \omega^i, e_j \rangle = \delta^i_j$). The exterior derivative on the Maurer Cartan forms is given by $d\omega^k = -\frac{1}{2}{C_{ij}}^k \omega^i \wedge \omega^j$ and the Jacobi identities on the Lie algebra are equivalent to the integrability condition $d^2 \equiv 0$. Immediately, this condition requires that the structure equations take the following form:

$$
\begin{aligned}
[e_1, e_2] &= a_3e_3 + a_4e_4 + a_5e_5 \\
[e_1, e_3] &= b_1e_1 + b_2e_2 \\
[e_2, e_3] &= b_2e_1 + b_1e_2 \\
[e_1, e_5] &= -e_2 \\
[e_1, e_4] &= d_1e_1 + d_2e_2 \\
[e_2, e_4] &= d_2e_1 + d_1e_2 \\
[e_2, e_5] &= -e_1 \\
[e_3, e_4] &= h_3e_3 + h_4e_4 + h_5e_5
\end{aligned} \tag{6.2}
$$

Applying the change of basis $(e_1, e_2, e_3 + b_2e_5, e_4 + d_2e_5, e_5)$, we can take $b_2 = d_2 = 0$ without loss of generality. Then $d^2\omega^1 = 0$ forces $h_5 = 0$ as well, yielding the following Lie brackets:

$$
\begin{aligned}
[e_1, e_2] &= a_3e_3 + a_4e_4 + a_5e_5 \\
[e_1, e_3] &= b_1e_1 \\
[e_2, e_3] &= b_1e_2 \\
[e_1, e_5] &= -e_2 \\
[e_1, e_4] &= d_1e_1 \\
[e_2, e_4] &= d_1e_2 \\
[e_2, e_5] &= -e_1 \\
[e_3, e_4] &= h_3e_3 + h_4e_4
\end{aligned} \tag{6.3}
$$

Let $\mathfrak{n}$ be the centralizer of the isotropy subalgebra (note from the structure constants that $\mathfrak{n}$ always forms a subalgebra). We seek the largest subalgebra $\mathfrak{i}$ of $\mathfrak{n}$ that is also an ideal in $\mathfrak{g}_{F13}$. Let $x = \alpha e_1 + \beta e_2$ be an arbitrary linear combination of e_1 and e_2 and let $y = \mu e_3 + \lambda e_4 + \nu e_5$ be a fixed vector in $\mathfrak{n}$. The Lie bracket is given by

$$[x,y] = \left(\mu b_1 + \lambda d_1 - \frac{\beta}{\alpha}\nu\right)\alpha e_1 + \left(\mu b_1 + \lambda d_1 - \frac{\alpha}{\beta}\nu\right)\beta e_2. \tag{6.4}$$

If $[x,y] \in \mathfrak{n}$, then $\beta\nu - \alpha\nu = 0$. Since α and β are arbitrary, we require $\nu = 0$. Thus, $\mathfrak{i}$ must take the form $\mathfrak{i} = \{\mu e_3 + \lambda e_4 : \mu b_1 = -\lambda d_1\}$. If $b_1 = d_1 = 0$, then $\mathfrak{i}$ is a two-dimensional subalgebra of $\mathfrak{n}$, and furthermore is an ideal in $\mathfrak{g}_{F13}$. If either b_1 or d_1 is non-zero, then $\mathfrak{i}$ is at most one-dimensional and $\mathfrak{n}$ has no two-dimensional subalgebra that is also an ideal in $\mathfrak{g}_{F12}$. Therefore, $b_1 = d_1 = 0$ if and only if $\dim \mathfrak{i} = 2$.

6.1. The Dimension of $\mathfrak{i}$ is Two

If $\dim \mathfrak{i} = 2$, then $b_1 = d_1 = 0$ the Jacobi identity requires that either $a_3 = a_4 = 0$ or $h_3 = h_4 = 0$; this requirement ensures that the Jacobi identity is completely satisfied. These can be distinguished by the dimension of the center.

6.1.1. The Center is Trivial. If the center is trivial, then either h_3 or h_4 is non-zero and $a_3 = a_4 = 0$ and there is a basis in which the structure constants take the following form, with the Jacobi identity completely satisfied:

$$\begin{aligned} [e_1,e_2] &= a_5 e_5 \\ [e_1,e_5] &= -e_2 \\ [e_2,e_5] &= -e_1 \\ [e_3,e_4] &= e_3 \end{aligned} \tag{6.5}$$

By scaling e_1 and e_2 by $|a_5|^{-1/2}$, then interchanging if necessary, we may take $a_5 \in \{0,1\}$. The derived algebra is three-dimensional if and only if $a_5 = 0$. For $a_5 = 0$, the change of basis $(-e_1+e_2, e_1+e_2, e_5, e_3, e_4)$ gives the algebra pair $(F13,0)$ in standard form. For $a_5 = 1$, the change of basis $(e_1 - e_2, 2e_5, -e_1 - e_2, e_3, e_4)$ gives the algebra pair $(F13,1)$ in standard form.

6.1.2. The Center is not Trivial. If the center is not trivial, then it is two dimensional and $h_3 = h_4 = 0$. By considering the derived series, we find that $a_5 = 0$ if and only if $\mathfrak{g}_{F13}$ is solvable. Furthermore, when $a_5 = 0$, the derived algebra is two-dimensional if and only if $a_3 = a_4 = 0$. If $a_5 \neq 0$, scaling e_1 and e_2 by $|a_5|^{-1/2}$, scaling e_3 by $\frac{1}{a_5}$, and interchanging e_1 and e_2 if necessary allow us to take $a_5 = 1$. In this case, the isotropy is in the derived algebra if and only if $a_3 = a_4 = 0$. If either a_3 or a_4 is non-zero then with either the change of basis $(e_1, e_2, a_3e_3 + a_4e_4, e_3, e_5)$ or $(e_1, e_2, a_3e_3 + a_4e_4, e_4, e_5)$, we can take $a_3 = 1$ and $a_4 = 0$ without loss of generality. Thus, there are four inequivalent algebra-subalgebra pairs in this case. The changes of basis are summarized in Table 6.1.

a_5	a_3	Change of Basis	Pair Designation
0	0	$(-e_1 + e_2, -e_1 - e_2, e_5, e_3, e_4)$	$(F13, 2)$
0	1	$\left(e_3, e_1 - e_2, \frac{1}{2}e_1 + \frac{1}{2}e_2, -e_5, e_4\right)$	$(F13, 3)$
1	0	$(e_1 - e_2, 2e_5, -e_1 - e_2, e_3, e_4)$	$(F13, 4)$
1	1	$(e_1 - e_2, 2e_3 + 2e_5, -e_1 - e_2, e_3, e_4)$	$(F13, 5)$

TABLE 6.1. Summary of changes of basis to standard form for $\mathfrak{g}_{F13}$ when $\dim \mathfrak{i} \geq 2$ and the center is non-empty.

6.2. The Dimension of $\mathfrak{i}$ is Less than Two

If the dimension of $\mathfrak{i}$ is less than two, than at least one of b_1 and d_1 is non-zero. Applying the change of basis $\left(e_1, e_2, \frac{1}{b_1}e_3, e_4 - \frac{d_1}{b_1}e_3, e_5\right)$ when $b_1 \neq 0$, and the change of basis $\left(e_1, e_2, \frac{1}{d_1}e_4, e_3 - \frac{b_1}{d_1}e_4, e_5\right)$ when $b_1 = 0$ (and $d_1 \neq 0$) eliminates d_1 and sets b_1 to one. The Jacobi identities then require that the structure equations take on the following form, with appropriate relabeling of arbitrary constants:

$$
\begin{aligned}
[e_1, e_2] &= a_4e_4 \\
[e_1, e_3] &= e_1 \\
[e_2, e_3] &= e_2 \\
[e_1, e_5] &= e_2 \\
[e_2, e_5] &= -e_1 \\
[e_3, e_4] &= h_4e_4
\end{aligned} \tag{6.6}
$$

The Jacobi identities are fully satisfied if either $a_4 = 0$ or $h_4 = -2$. The derived series is given by $\mathfrak{g}^{(1)}_{F13} = \text{span}\{e_1, e_2, a_4e_4, h_4e_4\}$, then $\mathfrak{g}^{(2)}_{F13} = \text{span}\{a_4e_4\}$, and finally, $\mathfrak{g}^{(3)}_{F13} = \{0\}$. Thus $a_4 = 0$ if and only if $\dim\mathfrak{g}^{(2)}_{F13} = 0$ and in that case, $h_4 = 0$ if and only if $\dim\mathfrak{g}^{(1)}_{F13} = 2$. The changes of basis are summarized in Table 6.2.

a_4	h_4	Change of Basis	Pair Designation
$a_4 \neq 0$	-2	$\left(-2a_4e_4, e_1 + e_2, e_1 - e_2, -\frac{1}{2}e_3 - \frac{1}{2}e_5, -e_5\right)$	$(F13, 6)$
0	0	$\left(2e_1 - 2e_2, \frac{1}{2}e_3 + \frac{1}{2}e_5, 2e_1 + 2e_2, \frac{1}{2}e_3 - \frac{1}{2}e_5, e_4\right)$	$(F13, 7)$
0	$h_4 \neq 0$	$\left(e_1 + e_2, -e_1 - e_2, -2e_4, -\frac{1}{2}e_3 - \frac{1}{2}e_5, -\frac{1}{2}e_3 + \frac{1}{2}e_5\right)$	$P\,(F13, 8)\ (a = b = -\frac{h_4}{2})$

TABLE 6.2. Summary of changes of basis to standard form for $\mathfrak{g}_{F13}$ when $\dim \mathfrak{i} < 2$.

CHAPTER 7

F14: NULL ROTATIONS

The F14 subalgebra of $\mathfrak{so}(3,1)$ in standard coordinates and basis is given by the null rotation $\{R_y + K_z\}$ where

$$R_y + K_z \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $\mathfrak{g}_{F14}$ be a generic five-dimensional Lie algebra with basis $\{e_i\}$ and $ad\,(e_5) = (R_y + K_z) \oplus (0)$ so that $\{e_5\}$ is a subalgebra of isotropy type $F14$. The basis given by $(e_1, e_2, e_1 + e_3, e_4, e_5)$ is more convenient than the standard basis and yields

$$ad\,(e_5) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that the centralizer of the isotropy, $\mathrm{cent}_{\mathfrak{g}_{F14}}(e_5)$, is three-dimensional and spanned by $\{e_3, e_4, e_5\}$. Since $[e_4, e_5] = [e_3, e_5] = 0$, the Jacobi identity on these three basis elements reduces to $[e_5, [e_3, e_4]] = 0$ and thus, $[e_3, e_4] \in \mathrm{cent}_{\mathfrak{g}_{F14}}(e_5)$. Furthermore, since $[e_5, e_2] = e_3$, the Jacobi identity on $\{e_2, e_4, e_5\}$ reduces to $[e_3, e_4] = [e_5, [e_2, e_4]]$. Since $[e_5, [e_2, e_4]]$ cannot have any e_5 component, neither can $[e_3, e_4]$, and thus $\{e_3, e_4\}$ forms a two-dimensional subalgebra. There are only two two-dimensional algebras: the abelian algebra and the non-abelian algebra. This yields two cases: either $\mathrm{cent}_{\mathfrak{g}_{F14}}(e_5)$ is abelian, or $\mathrm{cent}_{\mathfrak{g}_{F14}}(e_5)$ is non-abelian.

7.1. The Centralizer of the Isotropy is Non-Abelian

If the centralizer of the isotropy is non-abelian, then there is a basis in which $[e_3, e_4] = e_3$ with the adjoint of e_5 left unchanged. The Jacobi identities then require that the structure constants are of the form

$$
\begin{aligned}
[e_1, e_2] &= a_2 e_2 + a_3 e_3 \\
[e_1, e_3] &= a_2 e_3 \\
[e_1, e_4] &= e_1 + d_2 e_2 + d_3 e_3 + a_2 e_4 - (a_2 d_2 + a_3)\, e_5 \\
[e_1, e_5] &= e_2 \\
[e_2, e_4] &= e_2 - d_2 e_3 \\
[e_2, e_5] &= -e_3 \\
[e_3, e_4] &= e_3
\end{aligned}
$$

with all other Lie brackets giving zero. The change of basis $(e_1 - a_3 e_5, e_2, e_3, e_4 - d_2 e_5, e_5)$ eliminates some extraneous structure constants and produces the following:

$$
\begin{aligned}
[e_1, e_2] &= a_2 e_2 \\
[e_1, e_3] &= a_2 e_3 \\
[e_1, e_4] &= e_1 + d_3 e_3 + a_2 e_4 \\
[e_1, e_5] &= e_2 \\
[e_2, e_4] &= e_2 \\
[e_2, e_5] &= -e_3 \\
[e_3, e_4] &= e_3
\end{aligned}
$$

Suppose $a_2 \neq 0$. Then the change of basis $\left(\frac{1}{a_2} e_1 + e_4, e_2, a_2 e_3, e_4, a_2 e_5\right)$ together with the relabeling $\frac{d_3}{a_2^2} \to d_3$ gives structure equations

$$\begin{aligned}
[e_1, e_4] &= e_1 + d_3 e_3 \\
[e_1, e_5] &= e_2 \\
[e_2, e_4] &= e_2 \\
[e_2, e_5] &= -e_3 \\
[e_3, e_4] &= e_3
\end{aligned}$$

and thus we may take $a_2 = 0$ without loss of generality. Consider the adjoint of any vector that is not in the nilradical, but is in the centralizer of the isotropy. In this basis, that is any vector with an e_4 component, but no e_1 or e_2 components. Restricted to the nilradical, the adjoint takes the form

$$A = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ \gamma & \alpha & 0 & 0 \\ d_3\alpha & \gamma & \alpha & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where α, and γ correspond to the components of e_4, and $e_5$respectively (so that $\alpha \neq 0$). Note that A can be chosen diagonalizable if and only if $d_3 = 0$ (by choosing $\gamma = 0$). If $d_3 = 0$, the change of basis $(-e_3, e_2, e_1, e_5, e_4)$ gives the algebra pair $(F14, 0)$ in standard form. If $d_3 \neq 0$, the change of basis
$\left(-|d_3|\, e_3, \sqrt{|d_3|}e_2, e_1, \sqrt{|d_3|}e_5, e_4\right)$ gives the algebra pair $(F14, 1)$ in standard form (where the sign of ϵ is the sign of d_3).

7.2. The Centralizer of the Isotropy is Abelian

If the centralizer of the isotropy is abelian, then $[e_3, e_4] = 0$. If $a_1 \neq 0$, the Jacobi identities require that $a_2 = d_3 = d_4 = d_5 = 0$ and the structure equations are of the following form:

$$
\begin{aligned}
[e_1, e_2] &= a_1 e_1 + a_3 e_3 + a_4 e_4 + a_5 e_5 \\
[e_1, e_3] &= a_1 e_2 \\
[e_1, e_4] &= d_2 e_2 \\
[e_1, e_5] &= e_2 \\
[e_2, e_3] &= a_1 e_3 \\
[e_2, e_4] &= -d_2 e_3 \\
[e_2, e_5] &= -e_3
\end{aligned}
$$

The change of basis

$$
\left(\frac{\sqrt{2}}{a_1}e_1 + \frac{a_3 a_1 + a_4 d_2 + a_5}{\sqrt{2}a_1^2}e_3 + \frac{a_4\sqrt{2}}{a_1^2}e_4 + \frac{a_5\sqrt{2}}{a_1^2}e_5, \frac{2}{a_1}e_2, \frac{\sqrt{2}}{a_1}e_3, -\frac{\sqrt{2}}{a_1}e_3 - \sqrt{2}e_5, e_4 - d_2 e_5\right)
$$

gives the algebra pair $(F14, 2)$ in standard form. This is the only $F14$ algebra pair with a two-dimensional abelian algebra that fully decomposes from the rest of the algebra. In all other cases, the Jacobi identities require that the structure constants are of the form

$$
\begin{aligned}
[e_1, e_2] &= a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 \\
[e_1, e_3] &= a_2 e_3 \\
[e_1, e_4] &= d_2 e_2 + d_3 e_3 + d_4 e_4 - d_2 d_4 e_5 \\
[e_1, e_5] &= e_2 \\
[e_2, e_4] &= -d_2 e_3 \\
[e_2, e_5] &= -e_3
\end{aligned}
$$

with all other Lie brackets giving zero. The change of basis $(e_1 - a_3 e_5, e_2, e_3, e_4 - d_2 e_5, e_5)$ together with the relabeling $a_4 d_2 + a_5 \to a_5$, eliminates some extraneous structure constants and produces the following:

$$
\begin{aligned}
[e_1, e_2] &= a_2 e_2 + a_4 e_4 + a_5 e_5 \\
[e_1, e_3] &= a_2 e_3 \\
[e_1, e_4] &= d_3 e_3 + d_4 e_4 \\
[e_1, e_5] &= e_2 \\
[e_2, e_5] &= -e_3
\end{aligned}
\tag{7.1}
$$

with all other Lie brackets giving zero. The derived algebra must contain e_2 and e_3, and so its dimension is given by two plus the rank of the matrix $\left(\begin{smallmatrix} a_4 & a_5 \\ d_4 & 0 \end{smallmatrix}\right)$. That is, the derived algebra is of two, three, or four dimensions, and the remaining algebra pairs are organized accordingly.

7.2.1. Derived Algebra is Two-Dimensional. If the derived algebra of $\mathfrak{g}_{F14}$ is two-dimensional, then $a_4 = a_5 = d_4 = 0$. The center, $C(\mathfrak{g}_{F14})$, cannot contain e_1, e_2, or e_5, but may contain e_3 or e_4 (or both), depending on the values of a_2 and d_3, respectively. Note that if the center does not contain e_3, then a_2 is non-zero and $\left[e_1, e_4 - \frac{d_3}{a_2} e_3\right] = 0$. Thus, the center must be either one-dimensional or two-dimensional.

If the center is one-dimensional, then either e_3 is in the center (and $a_2 \neq 0$) or it is (and $a_2 = 0$). These can be distinguished by the second algebra in the lower central series, given by $\mathfrak{g}^2_{F14} = [\mathfrak{g}_{F14}, [\mathfrak{g}_{F14}, \mathfrak{g}_{F14}]]$, which contains $a_2 e_2$ and $a_2 e_3$. The dimension of $\mathfrak{g}^2_{F14}$ is two if and only if $a_2 \neq 0$ (in which case $\mathfrak{g}^2_{F14} = \mathfrak{g}^1_{F14}$ and $\mathfrak{g}_{F14}$ is not nilpotent). In this case, the change of basis $\left(-\frac{1}{a_2} e_3, \frac{1}{a_2} e_2, -\frac{1}{a_2} e_2 + e_5, -\frac{1}{a_2} e_1, -e_4 - \frac{d_3}{a_2} e_3\right)$ gives the algebra pair $(F14, 3)$ in standard form. The dimension of $\mathfrak{g}^2_{F14}$ is zero if and only if $a_2 = 0$ (in which case $\mathfrak{g}_{F14}$ is nilpotent). The change of basis $\left(-e_3, e_2 + e_3, \frac{1}{d_3} e_4, e_1 - e_2, e_5\right)$ gives the algebra pair $(F14, 4)$ in standard form.

If, on the other hand, the center is two-dimensional, then the change of basis $(-e_3, e_2 + e_3, e_1 - e_2, e_4, e_5)$ gives the algebra pair $(F14, 5)$ in standard form.

7.2.2. Derived Algebra is Three-Dimensional. If the derived algebra of $\mathfrak{g}_{F14}$ is three-dimensional, then the rank of $\left(\begin{smallmatrix} a_4 & a_5 \\ d_4 & 0 \end{smallmatrix}\right)$ is one and either $a_5 = 0$ and at least one of a_4 and d_4 is non-zero or $a_5 \neq 0$ and $d_4 = 0$. If $a_5 = 0$, then the derived algebra of $\mathfrak{g}_{F14}$ is spanned by

$\{e_2, e_3, e_4\}$, which is abelian (see Equation 7.1). If $a_5 \neq 0$, then the derived algebra is spanned by $\{e_2, e_3, a_4e_4 + a_5e_5\}$, which is non-abelian because $[a_4e_4 + a_5e_5, e_2] = -a_5e_2$. Thus, the derived algebra is abelian if and only if $a_5 = 0$.

7.2.2.1. *Derived Algebra is Abelian.* If the derived algebra is abelian, $a_5 = 0$ and the structure equations are given by

$$\begin{aligned} [e_1, e_2] &= a_2e_2 + a_4e_4 \\ [e_1, e_3] &= a_2e_3 \\ [e_1, e_4] &= d_3e_3 + d_4e_4 \\ [e_1, e_5] &= e_2 \\ [e_2, e_5] &= -e_3 \end{aligned} \tag{7.2}$$

where one or both of a_4 and d_4 is nonzero. The center, $C(\mathfrak{g}_{F14})$, cannot contain e_1, e_2, or e_5 and may be two-dimensional, one-dimensional, or trivial. If e_3 and e_4 are both in the center, then $a_2 = d_3 = d_4 = 0$ and a_4 is necessarily nonzero, so scaling e_4 by a_4 gives the following structure equations

$$\begin{aligned} [e_1, e_2] &= e_4 \\ [e_1, e_5] &= e_2 \\ [e_2, e_5] &= -e_3 \end{aligned}$$

The change of basis $(e_3, -e_4, -e_2, -e_1, e_5)$ then gives the algebra pair $(F14, 6)$ in standard form.

If the center is one-dimensional, then we consider the lower central series: The second and third algebras in the lower central series are $\mathfrak{g}^2_{F14} \equiv [\mathfrak{g}_{F14}, [\mathfrak{g}_{F14}, \mathfrak{g}_{F14}]] = \operatorname{span}\{a_2e_2 + a_4e_4, e_3, d_4e_4\}$. This algebra commutes with the isotropy if and only if $a_2 = 0$.

Consider first the case in which the center is one-dimensional and $\mathfrak{g}^2_{F14}$ does not commute with the isotropy (i.e., $a_2 \neq 0$). Since the center is one-dimensional, there is a one-dimensional subspace, $Se_3 + Te_4$, that commutes with e_1, i.e., $(Sa_2 + Td_3)\, e_3 + Td_4e_4 = 0$, so that $S = -\frac{d_3}{a_2}T$ and $Td_4 = 0$.

Thus, if $d_4 \neq 0$, then the center is trivial, a case considered elsewhere. Therefore, here we require $d_4 = 0$ (and then $a_4 \neq 0$ since the derived algebra is three-dimensional and abelian). In this case, the change of basis

$$\left(-\frac{1}{a_2}e_3, \frac{1}{a_2}e_2 - \frac{a_4 d_3}{a_2^3}e_3 + \frac{a_4}{a_2^2}e_4, -\frac{1}{a_2}e_2 + \frac{a_4 d_3 - a_2^2}{a_2^3}e_3 - \frac{a_4}{a_2^2}e_4 + e_5, \frac{1}{a_2}e_1 - \frac{1}{a_2}e_2 - \frac{a_4 d_3}{a_2^2}e_5\right)$$

gives the algebra pair $(F14, 7)$ in standard form.

If the center is one-dimensional and $\mathfrak{g}^2_{F14}$ commutes with the isotropy, $\mathfrak{g}^3_{F14}$ is given by the span of $\{d_4 e_4 + d_3 e_3\}$ (since $a_2 = 0$ in this case). Then $\mathfrak{g}^4_{F14} \equiv \left[\mathfrak{g}_{F14}, \mathfrak{g}^3_{F14}\right] = \operatorname{span}\left\{d_3 d_4 e_3 + d_4^2 e_4\right\}$ and $\mathfrak{g}^4_{F14}$ is trivial if and only if $d_4 = 0$, and one-dimensional otherwise (and again, $a_4 \neq 0$ since the derived algebra is three-dimensional and abelian). If $\mathfrak{g}^4_{F14}$ is trivial, then d_3 is nonzero; otherwise we have the case of two-dimensional center treated earlier. Applying the change of basis $\left(\frac{1}{\sqrt{a_4 d_3}}e_3, -\frac{1}{d_3}e_4, -\frac{1}{\sqrt{a_4 d_3}}e_2, e_5, \frac{1}{\sqrt{a_4 d_3}}e_1\right)$ for $a_4 d_3$ positive or $\left(\frac{1}{\sqrt{-a_4 d_3}}e_3, -\frac{1}{d_3}e_4, \frac{1}{\sqrt{-a_4 d_3}}e_2, -e_5, \frac{1}{\sqrt{-a_4 d_3}}e_1\right)$ for $a_4 d_3$ negative gives the algebra in standard form. In both cases, the algebra pair has identification $(F14, 8)$. If, on the other hand, $\mathfrak{g}^4_{F14}$ is one-dimensional, then $d_4 \neq 0$. The parameter a_4 may be zero or non-zero, which determines whether or not the isotropy is in the terminal algebra of the upper central series. First consider the case in which $a_4 \neq 0$. The change of basis

$$\left(\frac{1}{d_4}e_3, \frac{1}{d_4}e_2 - \frac{a_4 d_3}{d_4^3}e_3 - \frac{a_4}{d_4^2}e_4, -\frac{1}{d_4}e_2 + e_5, \frac{a_4 d_3}{d_4^3}e_3 + \frac{a_4}{d_4^2}e_4, \frac{1}{d_4}e_1 + \frac{a_4 d_3}{d_4^2}e_5\right)$$

gives the algebra pair $(F14, 9)$ in standard form. If $a_4 = 0$, the algebra pair can be written in standard form with identification $(F14, 10)$ using the change of basis $\left(\frac{1}{d_4}e_3, \frac{1}{d_4}e_2, \frac{1}{d_4}e_3 + e_5, \frac{d_3}{d_4^2}e_3 + \frac{1}{d_4}e_4, \frac{1}{d_4}e_1\right)$.

Next, if the center is trivial, d_4 and a_2 are both nonzero and we consider vectors not in the nilradical. In the basis given by the structure equations in Equations 7.2 these vectors, $X(\alpha, \beta, \gamma)$, have adjoint matrices restricted to the nilradical of the form

$$ad\left(X(\alpha, \beta, \gamma)\right) = \begin{pmatrix} \alpha a_2 & 0 & 0 & \alpha \\ \gamma & \alpha a_2 & \alpha d_3 & -\beta \\ \alpha a_4 & 0 & \alpha d_4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where α, β, and γ are parameters determined choice of vector (α is the e_1 component and may not be zero). The eigenvalues of $ad\left(X(\alpha, \beta, \gamma)\right)$ are zero with multiplicity one, αa_2 with multiplicity two, and αd_4 with multiplicity one. We classify the algebra pair according to properties of this family of matrices, as summarized in Table 7.1.

Pair Designation	Properties of $ad\left(X\left(\alpha,\beta,\gamma\right)\right)$	Parameters
$(F14,11)$	Two distinct nonzero eigenvalues. X can be chosen such that $ad\left(X\right)$ on the isotropy is an eigenvector of $ad\left(X\left(\alpha,\beta,\gamma\right)\right)$	$a_2 \neq d_4,\ a_4 \neq 0$
$(F14,12)$	Two distinct nonzero eigenvalues. X cannot be chosen such that $ad\left(X\right)$ on the isotropy is an eigenvector of $ad\left(X\left(\alpha,\beta,\gamma\right)\right)$	$a_2 \neq d_4,\ a_4 = 0$
$(F14,13)$	One nonzero eigenvalue, αa_2. Rank of $ad\left(X\right) - \alpha a_2 I$ is three, regardless of (α,β,γ).	$a_2 = d_4,\ a_4, d_3 \neq 0$
$(F14,14)$	One nonzero eigenvalue, αa_2. Rank of $ad\left(X\right) - \alpha a_2 I$ is two or three, depending on choice of (α,β,γ).	$a_2 = d_4,\ a_4 = 0,\ d_3 \neq 0$
$(F14,15)$	One nonzero eigenvalue, αa_2. Rank of $ad\left(X\right) - \alpha a_2 I$ is two, regardless of (α,β,γ).	$a_2 = d_4,\ a_4 \neq 0,\ d_3 = 0$
$(F14,16)$	One nonzero eigenvalue, αa_2. Rank of $ad\left(X\right) - \alpha a_2 I$ is one or two, depending on choice of (α,β,γ).	$a_2 = d_4,\ a_4 = d_3 = 0$

Pair Designation	Change of Basis
$(F14,11)$	$\left(-\frac{1}{a_2}e_3, \frac{1}{a_2}e_2 - \frac{d_4}{a_2}\tilde{e}_4, -\frac{1}{a_2}e_2 + \frac{d_4}{a_2}\tilde{e}_4 + e_5, \tilde{e}_4, \frac{1}{a_2}e_1 - \frac{a_4 d_3}{a_2^2 - a_2 d_4}e_5\right)$ $\tilde{e}_4 = \frac{a_4 d_3}{d_4(a_2-d_4)^2}e_3 - \frac{a_4}{d_4(a_2-d_4)}e_4$
$(F14,12)$	$\left(-\frac{1}{a_2}e_3, \frac{1}{a_2}e_2, -\frac{1}{a_2}e_2 + e_5, \frac{1}{d_4(a_2-d_4)}\left(\frac{d_3}{a_2-d_4}e_3 - e_4\right), \frac{1}{a_2}e_1\right)$
$(F14,13)$	$\left(\frac{a_4^2 d_3^2}{a_2^5}e_3, -\tilde{e}_3 + \frac{a_4 d_3}{a_2^2}e_5, \tilde{e}_3, \frac{a_4 d_3}{a_2^3}e_2 - \tilde{e}_3, \frac{1}{a_2}e_1 - \frac{a_4 d_3}{a_2^2}e_5\right)$ $\tilde{e}_3 = \frac{a_4 d_3}{a_2^3}e_2 + \frac{d_3^2 a_4^2(a_2+1)}{a_2^5}e_3 - \frac{a_4^2 d_3}{a_2^4}e_4$
$(F14,14)$	$\left(-\frac{1}{a_2}e_3, \frac{1}{a_2}e_2, -\frac{1}{a_2}e_2 + e_5, \frac{a_2}{d_3}e_3 - \frac{1}{d_3}e_4, \frac{1}{a_2}e_1\right)$
$(F14,15)$	$\left(-\frac{1}{a_2}e_3, -\frac{1}{a_2}e_2 + \frac{a_4}{a_2^2}e_4 + e_5, \frac{1}{a_2}e_2 - \frac{a_4}{a_2^2}e_4, \frac{a_4}{a_2^2}e_4, \frac{1}{a_2}e_1\right)$
$(F14,16)$	$\left(-\frac{1}{a_2}e_3, \frac{1}{a_2}e_2, -\frac{1}{a_2}e_2 + e_5, e_4, \frac{1}{a_2}e_1\right)$

TABLE 7.1. Summary of invariants and changes of basis to standard form for $\mathfrak{g}_{F14}$ when the derived algebra is three-dimensional abelian with trivial center.

7.2.2.2. *Derived Algebra is Non-Abelian.* If the derived algebra is three-dimensional and abelian, $a_5 \neq 0$ and $d_4 = 0$. The structure equations are given by

$$\begin{aligned} [e_1, e_2] &= a_2 e_2 + a_4 e_4 + a_5 e_5 \\ [e_1, e_3] &= a_2 e_3 \\ [e_1, e_4] &= d_3 e_3 \\ [e_1, e_5] &= e_2 \\ [e_2, e_5] &= -e_3. \end{aligned} \tag{7.3}$$

In this case, the center, $C(\mathfrak{g}_{F14})$, cannot contain e_1, e_2, or e_5. The vector $Se_3 + Te_4$ commutes with all basis vectors except perhaps e_1 and $[e_1, Se_3 + Te_4] = (Sa_2 + Td_3)\, e_3$. Note that if $a_2 = d_3 = 0$, the center is two-dimensional, and otherwise is one-dimensional.

If the center is two-dimensional, the change of basis

$$\left(|a_5|^{-1/2}\, e_1, a_5\, |a_5|^{-1/2}\, e_2, |a_5|^{3/2}\, e_3, e_4, a_4 e_4 + a_5 e_5\right)$$

yields the following structure constants:

$$\begin{aligned} [e_1, e_2] &= \pm e_5 \\ [e_1, e_5] &= e_2 \\ [e_2, e_5] &= -e_3 \end{aligned} \tag{7.4}$$

with the $\pm$ corresponding to the sign of a_5.

Claim 69. The sign of a_5 is essential (i.e., the positive branch cannot be related to the negative branch by a real isomorphism).

Proof. Note that $\mathfrak{g}_{F14} = \tilde{\mathfrak{g}}_{F14} \oplus \text{span}\,\{e_4\}$ where $\tilde{\mathfrak{g}}_{F14} \equiv \text{span}\,\{e_1, e_2, e_3, e_5\}$. The center of $\tilde{\mathfrak{g}}_{F14}$ is spanned by $\{e_3\}$ and the quotient of $\tilde{\mathfrak{g}}_{F14}$ by $\mathfrak{h} \equiv \text{span}\,\{e_3\}$ is spanned by $\{e_1 + \mathfrak{h}, e_2 + \mathfrak{h}, e_5 + \mathfrak{h}\}$. For convenience, let $(\epsilon_1, \epsilon_2, \epsilon_3) \equiv (e_1 + \mathfrak{h}, e_2 + \mathfrak{h}, e_5 + \mathfrak{h})$ so that the only non-zero Lie brackets are $[\epsilon_1, \epsilon_2] = \pm\epsilon_3$ and $[\epsilon_1, \epsilon_3] = \epsilon_2$. In this basis, the Killing form is given by $B = \left(\begin{smallmatrix} \pm 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ and thus the signature of B is determined by the sign of a_5. □

In the basis given with structure equations given by Equations 7.4, the isotropy subalgebra is spanned by $-a_4e_4 + e_5$ and is a subalgebra of the derived algebra if and only if $a_4 = 0$, in which case the isotropy subalgebra is given by e_5. Otherwise, the automorphism $(e_1, e_2, e_3, -a_4e_4, e_5) \to (e_1, e_2, e_3, e_4, e_5)$ is non-degenerate and leaves the structure equations unchanged, yielding the isotropy $e_4 + e_5$. For the case where a_5 is positive, the change of basis $\left(-e_3, -e_2 + e_5, -\frac{1}{2}e_2 - \frac{1}{2}e_5, -e_1, e_4\right)$ (on the basis given in Equations 7.4) gives $\mathfrak{g}_{F14}$ and the isotropy in standard form with identification $(F14, 17)$ if the isotropy is in the derived algebra and $(F14, 18)$ otherwise. For the case where a_5 is negative, the change of basis $(e_3, e_2, -e_5, -e_1, e_4)$ (on the basis given in Equations 7.4) gives $\mathfrak{g}_{F14}$ and the isotropy in standard form with identification $(F14, 19)$ if the isotropy is in the derived algebra and $(F14, 20)$ otherwise.

If the center of $\mathfrak{g}_{F14}$ is one-dimensional, then at least one of a_2 and d_3 is non-zero. The center is spanned by $Se_3 + Te_4$ where $(Sa_2 + Td_3) = 0$. Suppose $a_2 = 0$. Then $T = 0$ and the center spanned by e_3, and the second algebra in the upper central series is spanned by e_3 and e_4. If $a_2 \neq 0$, then the center has an e_4 component and the upper central series terminates with the center. Therefore, the second algebra in the upper central series is two-dimensional (spanned by e_3 and e_4) if and only if $a_2 = 0$, (requiring $d_3 \neq 0$), and the change of basis

$$\left(\frac{1}{\sqrt{|a_5|}}e_3, \frac{1}{\sqrt{|a_5|}}e_2, -\frac{a_4}{a_5}e_4 - e_5, \frac{1}{d_3}e_4, \frac{1}{\sqrt{|a_5|}}e_1 + \frac{a_4d_3}{a_5\sqrt{|a_5|}}e_2\right)$$

gives structure equations

$$\begin{aligned}
[e_2, e_3] &= e_1 \\
[e_2, e_5] &= \pm e_3 \\
[e_3, e_5] &= e_2 \\
[e_4, e_5] &= e_1
\end{aligned}$$

with the $\pm$ corresponding to the sign of a_5 and isotropy now spanned by $e_3 + \frac{a_4d_3}{a_5}e_4$. Note that the isotropy is in the derived algebra if and only if $a_4 = 0$, in which case the isotropy is spanned by e_3. If $a_4 \neq 0$, then the isotropy to be taken to be $e_3 + e_4$ via the automorphism given by the change of basis $\left(\left(\frac{a_4 d_3}{a_5}\right)^2 e_1, \frac{a_4 d_3}{a_5} e_2, \frac{a_4 d_3}{a_5} e_3, \left(\frac{a_4 d_3}{a_5}\right)^2 e_4, e_5\right)$. The sign of a_5 determines the signature of the Killing form, which has only one non-zero eigenvalue. If $a_5 < 0$, the algebra is in standard form with identification $(F14, 21)$ if the isotropy is in the derived algebra, and $(F14, 22)$ otherwise. If $a_5 > 0$, then the change of basis $(-2e_1, e_2 + e_3, e_2 - e_3, -2e_4, e_5)$ gives the algebra pair in standard form with identification $(F14, 23)$ if the isotropy is in the derived algebra, and $(F14, 24)$ otherwise.

If the second term in the upper central series is not two-dimensional, then $a_2 \neq 0$. The isotropy is in the derived algebra if and only if $a_4 = 0$, in which case the change of basis $\left(\frac{1}{a_2} e_1, \frac{1}{a_2} e_2, \frac{1}{a_2} e_3, \frac{d_3}{a_2} e_3 - e_4, e_5\right)$ applied to the basis given by Equations 7.3 together with relabeling $\frac{a_4}{a_2^2} \to a_4$ and $\frac{a_5}{a_2^2} \to a_5$ gives the structure equations presented in Equations 7.5. Otherwise, the change of basis $\left(\frac{1}{a_2} e_1 - \frac{d_3 a_4}{a_2^2} e_5, \frac{1}{a_2} e_2, \frac{1}{a_2} e_3, \frac{a_4 d_3}{a_2 a_5} e_3 - \frac{a_4}{a_5} e_4, -\frac{a_4 d_3}{a_2 a_5} e_3 + \frac{a_4}{a_5} e_4 + e_5\right)$ yields the same structure equations with isotropy $e_4 + e_5$.

$$\begin{aligned} [e_1, e_2] &= e_2 + a_5 e_5 \\ [e_1, e_3] &= e_3 \\ [e_1, e_5] &= e_2 \\ [e_2, e_5] &= -e_3. \end{aligned} \tag{7.5}$$

The nilradical is spanned by $\{e_2, e_3, e_4, e_5\}$ so any vector not in the nilradical has an e_1 component of magnitude λ and its adjoint restricted to the nilradical has eigenvalues zero, λ, and $\lambda\left(\frac{1 \pm \sqrt{1+4a_5}}{2}\right)$, each with multiplicity one, since $a_5 \neq 0$ in this case. Note that if $a_5 = -\frac{1}{4}$, then $\frac{\lambda}{2}$ is an eigenvalue of multiplicity two. In this case, the change of basis $\left(-e_3, \sqrt{2} e_2 - \frac{\sqrt{2}}{2} e_5, \frac{1}{\sqrt{2}} e_5, 2e_1, \frac{1}{\sqrt{2}} e_4\right)$ gives the algebra pair in standard form with identification $(F14, 25)$ if the isotropy is in the derived algebra and $(F14, 26)$ otherwise. If the eigenvalues are real and distinct, then $a_5 > -\frac{1}{4}$ and the change of basis $\left(\frac{a-1}{1+a} e_3, e_2 - \frac{a}{1+a} e_5, e_2 - \frac{1}{1+a} e_5, (1+a)\, e_1, \frac{1-a}{1+a} e_4\right)$, with $a_5 = \frac{-a}{(1+a)^2}$, gives the algebra pair in standard form with identification $(F14, 27)$ if the isotropy is in the derived algebra and $(F14, 28)$ otherwise. Finally, if $\lambda\left(\frac{1 \pm \sqrt{1+4a_5}}{2}\right)$ are non-real, then $a_5 < -\frac{1}{4}$ and the change of basis $\left(-\frac{(\alpha^2+1)^2}{2\alpha^3} e_3, -\frac{\alpha^2+1}{2\alpha^2} e_5, \frac{\alpha^2+1}{\alpha} e_2 - \frac{\alpha^2+1}{2\alpha} e_5, 2\alpha e_1, -\frac{\alpha^2+1}{2\alpha^2} e_4\right)$ with $\alpha = \frac{1}{\sqrt{1+4a_5}}$ gives the algebra pair in standard form with identification $(F14, 29)$ if the isotropy is in the derived algebra and $(F14, 30)$ otherwise.

7.2.3. Derived Algebra is Four-Dimensional. If the derived algebra of $\mathfrak{g}_{F14}$ is four-dimensional, then the rank of $\left(\begin{smallmatrix} a_4 & a_5 \\ d_4 & 0 \end{smallmatrix}\right)$ is two, so a_5 and d_4 are both non-zero. From Equations 7.1, it is clear that e_3 is the only basis vector that could be in the center of $\mathfrak{g}_{F14}$, depending on the value of a_2. That is, the center is trivial if and only if $a_2 \neq 0$.

7.2.3.1. *The Center is One-Dimensional.* If the center is one-dimensional, then $a_2 = 0$ and the structure equations are given by

$$\begin{aligned} [e_1, e_2] &= a_4 e_4 + a_5 e_5 \\ [e_1, e_4] &= d_3 e_3 + d_4 e_4 \\ [e_1, e_5] &= e_2 \\ [e_2, e_5] &= -e_3 \end{aligned} \tag{7.6}$$

where again, a_5 and d_4 are non-zero. Consider an arbitrary vector X not in the nilradical (which is spanned by e_2 through e_4). Its adjoint on the nilradical is of the form

$$ad\left(X\left(\alpha, \beta, \gamma\right)\right) = \begin{pmatrix} 0 & 0 & 0 & \lambda \\ \gamma & 0 & \lambda d_3 & -\beta \\ \lambda a_4 & 0 & \lambda d_4 & 0 \\ \lambda a_5 & 0 & 0 & 0 \end{pmatrix}$$

where λ, β, and γ are the components in the e_1, e_2, and e_5 directions respectively. Then $[X, [X, e_5]] = \lambda\left(\gamma e_3 + \lambda a_4 e_4 + \lambda a_5 e_5\right)$. Thus there is a choice for X such that its bracket with the isotropy is an eigenvector of its adjoint if and only if $a_4 = 0$ (i.e., choose $\gamma = 0$). The eigenvalues of its adjoint are independent of β and γ and are 0,λd_4, and $\pm\lambda\sqrt{a_5}$.

7.2.3.2. *The Center is Trivial.* If the center is trivial, then the structure equations are given by Equations 7.1 with a_2, d_4, and a_5 non-zero. Using the change of basis $\left(\frac{1}{a_2}e_1, \frac{1}{a_2^2}e_2, \frac{1}{a_2^3}e_3, \frac{1}{a_2^3}e_4, \frac{1}{a_2}e_5\right)$ together with the relabeling of constants $\frac{a_5}{a_2^2} \to a_5$, $\frac{d_3}{a_2} \to d_3$, and $\frac{d_4}{a_2} \to d_4$, the parameter a_2, without

Pair Designation	Properties of $ad\,(X\,(\lambda,\beta,\gamma))$	Parameters
$(F14,31)$	Has two imaginary eigenvalues. X can be chosen such that its bracket with the isotropy gives an eigenvector of $ad\,(X)$.	$a_4=0,\ a_5<0$
$(F14,32)$	Has two imaginary eigenvalues. X cannot be chosen such that its bracket with the isotropy gives an eigenvector of $ad\,(X)$.	$a_4\neq 0,\ a_5<0$
$(F14,33)$	Has four real distinct eigenvalues. X can be chosen such that its bracket with the isotropy gives an eigenvector of $ad\,(X)$.	$a_4=0,\ a_5>0$
$(F14,34)$	Has four real distinct eigenvalues. X cannot be chosen such that its bracket with the isotropy gives an eigenvector of $ad\,(X)$.	$a_4\neq 0,\ a_5>0$
$(F14,35)$	Has repeated eigenvalues. X can be chosen such that its bracket with the isotropy gives an eigenvector of $ad\,(X)$.	$a_4=0,\ a_5=0$
$(F14,36)$	Has repeated eigenvalues. X cannot be chosen such that its bracket with the isotropy gives an eigenvector of $ad\,(X)$.	$a_4\neq 0,\ a_5=0$

Pair Designation	Change of Basis
$(F14,31)$	$\left(-\frac{d_4^4}{a_5\sqrt{-a_5}}e_3, \frac{d_4^2}{a_5}e_2, \frac{d_4^2}{\sqrt{-a_5}}e_5, -d_3e_3-d_4e_4, \frac{1}{\sqrt{-a_5}}e_1\right),\ \alpha=\frac{d_4}{\sqrt{-a_5}}$
$(F14,32)$	$\left(-\frac{d_4^4}{a_5\sqrt{-a_5}}e_3, \frac{d_4^2}{a_5}e_2-\frac{d_4}{\sqrt{-a_5}}\tilde{e}_4, \tilde{e}_4+\frac{d_4^2}{\sqrt{-a_5}}e_5, \tilde{e}_4, \frac{1}{\sqrt{-a_5}}e_1+\frac{a_4d_3}{d_4\sqrt{-a_5}}e_5\right),$ $\tilde{e}_4=\frac{a_4d_4}{\left(a_5-d_4^2\right)\sqrt{-a_5}}\left(d_3e_3+d_4e_4\right),\ \alpha=\frac{d_4}{\sqrt{-a_5}}$
$(F14,33)$	$\left(\frac{2d_4^4}{a_5\sqrt{a_5}}e_3, \frac{d_4^2}{a_5}e_2+\frac{d_4^2}{\sqrt{a_5}}e_5, \frac{d_4^2}{a_5}e_2-\frac{d_4^2}{\sqrt{a_5}}e_5, -d_3e_3-d_4e_4, \frac{1}{\sqrt{a_5}}e_1\right),\ \alpha=\frac{d_4}{\sqrt{a_5}}$
$(F14,34)$	$\left(\frac{2}{\sqrt{a_5}}e_3, \frac{1}{\sqrt{a_5}}e_2+\alpha\tilde{e}_4+e_5, \frac{1}{\sqrt{a_5}}e_2+(\alpha-1)\,\tilde{e}_4-e_5, \tilde{e}_4, \frac{1}{\sqrt{a_5}}e_1+\frac{a_4d_3}{d_4\sqrt{a_5}}e_5\right),$ $\tilde{e}_4=\frac{2a_4}{\left(a_5-d_4^2\right)d_4}\left(d_3e_3+d_4e_4\right),\ \alpha=\frac{d_4}{\sqrt{a_5}}$
$(F14,35)$	$\left(2d_4e_3, e_2+d_4y_5, e_2-d_4y_5, -d_3e_3-d_4e_4, \frac{1}{d_4}e_1\right)$
$(F14,36)$	$\left(-d_4e_3, e_2-\tilde{e}_4-d_4y_5, e_2+d_4y_5, \tilde{e}_4, \frac{1}{d_4}e_1+\frac{a_4d_3}{d_4^2}e_5\right),$ $\tilde{e}_4=\frac{a_4}{2d_4^2}\left(d_3e_3+d_4e_4\right)$

TABLE 7.2. Summary of invariants and changes of basis to standard form for $\mathfrak{g}_{F14}$ when the derived algebra is four-dimensional with one-dimensional center.

loss of generality, can be set to one, giving the following structure equations:

$$\begin{aligned}
[e_1, e_2] &= e_2 + a_4 e_4 + a_5 e_5 \\
[e_1, e_3] &= e_3 \\
[e_1, e_4] &= d_3 e_3 + d_4 e_4 \\
[e_1, e_5] &= e_2 \\
[e_2, e_5] &= -e_3.
\end{aligned} \tag{7.7}$$

The nilradical is spanned by $\{e_2, e_3, e_4, e_5\}$. The parameter a_4 may or may not affect the algebra classification, depending on other parameter values, but it does determine an invariant of the isotropy subalgebra. Consider the isotropy, spanned by e_5, together with any vector not in the nilradical, w_1. Then $w_2 = [e_5, w_1]$ is necessarily in the nilradical and has an e_2 component (possibly also an e_3 component, but no others); therefore it is not in the span of w_1 and e_5. Since $[e_5, w_2]$ is proportional to e_3, any subalgebra containing the isotropy that is not a subalgebra of the nilradical must contain span $\{e_5, w_1, w_2, e_3\}$. Since $[w_1, e_3]$ is always proportional to e_3, this vector space forms an algebra if and only if $[w_1, e_2]$ stays within span $\{e_5, w_1, w_2, e_3\}$, which requires $a_4 = 0$. Therefore, there is a four-dimensional subalgebra containing the isotropy and not equal to the nilradical if and only if $a_4 = 0$.

Let A be the adjoint matrix of any vector not in the nilradical restricted to the nilradical itself. This matrix is necessarily similar to

$$A = \lambda \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & d_3 & 0 \\ a_4 & 0 & d_4 & 0 \\ a_5 & 0 & 0 & 0 \end{pmatrix}$$

where λ is a proportionality parameter determined by the e_1 component. (The inclusion of components other than e_1 produce non-zero entries in the first and fourth columns of the second row, but these can be removed via a straightforward similarity transformation, see Appendix B.5 for details.) Note that A has eigenvalues in $\left\{\lambda, \lambda d_4, \lambda\left(\frac{1\pm\sqrt{1+4a_5}}{2}\right)\right\}$. First note that there are non-real eigenvalues of A if and only if $a_5 < -\frac{1}{4}$; we shall use this fact to identify invariant algebra pairs. The eigenvalues are distinct only if $d_4 \neq 1$, $a_5 \neq -1/4$, and $a_5 \neq d_4^2 - d_4$. There are two distinct and two repeated eigenvalues when one of the following holds:

(1) $d_4 \neq 1, \frac{1}{2}$, $a_5 = -\frac{1}{4}$

(2) $a_5 = d_4^2 - d_4 \neq -\frac{1}{4}$ (i.e., $d_4 \neq \frac{1}{2}$).

(3) $d_4 = 1$, $a_5 \neq -\frac{1}{4}$

In the second case, the repeated eigenvalue is $\mu_2 = \frac{\lambda}{2}$ and has associated with it only one eigenvector, $2e_2 + \frac{8d_3a_4}{2d_4-1}e_3 - \frac{4a_4}{2d_4-1}e_4 - e_5$. In the second case, the repeated eigenvalue is $\mu_3 = \lambda d_4$and $\frac{d_3}{d_4-1}e_3 + e_4$ is always an eigenvector associated with μ_3. In the case when $a_4 = 0$ (as determined above), $e_2 + (d_4 - 1)\, e_5$ is also an eigenvector associated with μ_3. In the third case, let the repeated eigenvalue be $\mu_1 = \lambda$. The vector e_3 is always an eigenvector associated with μ_1, and in the case when $d_3 = 0$, e_4 is also an eigenvector associated with μ_1. Note especially that the derived algebra of the nilradical (spanned by e_3) is always in the eigenspace.

Since a_5 is nonzero, the only case in which and eigenvalue has algebraic multiplicity three is when $d_4 = \frac{1}{2}$ and $a_5 = -1/4$ and there are two eigenvalues of multiplicity two only if $d_4 = 1$ and $a_5 = -\frac{1}{4}$. In the latter case, the eigenspace is three-dimensional if $d_3 = 0$ and two-dimensional otherwise. When the eigenspace is three dimensional, it contains the center of the nilradical, otherwise, it does not. In both cases, the eigenspace contains the derived algebra of the center. It is not possible for all eigenvalues to be equal. We summarize the invariants in Table 7.3 with $\mathfrak{d} = \text{span}\{e_3\}$ and $\mathfrak{c} = \text{span}\{e_3, e_4\}$ being the derived algebra of the nilradical and the center of the nilradical respectively, for conciseness.

TABLE 7.3. Summary of invariant characteristics for the $F14$ algebra-subalgebra pairs with four-dimensional derived algebras and trivial centers.

Case ID	Invariant Characteristics	Parameter Values
$(F14, 37)$	Two non-real eigenvalues, two distinct, $a_4 \neq 0$	$d_4 \neq 1$, $a_5 < -\frac{1}{4}$, $a_4 \neq 0$
$(F14, 38)$	Two non-real eigenvalues, two distinct, $a_4 = 0$	$d_4 \neq 1$, $a_5 < -\frac{1}{4}$, $a_4 = 0$
$(F14, 39)$	Four distinct real eigenvalues, $a_4 \neq 0$	$d_4 \neq 1$, $a_5 > -\frac{1}{4}$, $a_5 \neq d_4^2 - d_4$, $a_4 \neq 0$
$(F14, 40)$	Four distinct real eigenvalues, $a_4 = 0$	$d_4 \neq 1$, $a_5 > -\frac{1}{4}$, $a_5 \neq d_4^2 - d_4$, $a_4 = 0$
$(F14, 41)$	One eigenvalue μ with algebraic multiplicity two and geometric multiplicity one. The eigenspace corresponding to μ is not in $\mathfrak{c}$ or $\mathfrak{d}$. $a_4 \neq 0$. (case 1)	$d_4 \neq 1, \frac{1}{2}$, $a_5 = -\frac{1}{4}$, $a_4 \neq 0$

Case ID	Invariant Characteristics	Parameter Values
$(F14,42)$	One eigenvalue μ with algebraic multiplicity two and geometric multiplicity one. The eigenspace corresponding to μ is not in $\mathfrak{c}$ or $\mathfrak{d}$. $a_4 = 0$. (case 1)	$d_4 \neq 1, \frac{1}{2}$, $a_5 = -\frac{1}{4}$, $a_4 = 0$
$(F14,43)$	One eigenvalue μ with algebraic multiplicity two and geometric multiplicity one. The eigenspace corresponding to μ is in $\mathfrak{c}$ but not $\mathfrak{d}$. $a_4 \neq 0$.	$d_4 \neq 1$, $a_5 = d_4^2 - d_4$, $a_4 \neq 0$
$(F14,44)$	One eigenvalue μ with algebraic multiplicity two and geometric multiplicity two. The eigenspace corresponding to μ does not contain $\mathfrak{d}$.	$d_4 \neq 1$, $a_5 = d_4^2 - d_4$, $a_4 = 0$
$(F14,45)$	Three repeated eigenvalues, $a_4 \neq 0$	$d_4 = \frac{1}{2}$, $a_5 = -\frac{1}{4}$, $a_4 \neq 0$
$(F14,46)$	Three repeated eigenvalues, $a_4 = 0$	$d_4 = \frac{1}{2}$, $a_5 = -\frac{1}{4}$, $a_4 = 0$
$(F14,47)$	One eigenvalue μ with algebraic multiplicity two and eigenspace given by $\mathfrak{d}$. No non-real eigenvalues. $a_4 \neq 0$.	$d_4 = 1$, $a_5 > -\frac{1}{4}$, $d_3 \neq 0$, $a_4 \neq 0$
$(F14,48)$	One eigenvalue μ with algebraic multiplicity two and eigenspace given by $\mathfrak{d}$. No non-real eigenvalues. $a_4 = 0$.	$d_4 = 1$, $a_5 > -\frac{1}{4}$, $d_3 \neq 0$, $a_4 = 0$
$(F14,49)$	One eigenvalue μ with algebraic multiplicity two and eigenspace given by $\mathfrak{d}$. Non-real eigenvalues. $a_4 \neq 0$.	$d_4 = 1$, $a_5 < -\frac{1}{4}$, $d_3 \neq 0$, $a_4 \neq 0$
$(F14,50)$	One eigenvalue μ with algebraic multiplicity two and eigenspace given by $\mathfrak{d}$. Non-real eigenvalues. $a_4 = 0$.	$d_4 = 1$, $a_5 < -\frac{1}{4}$, $d_3 \neq 0$, $a_4 = 0$
$(F14,51)$	Two sets of repeated eigenvalues μ_1 and μ_2. The eigenspace of μ_1 and μ_2 does not contain $\mathfrak{c}$. $a_4 \neq 0$.	$d_4 = 1$, $a_5 = -\frac{1}{4}$, $d_3 \neq 0$, $a_4 \neq 0$
$(F14,52)$	Two sets of repeated eigenvalues μ_1 and μ_2. The eigenspace of μ_1 and μ_2 does not contain $\mathfrak{c}$. $a_4 = 0$.	$d_4 = 1$, $a_5 = -\frac{1}{4}$, $d_3 \neq 0$, $a_4 = 0$
$(F14,53)$	One eigenvalue μ with algebraic multiplicity two and eigenspace given by $\mathfrak{c}$. No non-real eigenvalues. $a_4 \neq 0$.	$d_4 = 1$, $a_5 > -\frac{1}{4}$, $d_3 = 0$, $a_4 \neq 0$
$(F14,54)$	One eigenvalue μ with algebraic multiplicity two and eigenspace given by $\mathfrak{c}$. No non-real eigenvalues. $a_4 = 0$.	$d_4 = 1$, $a_5 > -\frac{1}{4}$, $d_3 = 0$, $a_4 = 0$
$(F14,55)$	One eigenvalue μ with algebraic multiplicity two and eigenspace given by $\mathfrak{c}$. Non-real eigenvalues. $a_4 \neq 0$.	$d_4 = 1$, $a_5 < -\frac{1}{4}$, $d_3 = 0$, $a_4 \neq 0$

Case ID	Invariant Characteristics	Parameter Values
$(F14,56)$	One eigenvalue μ with algebraic multiplicity two and eigenspace given by $\mathfrak{c}$. Non-real eigenvalues. $a_4 = 0$.	$d_4 = 1,\ a_5 < -\frac{1}{4},\ d_3 = 0,\ a_4 = 0$
$(F14,57)$	Two sets of repeated eigenvalues μ_1 and μ_2. The eigenspace of μ_1 and μ_2 contains $\mathfrak{c}$. $a_4 \neq 0$.	$d_4 = 1,\ a_5 = -\frac{1}{4},\ d_3 = 0,\ a_4 \neq 0$
$(F14,58)$	Two sets of repeated eigenvalues μ_1 and μ_2. The eigenspace of μ_1 and μ_2 contains $\mathfrak{c}$. $a_4 = 0$.	$d_4 = 1,\ a_5 = -\frac{1}{4},\ d_3 = 0,\ a_4 = 0$

To construct changes of basis for each of these cases, we begin with the structure equations given by Equation 7.7. First, consider the cases in which $d_4 \neq 1$ and $a_5 \neq d_4^2 - d_4$ ($(F14,37)$ through $(F14,42)$). In these cases, the change of basis $\left(e_1 + \frac{a_4 d_3}{d_4 - 1} e_5, e_2 - a_4 (d_4 - 1)\tilde{e}_4, e_3, \tilde{e}_4, -a_4\tilde{e}_4 + e_5\right)$ where $\tilde{e}_4 = \frac{d_3}{(d_4-1)\left(d_4^2 - d_4 - a_5\right)} e_3 + \frac{1}{d_4^2 - d_4 - a_5}$ produces structure equations as follows:

$$
\begin{aligned}
[e_1, e_2] &= a_5 e_5 \\
[e_1, e_3] &= e_3 \\
[e_1, e_4] &= d_4 e_4 \\
[e_1, e_5] &= e_2 + e_5 \\
[e_2, e_5] &= -e_3.
\end{aligned} \tag{7.8}
$$

The isotropy in this basis is given by $a_4 e_4 + e_5$, but since scaling e_4 does not change the structure equations, the isotropy is spanned by either $e_4 + e_5$ or e_5. In cases $(F14,37)$ and $(F14,38)$, the change of basis $(-2\alpha e_3, e_5, -2\alpha e_2 - \alpha e_5, e_4, 2\alpha e_1)$ where $\alpha = \frac{-1}{\sqrt{-1-4a_5}}$ gives the algebra pairs with $\beta = 2d_4\alpha$. Note that in these two cases, $\beta = 2\alpha$ is not possible since $d_4 \neq 1$. In cases $(F14,39)$ and $(F14,40)$, the change of basis $\left(-4\alpha e_3, 2\alpha e_2 + (1+\alpha) e_5, -2\alpha e_2 + (1-\alpha) e_5, e_4, \frac{2\alpha}{1+\alpha} e_1\right)$ with $\alpha = \frac{-1}{\sqrt{1+4a_5}}$ gives the algebra pairs in standard form with $a = \frac{\alpha - 1}{\alpha + 1}$ and $b = \frac{2d_4\alpha}{\alpha+1}$. Note that in these cases $b = 1$ is not possible since this would require $\frac{2d_4}{1-\sqrt{1+4a_5}} = 1$, implying $a_5 = d_4^2 - d_4$, which is excluded here. Furthermore, $a + 1 = b$ is excluded since this would require $d_4 = 1$. In cases $(F14,41)$

and $(F14,42)$, the change of basis $\left(\frac{1}{2}e_3, \frac{1}{2}e_5, e_2 + \frac{1}{2}e_5, \frac{1}{2}e_4, 2e_1\right)$ gives the algebra pairs in standard form with $a = 2d_4$. Note that $a \neq 1$ and $a \neq 2$ in these cases.

For case $(F14,43)$, applying the change of basis $\left(e_1 + \frac{d_3 a_4}{d_4-1}e_5, e_2 - e_5, e_3, \frac{d_3 a_4}{d_4-1}e_3 + a_4 e_4, e_5\right)$ to the basis given by Equations 7.7 and applying the relationship $a_5 = d_4^2 - d_4$ eliminates all parameters except d_4 and gives structure equations as follows:

$$\begin{aligned}
[e_1, e_2] &= e_4 + \left(d_4^2 - d_4\right) e_5 \\
[e_1, e_3] &= e_3 \\
[e_1, e_4] &= d_4 e_4 \\
[e_1, e_5] &= e_2 + e_5 \\
[e_2, e_5] &= -e_3.
\end{aligned} \tag{7.9}$$

Applying $\left(-\alpha^3 e_3, \alpha e_2 + e_4 + \alpha d_4 e_5, -\alpha e_2 - e_4 + \left(\alpha^2 + d_4\right) e_5, \frac{\alpha}{d_4}e_4, \frac{1}{d_4}e_1\right)$ with $\alpha = 2d_4 - 1$ then gives the algebra pair in standard form with $a = \frac{1}{d_4} - 1$.

For case $(F14,44)$, applying the change of basis $\left(e_1, e_2 - e_5, e_3, \frac{d_3}{d_4-1}e_3 + e_4, e_5\right)$ to the basis given by Equations 7.7 and applying the relationship $a_5 = d_4^2 - d_4$ eliminates all parameters except d_4 and gives structure equations as follows:

$$\begin{aligned}
[e_1, e_2] &= \left(d_4^2 - d_4\right) e_5 \\
[e_1, e_3] &= e_3 \\
[e_1, e_4] &= d_4 e_4 \\
[e_1, e_5] &= e_2 + e_5 \\
[e_2, e_5] &= -e_3.
\end{aligned} \tag{7.10}$$

Applying $\left(-\alpha^3 e_3, \alpha e_2 + \alpha d_4 e_5, -\alpha e_2 + \left(\alpha^2 + d_4\right) e_5, \frac{\alpha}{d_4}e_4, \frac{1}{d_4}e_1\right)$ with $\alpha = 2d_4 - 1$ then gives the algebra pair in standard form with $a = \frac{1}{d_4} - 1$.

For case $(F14,45)$, applying the change of basis $\left(\frac{1}{2}e_3, e_2, e_2 + \tilde{e}_4 - \frac{1}{2}e_5, \tilde{e}_4, 2e_1 - 4a_4 d_3 e_5\right)$ with $\tilde{e}_4 = -4a_4 d_3 e_3 + 2a_4 e_4$to the basis given by Equations 7.7 gives the algebra pair in standard form.

For case $(F14,46)$, applying the change of basis $\left(\frac{1}{2}e_3, \frac{1}{2}e_5, e_2 - \frac{1}{2}e_5, -4d_3e_3 + 2e_4, 2e_1\right)$ to the basis given by Equations 7.7 gives the algebra pair in standard form.

Now consider the cases for which $d_4 = 1$ and $d_3 \neq 0$ ($(F14,47)$ through $(F14,52)$). The change of basis $\left(e_1 + \frac{a_4 d_3}{a_5}(e_2 - e_5), e_2 - e_5, e_3, \frac{1}{d_3}e_4, \frac{a_4}{a_5}e_4 + e_5\right)$ applied to the basis given by Equations 7.7 gives the following structure equations with isotropy $-\frac{a_4}{a_5 d_3}e_4 + e_5$:

$$
\begin{aligned}
[e_1, e_2] &= a_5 e_5 \\
[e_1, e_3] &= e_3 \\
[e_1, e_4] &= e_3 + e_4 \\
[e_1, e_5] &= e_2 + e_5 \\
[e_2, e_5] &= -e_3.
\end{aligned} \tag{7.11}
$$

If $a_4 \neq 0$, the automorphism given by $\left(e_1, \frac{a_4}{a_5 d_3}e_2, \left(\frac{a_4}{a_5 d_3}\right)^2 e_3, e_4, \frac{a_4}{a_5 d_3}e_5\right)$ allows us to take the isotropy to be $-e_4 + e_5$ without loss of generality. For $(F14,47)$ and $(F14,48)$, having eliminated all parameters except a_5, we now apply the change of basis

$$\left(\left(a^2 - 1\right)e_3, (a+1)e_2 + e_5, -(a+1)e_2 - ae_5, (a-1)e_4, (a+1)e_1\right)$$

where $a = \frac{\sqrt{1+4a_5} - 1 - 2a_5}{2a_5}$ (and $a_5 = \frac{-a}{(1+a)^2}$). This gives the algebra pairs in standard form. For $(F14,49)$ and $(F14,50)$, the change of basis $(2ae_3, -2ae_2 - ae_5, e_5, e_4, 2ae_1)$ with $a = \frac{1}{\sqrt{-1-4a_5}}$ applied to the basis given by the structure equations in Equations 7.11 gives the algebra pairs in standard form. In cases $(F14,51)$ and $(F14,52)$, the change of basis $(2e_3, 2e_2, -2e_2 - e_5, e_4, 2e_1)$ applied to the basis given by Equations 7.11 gives the algebra pairs in standard form.

Now consider the cases for which $d_4 = 1$ and $d_3 = 0$ ($(F14,53)$ through $(F14,58)$). The change of basis $\left(e_1, e_2 - e_5, e_3, e_4, \frac{a_4}{a_5}e_4 + e_5\right)$ applied to the basis given by Equations 7.7 gives the following structure equations with isotropy $-\frac{a_4}{a_5}e_4 + e_5$:

$$
\begin{aligned}
[e_1, e_2] &= a_5 e_5 \\
[e_1, e_3] &= e_3 \\
[e_1, e_4] &= e_4 \\
[e_1, e_5] &= e_2 + e_5 \\
[e_2, e_5] &= -e_3.
\end{aligned} \tag{7.12}
$$

If $a_4 \neq 0$, the automorphism given by scaling e_4 allows us to take the isotropy to be $-e_4 + e_5$ without loss of generality. For $(F14, 53)$ and $(F14, 54)$, having eliminated all parameters except a_5, we now apply the change of basis

$$\left(\left(a^2 - 1\right) e_3, (a+1)\, e_2 + e_5, -\,(a+1)\, e_2 - a e_5, (a-1)\, e_4, (a+1)\, e_1\right)$$

where $a = \frac{\sqrt{1+4a_5}-1}{\sqrt{1+4a_5}+1} \in (-1, 1)$. This gives the algebra pairs in standard form. For $(F14, 55)$ and $(F14, 56)$, the change of basis $(2ae_3, -2ae_2 - ae_5, e_5, e_4, 2ae_1)$ with $a = \frac{1}{\sqrt{-1-4a_5}}$ applied to the basis given by the structure equations in Equations 7.12 gives the algebra pairs in standard form. In cases $(F14, 57)$ and $(F14, 58)$, the change of basis $(2e_3, 2e_2, -2e_2 - e_5, e_4, 2e_1)$ applied to the basis given by Equations 7.12 gives the algebra pairs in standard form.

CHAPTER 8

Application: Verification of Space-Time Classifications

A. Z. Petrov [**10**], provides a classification of Lorentzian metrics of dimension four according to isometry dimension and orbit type. In particular, the Killing vectors are given, allowing straightforward comparison to the isometry-isotropy subalgebra pair lists generated in previous chapters. Any entry in [**10**] with five-dimensional isometry admitting a slice and having reductive isotropy should have an isometry-isotropy algebra-subalgebra pair corresponding to one of the algebra-subalgebra pairs in this thesis. For the case of degenerate orbits, an explicit reductive complement must either be found or shown not to exist, and a local slice at an arbitrary point must also be found. In principle, this can be difficult, but in practice, the bases chosen by Petrov are well-adapted to the calculation or exclusion of local slices (see for instance Example 34).

The Schmidt method only guarantees that the algebra-subalgebra pairs we have constructed are realizable as Killing vectors on a pseudo-Riemannian manifold $(\mathcal{M}, g)$ where the isotropy contains a subgroup generated by the isotropy we have designated. In most cases, the metrics that realize an algebra pair $(\mathfrak{g}, \mathfrak{h})$ as Killing vectors necessarily admit additional Killing vectors which act reductively on $\mathfrak{g}$. In this situation, $(\mathfrak{g}, \mathfrak{h})$ would be present among Petrov's vector fields only as a reductive subalgebra of a system of vector fields of dimension six or greater (e.g., Example 68). Therefore, our list of algebra-subalgebra pairs is inherently more inclusive than the vector fields given by Petrov.

By building the most general $\mathfrak{h}$-invariant metric on a reductive complement in $\mathfrak{g}$ for each algebra pair we have generated and directly calculating the isometries, we can determine whether or not these extra symmetries arise. We find that in all but eleven of the one-dimensional isotropy cases, additional symmetries must exist (see Appendix B.6). For the eleven cases which capture the entire isometry algebra, we have found a change of basis matching Petrov's Killing fields to our algebra-subalgebra pairs (Table 8.1). All of the reductive simple-G spaces in Petrov's classification with five-dimensional isometry and one-dimensional isotropy are accounted for in this list (simple-G for these dimensions reduces to homogeneous).

TABLE 8.1. Algebra pairs and the corresponding Petrov vector fields. Unless otherwise specified, the isotropy for these vector fields is computed at the origin and generic, non-zero parameters are used. The change of basis given, when applied to the Killing fields in the indicated equation of Petrov's classification [**10**], aligns the Killing fields with the algebra pair given in the leftmost column. The absence of an algebra pair among Petrov's Killing fields is indicative of the presence of necessary extra symmetries.

Pair ID	Petrov Eq.	Parameters	Change of Basis (on Petrov System)
$(F12,4)$	(33.17)	$\epsilon=-1$	$\left(-\sqrt{2}X_1, 2X_2, \sqrt{2}X_3, \frac{\sqrt{2}}{2}X_4, -X_5\right)$
$(F12,6)$	(33.19)[1]		$(X_3, -X_1, -X_2, X_4, -X_5)$
$(F12,6)$	(33.20)		$(X_2, -X_1, X_3, X_4, X_5)$
$(F12,8)$	(33.23)		$(-X_1, X_2, -X_3, -X_4, X_5)$
$(F12,9)$	(33.22)		$(X_1, X_2, X_3, -X_5, -X_4)$
$(F12,11)$	(33.31)		$\left(-X_3, -X_2, -X_1, -\frac{1}{k}X_5, -X_4\right)$
$(F13,3)$	(33.21)	$c=0$	$(-X_1, X_2, X_3, -X_4, X_5)$
$(F13,5)$	(33.17)	$\epsilon=1$	$\left(\frac{X_1-2X_2+X_3}{2}, X_1-X_3, \frac{X_1+2X_2+X_3}{2}, \frac{1}{2}X_4, X_5\right)$
$(F13,6)$	(33.21)	$c\neq 0$	$\left(X_1, -X_3, X_2, -X_4-\frac{1}{c}X_5, -X_4\right)$
$(F13,8)$	(33, 28)	$k+\epsilon\neq 0$	$\left(X_1, X_2, X_3, -\frac{1}{k+\epsilon}X_5, -X_4-\frac{1}{k+\epsilon}X_5\right)$
$(F14,1)$	(33.14)[2]		$\left(-\lvert k\rvert X_1, \sqrt{\lvert k\rvert}X_2, X_5, -\sqrt{\lvert k\rvert}X_2-\sqrt{\lvert k\rvert}X_3, X_4-X_5\right)$
$(F14,2)$	(33.16)		$(-X_2, -2X_3, -X_5, X_1, X_4)$

[1]Isotropy computed at $\left(\frac{\pi}{2}, 0, 0, 0\right)$.

[2]When $k<0$, this isometry algebra includes (33.18) as a special case.

Bibliography

[1] W. M. Boothby. *An introduction to differentiable manifolds and Riemannian geometry*. Academic Press, 2 (rev.) edition, 2003.

[2] A. Bowers. An algebraic construction of Lorentz homogeneous spaces of low dimension . *J. Lie Theory*, 23(3):887–906, 2012.

[3] M. E. Fels and A. G. Renner. Non-reductive homogeneous pseudo-Riemannian manifolds of dimension four. *Canadian Journal of Mathematics*, 12(1), 2006.

[4] J. Hicks. The Riemann curvature tensor, its invariants, and their use in the classification of spacetimes. Presentation at Mathematical Association of America Conference, March 2015.

[5] B. Komrakov. Einstein-Maxwell equations on four-dimensional homogeneous spaces. *Lobachevskii Jornal of Mathematics*, 8:33–165, 2001.

[6] M. A. H. MacCallum. Locally isotropy space-times with nonnull homogeneous hypersurfaces. In *Essays In general relativity: A festschrift for Abraham Taub*, pages 121–138. 1980.

[7] P.J Olver. *Applications of Lie groups to differential equations*. Springer, 2 edition, 1993.

[8] R. S. Palais. On the existence of slices for actions of non-compact Lie groups. *Annals of Mathematics*, 73(2):295–323, March 1961.

[9] J. Patera, P. Winternitz, and H. Zassenhaus. Continuous subgroups of the fundamental groups of physics I. General method and the Poincare group. *J. Math Physics*, 16(1):1597–1614, January 1975.

[10] A. Z. Petrov. *Einstein spaces*. Pergamon Press, 1969. Translated from the 1961 Russian version by Kelleher, R. F.

[11] B. G. Schmidt. Homogeneous Riemannian spaces and Lie algebras of Killing fields. *General Relativity and Gravitation*, 2(2):105–120, 1971.

[12] L. Snobl and P. Winternitz. *Classification and Identification of Lie Algebras*, volume 33 of *CRM Monograph Series*. 2014.

APPENDIX A

Lie Algebra Structure Equations

	e_1	e_2	e_3	e_4	e_5
e_1	.	$2\,e_1$	$-e_2$	.	.
e_2		.	$2\,e_3$	.	.
e_3			.	.	.
e_4				.	e_4
e_5					.

$(F8,0)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$-e_1$
e_2		.	.	e_1	.
e_3			.	e_2	e_3
e_4				.	$-e_4$
e_5					.

$(F8,1)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	ae_1
e_2		.	.	.	$-ae_2$
e_3			.	.	e_4
e_4				.	$-e_3$
e_5					.

$0 < a$

$(F11,0)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	e_2	.	.
e_2		.	$-e_1$	.	.
e_3			.	.	.
e_4				.	$-e_4$
e_5					.

$(F12,0)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	$2\,e_1$	$-e_2$	.	.
e_2		.	$2\,e_3$	.	.
e_3			.	.	.
e_4				.	e_4
e_5					.

$(F12,1)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	e_3	$-e_2$	.	.
e_2		.	e_1	.	.
e_3			.	.	.
e_4				.	e_4
e_5					.

$(F12,2)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	$2\,e_1$	$-2\,e_2$	.	.
e_2		.	$2\,e_3$	.	.
e_3			.	.	.
e_4				.	.
e_5					.

$(F12,3)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	$2\,e_1$	$-2\,e_2$	.	.
e_2		.	$2\,e_3$	.	.
e_3			.	.	.
e_4				.	.
e_5					.

$(F12,4)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	e_3	$-e_2$	.	.
e_2		.	e_1	.	.
e_3			.	.	.
e_4				.	.
e_5					.

$(F12,5)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	e_3	$-e_2$	.	.
e_2		.	e_1	.	.
e_3			.	.	.
e_4				.	.
e_5					.

$(F12,6)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	e_2	.	.
e_2		.	$-e_1$	.	.
e_3			.	.	.
e_4				.	.
e_5					.

$(F12,7)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	.
e_2		.	e_1	e_3	.
e_3			.	$-e_2$	.
e_4				.	.
e_5					.

$(F12,8)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	$-2\,e_1$	.
e_2		.	e_1	$-e_2$	$-e_3$
e_3			.	$-e_3$	e_2
e_4				.	.
e_5					.

$(F12,9)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	$-e_1$	$-e_2$	.
e_2		.	$-e_2$	e_1	.
e_3			.	.	.
e_4				.	.
e_5					.

$(F12,10)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	$\beta\,e_1$	.
e_2		.	.	$-e_2$	e_3
e_3			.	$-e_3$	$-e_2$
e_4				.	.
e_5					.

$\beta \neq 0$

$(F12,11)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	e_1	.	.
e_2		.	$-e_2$	.	.
e_3			.	.	.
e_4				.	e_4
e_5					.

$(F13,0)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	$2\,e_1$	$-e_2$	.	.
e_2		.	$2\,e_3$	.	.
e_3			.	.	.
e_4				.	e_4
e_5					.

$(F13,1)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	e_1	.	.
e_2		.	$-e_2$	.	.
e_3			.	.	.
e_4				.	.
e_5					.

$(F13,2)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	.
e_2		.	$-e_1$	$-e_2$	.
e_3			.	e_3	.
e_4				.	.
e_5					.

$(F13,3)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	$2\,e_1$	e_2	.	.
e_2		.	$2\,e_3$	.	.
e_3			.	.	.
e_4				.	.
e_5					.

$(F13,4)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	$2\,e_1$	e_2	.	.
e_2		.	$2\,e_3$	.	.
e_3			.	.	.
e_4				.	.
e_5					.

$(F13,5)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	e_1
e_2		.	.	e_1	$2\,e_2$
e_3			.	e_2	$-\epsilon\,e_1+e_3$
e_4				.	.
e_5					.

$(F14,1)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	$-e_1$	.
e_2		.	e_1	.	e_2
e_3			.	$-e_3$	$-e_3$
e_4				.	.
e_5					.

$(F13,6)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	$2\,e_1$	$-e_2$	.	.
e_2		.	$2\,e_3$	.	.
e_3			.	.	.
e_4				.	.
e_5					.

$(F14,2)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	e_1	.	.	.
e_2		.	.	.	.
e_3			.	e_3	.
e_4				.	.
e_5					.

$(F13,7)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	e_1	.
e_2		.	e_1	e_2	.
e_3			.	.	.
e_4				.	.
e_5					.

$(F14,3)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$-e_1$
e_2		.	.	$-e_2$	.
e_3			.	$-ae_3$	$-ae_3$
e_4				.	.
e_5					.

$0 < a, a \leq 1$

$(F13,8)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	.
e_2		.	.	.	e_1
e_3			.	e_1	.
e_4				.	e_2
e_5					.

$(F14,4)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	e_1
e_2		.	.	e_1	e_2
e_3			.	e_2	e_3
e_4				.	.
e_5					.

$(F14,0)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	.
e_2		.	.	.	e_1
e_3			.	.	e_2
e_4				.	.
e_5					.

$(F14,5)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	.
e_2		.	.	.	.
e_3			.	e_2	e_1
e_4				.	e_3
e_5					.

$(F14,6)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$-e_1$
e_2		.	e_1	.	$-e_2$
e_3			.	.	.
e_4				.	$-ae_4$
e_5					.

$a \neq 1, a \neq 0$

$(F14,11)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$-e_1$
e_2		.	e_1	.	$-e_2$
e_3			.	.	e_4
e_4				.	.
e_5					.

$(F14,7)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$-e_1$
e_2		.	e_1	.	$-e_2$
e_3			.	.	.
e_4				.	$-ae_4$
e_5					.

$a \neq 1, a \neq 0$

$(F14,12)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	.
e_2		.	.	.	e_1
e_3			.	e_1	$-e_2$
e_4				.	e_3
e_5					.

$(F14,8)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$-e_1$
e_2		.	e_1	.	.
e_3			.	.	$-e_3-e_4$
e_4				.	$-e_1-e_4$
e_5					.

$(F14,13)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	.
e_2		.	$-e_1$	.	.
e_3			.	.	$-e_2$
e_4				.	$-e_4$
e_5					.

$(F14,9)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$-e_1$
e_2		.	e_1	.	$-e_2$
e_3			.	.	.
e_4				.	$-e_1-e_4$
e_5					.

$(F14,14)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	.
e_2		.	$-e_1$	.	.
e_3			.	.	$-e_2$
e_4				.	$-e_4$
e_5					.

$(F14,10)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$-e_1$
e_2		.	$-e_1$	.	.
e_3			.	.	$-e_3-e_4$
e_4				.	$-e_4$
e_5					.

$(F14,15)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$-e_1$
e_2		.	e_1	.	$-e_2$
e_3			.	.	.
e_4				.	$-e_4$
e_5					.

$(F14, 16)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	.
e_2		.	e_1	$-e_2$	.
e_3			.	e_3	.
e_4				.	.
e_5					.

$(F14, 17)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	.
e_2		.	e_1	$-e_2$	.
e_3			.	e_3	.
e_4				.	.
e_5					.

$(F14, 18)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	.
e_2		.	e_1	e_3	.
e_3			.	$-e_2$	.
e_4				.	.
e_5					.

$(F14, 19)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	.
e_2		.	e_1	e_3	.
e_3			.	$-e_2$	.
e_4				.	.
e_5					.

$(F14, 20)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	.
e_2		.	e_1	.	$-e_3$
e_3			.	.	e_2
e_4				.	$-e_1$
e_5					.

$(F14, 21)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	.
e_2		.	e_1	.	$-e_3$
e_3			.	.	e_2
e_4				.	$-e_1$
e_5					.

$(F14, 22)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	.
e_2		.	e_1	.	e_2
e_3			.	.	$-e_3$
e_4				.	$-e_1$
e_5					.

$(F14, 23)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	.
e_2		.	e_1	.	e_2
e_3			.	.	$-e_3$
e_4				.	$-e_1$
e_5					.

$(F14, 24)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	$-2\,e_1$	.
e_2		.	e_1	$-e_2$	.
e_3			.	$-e_2-e_3$	.
e_4				.	.
e_5					.

$(F14, 25)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	$-2\,e_1$	.
e_2		.	e_1	$-e_2$	.
e_3			.	$-e_2-e_3$	.
e_4				.	.
e_5					.

$(F14, 26)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	$(-a-1)e_1$	.
e_2		.	$-e_1$	$-e_2$	.
e_3			.	$-ae_3$	.
e_4				.	.
e_5					.

$a \neq 0, a \leq 1, -1 < a$

$(F14, 27)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	$(-a-1)e_1$	.
e_2		.	$-e_1$	$-e_2$	.
e_3			.	$-ae_3$	.
e_4				.	.
e_5					.

$a \neq 0, a \leq 1, -1 < a$

$(F14, 28)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	$-2\,\alpha\,e_1$	.
e_2		.	e_1	$-\alpha\,e_2+e_3$	.
e_3			.	$-\alpha\,e_3-e_2$	.
e_4				.	.
e_5					.

$0 < \alpha$

$(F14, 29)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	$-2\,\alpha\,e_1$	.
e_2		.	e_1	$-\alpha\,e_2+e_3$	.
e_3			.	$-\alpha\,e_3-e_2$	.
e_4				.	.
e_5					.

$0 < \alpha$

$(F14, 30)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	.
e_2		.	e_1	.	$-e_3$
e_3			.	.	e_2
e_4				.	$-\alpha\,e_4$
e_5					.

$\alpha \neq 1, 0 < \alpha$

$(F14, 31)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	.
e_2		.	e_1	.	$-e_3$
e_3			.	.	e_2
e_4				.	$-\alpha\,e_4$
e_5					.

$\alpha \neq 1, 0 < \alpha$

$(F14, 32)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	.
e_2		.	e_1	.	$-e_2$
e_3			.	.	e_3
e_4				.	$-ae_4$
e_5					.

$a \neq 1$

$(F14, 33)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	.
e_2		.	e_1	.	$-e_2$
e_3			.	.	e_3
e_4				.	$-ae_4$
e_5					.

$a \neq 1$

$(F14, 34)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	.
e_2		.	e_1	.	$-e_2$
e_3			.	.	e_3
e_4				.	$-e_4$
e_5					.

$(F14, 35)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	.
e_2		.	e_1	.	$-e_2$
e_3			.	.	e_3
e_4				.	$-e_4$
e_5					.

$(F14, 36)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$-2\,\alpha\,e_1$
e_2		.	e_1	.	$-\alpha\,e_2+e_3$
e_3			.	.	$-\alpha\,e_3-e_2$
e_4				.	$-\beta\,e_4$
e_5					.

$\alpha \neq 0, 0 < \beta$

$(F14, 37)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$-2\,\alpha\,e_1$
e_2		.	e_1	.	$-\alpha\,e_2+e_3$
e_3			.	.	$-\alpha\,e_3-e_2$
e_4				.	$-\beta\,e_4$
e_5					.

$\alpha \neq 0, 0 < \beta$

$(F14, 38)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$(-a-1)e_1$
e_2		.	e_1	.	$-e_2$
e_3			.	.	$-ae_3$
e_4				.	$-be_4$
e_5					.

$a \leq 1, -1 < a, b \neq 0$

$(F14, 39)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$(-a-1)e_1$
e_2		.	e_1	.	$-e_2$
e_3			.	.	$-ae_3$
e_4				.	$-be_4$
e_5					.

$a \leq 1, -1 < a, b \neq 0$

$(F14, 40)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$-2\,e_1$
e_2		.	e_1	.	$-e_2-e_3$
e_3			.	.	$-e_3$
e_4				.	$-ae_4$
e_5					.

$a \neq 0$

$(F14, 41)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$-2\,e_1$
e_2		.	e_1	.	$-e_2-e_3$
e_3			.	.	$-e_3$
e_4				.	$-ae_4$
e_5					.

$a \neq 0$

$(F14, 42)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$(-a-1)e_1$
e_2		.	e_1	.	$-ae_2$
e_3			.	.	$-e_3-e_4$
e_4				.	$-e_4$
e_5					.

$(F14, 43)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$(-a-1)e_1$
e_2		.	e_1	.	$-ae_2$
e_3			.	.	$-e_3$
e_4				.	$-e_4$
e_5					.

$(F14, 44)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$-2\,e_1$
e_2		.	e_1	.	$-e_2-e_3$
e_3			.	.	$-e_3-e_4$
e_4				.	$-e_4$
e_5					.

$(F14, 45)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$-2\,e_1$
e_2		.	e_1	.	$-e_2-e_3$
e_3			.	.	$-e_3$
e_4				.	$-e_4$
e_5					.

$(F14, 46)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$(-a-1)e_1$
e_2		.	e_1	.	$-e_2$
e_3			.	.	$-ae_3$
e_4				.	$-e_1+(-a-1)e_4$
e_5					.

$a \leq 1, -1 < a,$

$(F14, 47)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$(-a-1)e_1$
e_2		.	e_1	.	$-e_2$
e_3			.	.	$-ae_3$
e_4				.	$-e_1+(-a-1)e_4$
e_5					.

$a \leq 1, -1 < a,$

$(F14, 48)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$-2\,\alpha\,e_1$
e_2		.	e_1	.	$-\alpha\,e_2-e_3$
e_3			.	.	$-\alpha\,e_3+e_2$
e_4				.	$-2\,\alpha\,e_4-e_1$
e_5					.

$0 < \alpha$

$(F14, 49)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$-2\,\alpha\,e_1$
e_2		.	e_1	.	$-\alpha\,e_2-e_3$
e_3			.	.	$-\alpha\,e_3+e_2$
e_4				.	$-2\,\alpha\,e_4-e_1$
e_5					.

$0 < \alpha$

$(F14, 50)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$-2\,e_1$
e_2		.	e_1	.	$-e_2-e_3$
e_3			.	.	$-e_3$
e_4				.	$-e_1-2\,e_4$
e_5					.

$(F14, 51)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$-2\,e_1$
e_2		.	e_1	.	$-e_2-e_3$
e_3			.	.	$-e_3$
e_4				.	$-e_1-2\,e_4$
e_5					.

$(F14, 52)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$(-a-1)e_1$
e_2		.	e_1	.	$-e_2$
e_3			.	.	$-ae_3$
e_4				.	$(-a-1)e_4$
e_5					.

$a \leq 1, -1 < a,$

$(F14, 53)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$(-a-1)e_1$
e_2		.	e_1	.	$-e_2$
e_3			.	.	$-ae_3$
e_4				.	$(-a-1)e_4$
e_5					.

$a \leq 1, -1 < a,$

$(F14, 54)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$-2\,\alpha\,e_1$
e_2		.	e_1	.	$-\alpha\,e_2-e_3$
e_3			.	.	$-\alpha\,e_3+e_2$
e_4				.	$-2\,\alpha\,e_4$
e_5					.

$0 < \alpha$

$(F14, 55)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$-2\,\alpha\,e_1$
e_2		.	e_1	.	$-\alpha\,e_2-e_3$
e_3			.	.	$-\alpha\,e_3+e_2$
e_4				.	$-2\,\alpha\,e_4$
e_5					.

$0 < \alpha$

$(F14, 56)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$-2\,e_1$
e_2		.	e_1	.	$-e_2-e_3$
e_3			.	.	$-e_3$
e_4				.	$-2\,e_4$
e_5					.

$(F14, 57)$

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$-2\,e_1$
e_2		.	e_1	.	$-e_2-e_3$
e_3			.	.	$-e_3$
e_4				.	$-2\,e_4$
e_5					.

$(F14, 58)$

APPENDIX B

Maple Worksheets

B.1. F8 Maple Worksheet

Maple Worksheet
Two-Dimensional Isotropy:

This worksheet steps through Chapter 3, serving to help validate the results. This chapter covers the only isotropies of dimension greater than one.

We begin by loading the packages needed.

```
> with(DifferentialGeometry):
  with(LieAlgebras):
```

As outlined in Chapter, in the case of isotropy dimension greater than one, we need only consider two-dimensional, nonabelian subalgebras of the Lorentz algebra, and may take the derived algebra of the isotropy, e4, to be a boost, rotation, or null rotation. We initialize these three possibilities in generic form:

```
> LD_Boost:=LieAlgebraData([
  '[e1,e2]=a1*e1+a2*e2+a3*e3+a4*e4+a5*e5',
  '[e1,e3]=b1*e1+b2*e2+b3*e3+b4*e4+b5*e5',
  '[e2,e3]=c1*e1+c2*e2+c3*e3+c4*e4+c5*e5',
  '[e1,e5]=d1*e1+d2*e2+d3*e3',
  '[e2,e5]=f1*e1+f2*e2+f3*e3',
  '[e3,e5]=g1*e1+g2*e2+g3*e3',
  '[e4,e1]=e2',
  '[e4,e2]=e1',
  '[e4,e5]=e4'
  ],[e1,e2,e3,e4,e5],alg_boost);

  LD_Rotation:=LieAlgebraData([
  '[e1,e2]=a1*e1+a2*e2+a3*e3+a4*e4+a5*e5',
  '[e1,e3]=b1*e1+b2*e2+b3*e3+b4*e4+b5*e5',
  '[e2,e3]=c1*e1+c2*e2+c3*e3+c4*e4+c5*e5',
  '[e1,e5]=d1*e1+d2*e2+d3*e3',
  '[e2,e5]=f1*e1+f2*e2+f3*e3',
  '[e3,e5]=g1*e1+g2*e2+g3*e3',
  '[e4,e1]=-e2',
  '[e4,e2]=e1',
  '[e4,e5]=e4'
  ],[e1,e2,e3,e4,e5],alg_rotation);

  LD_Null:=LieAlgebraData([
  '[e1,e2]=a1*e1+a2*e2+a3*e3+a4*e4+a5*e5',
  '[e1,e3]=b1*e1+b2*e2+b3*e3+b4*e4+b5*e5',
  '[e2,e3]=c1*e1+c2*e2+c3*e3+c4*e4+c5*e5',
  '[e1,e5]=d1*e1+d2*e2+d3*e3',
  '[e2,e5]=f1*e1+f2*e2+f3*e3',
  '[e3,e5]=g1*e1+g2*e2+g3*e3',
  '[e4,e1]=e2',
  '[e4,e2]=-e3',
  '[e4,e5]=e4'
  ],[e1,e2,e3,e4,e5],alg_null);
```

$$LD\_Boost := [e1, e2] = a1\,e1 + a2\,e2 + a3\,e3 + a4\,e4 + a5\,e5, [e1, e3] = b1\,e1 + b2\,e2 + b3\,e3 + b4\,e4 + b5\,e5, [e1, e4] = -e2, [e1, e5] = d1\,e1 + d2\,e2 + d3\,e3, [e2, e3] = c1\,e1 + c2\,e2 + c3\,e3 + c4\,e4 + c5\,e5, [e2, e4] = -e1, [e2, e5] = f1\,e1 + f2\,e2$$

$+ f3\, e3, [e3, e4] = 0, [e3, e5] = g1\, e1 + g2\, e2 + g3\, e3, [e4, e5] = e4$

$LD\_Rotation := [e1, e2] = a1\, e1 + a2\, e2 + a3\, e3 + a4\, e4 + a5\, e5, [e1, e3] = b1\, e1 + b2\, e2 + b3\, e3 + b4\, e4 + b5\, e5, [e1, e4] = e2, [e1, e5] = d1\, e1 + d2\, e2 + d3\, e3, [e2, e3] = c1\, e1 + c2\, e2 + c3\, e3 + c4\, e4 + c5\, e5, [e2, e4] = -e1, [e2, e5] = f1\, e1 + f2\, e2 + f3\, e3, [e3, e4] = 0, [e3, e5] = g1\, e1 + g2\, e2 + g3\, e3, [e4, e5] = e4$

$LD\_Null := [e1, e2] = a1\, e1 + a2\, e2 + a3\, e3 + a4\, e4 + a5\, e5, [e1, e3] = b1\, e1 + b2\, e2 + b3\, e3 + b4\, e4 + b5\, e5, [e1, e4] = -e2, [e1, e5] = d1\, e1 + d2\, e2 + d3\, e3, [e2, e3] = c1\, e1 + c2\, e2 + c3\, e3 + c4\, e4 + c5\, e5, [e2, e4] = e3, [e2, e5] = f1\, e1 + f2\, e2 + f3\, e3, [e3, e4] = 0, [e3, e5] = g1\, e1 + g2\, e2 + g3\, e3, [e4, e5] = e4$ **(1.1)**

```
> DGsetup(LD_Boost,[x],[o]);
  DGsetup(LD_Rotation,[y],[p]);
  DGsetup(LD_Null,[z],[q]);
```

Lie algebra: alg_boost

Lie algebra: alg_rotation

Lie algebra: alg_null **(1.2)**

We now demonstrate that the boost and rotation cases cannot satisfy the Jacobi identities.

First, the boost case.

```
> evalDG(
  LieBracket(x4,LieBracket(x5,x1))+
  LieBracket(x1,LieBracket(x4,x5))+
  LieBracket(x5,LieBracket(x1,x4)));

  evalDG(
  LieBracket(x4,LieBracket(x5,x2))+
  LieBracket(x2,LieBracket(x4,x5))+
  LieBracket(x5,LieBracket(x2,x4)));
```

$$-(d2 - f1)\, x1 - (d1 + 1 - f2)\, x2 + f3\, x3$$

$$(-f2 - 1 + d1)\, x1 + (d2 - f1)\, x2 + d3\, x3 \qquad \textbf{(1.3)}$$

Note that d1-f2 = -1 and d1-f2 = 1 are both required -- a contradiction.

Now, the rotation case.

```
> evalDG(
  LieBracket(y4,LieBracket(y5,y1))+
  LieBracket(y1,LieBracket(y4,y5))+
  LieBracket(y5,LieBracket(y1,y4)));

  evalDG(
  LieBracket(y4,LieBracket(y5,y2))+
  LieBracket(y2,LieBracket(y4,y5))+
  LieBracket(y5,LieBracket(y2,y4)));
```

$$-(d2 + f1)\, y1 + (d1 + 1 - f2)\, y2 - f3\, y3$$

$$(-f2 - 1 + d1)\, y1 + (d2 + f1)\, y2 + d3\, y3 \qquad \textbf{(1.4)}$$

Note that d1-f2 = -1 and d1-f2 = 1 are both required -- a contradiction.

Thus, only the null case remains. We apply the Jacobi identities in terms of the Maurer-Cartan forms.

```
> ddq1:=ExteriorDerivative(ExteriorDerivative(q1));
  ddq2:=ExteriorDerivative(ExteriorDerivative(q2));
  ddq3:=ExteriorDerivative(ExteriorDerivative(q3));
  ddq4:=ExteriorDerivative(ExteriorDerivative(q4));
  ddq5:=ExteriorDerivative(ExteriorDerivative(q5));
```

$$ddq1 := -(c2\,a1 - a2\,c1 + a5\,g1 + c3\,b1 - b3\,c1 - b5\,f1 + c5\,d1)\; q1 \wedge q2 \wedge q3 - b1\; q1 \wedge q2 \wedge q4 - (f2\,a1 - a2\,f1 - a3\,g1 + f3\,b1 - d3\,c1)\; q1 \wedge q2 \wedge q5 + c1\; q1 \wedge q3 \wedge q4 - (g2\,a1 + g3\,b1 - b2\,f1 - b3\,g1 + d2\,c1)\; q1 \wedge q3 \wedge q5 - f1\; q1 \wedge q4 \wedge q5 + (g1\,a1 - f1\,b1 + c1\,d1 - f2\,c1 - g3\,c1 + c2\,f1 + c3\,g1)\; q2 \wedge q3 \wedge q5 + g1\; q2 \wedge q4 \wedge q5$$

$$ddq2 := (a1\,b2 - b1\,a2 - a5\,g2 - c3\,b2 + b3\,c2 + b5\,f2 - c5\,d2 + c4)\; q1 \wedge q2 \wedge q3 - (a1 + b2)\; q1 \wedge q2 \wedge q4 + (a1\,d2 - d1\,a2 + a3\,g2 - f3\,b2 + d3\,c2)\; q1 \wedge q2 \wedge q5 - (b1 - c2)\; q1 \wedge q3 \wedge q4 - (g2\,a2 - b1\,d2 + d1\,b2 - b2\,f2 + g3\,b2 - b3\,g2 + d2\,c2)\; q1 \wedge q3 \wedge q5 + (d1 + 1 - f2)\; q1 \wedge q4 \wedge q5 - c1\; q2 \wedge q3 \wedge q4 + (g1\,a2 - b2\,f1 + d2\,c1 - g3\,c2 + c3\,g2)\; q2 \wedge q3 \wedge q5 + (f1 + g2)\; q2 \wedge q4 \wedge q5 + g1\; q3 \wedge q4 \wedge q5$$

$$ddq3 := (a1\,b3 + a2\,c3 - b1\,a3 - c2\,a3 - a5\,g3 + b5\,f3 - c5\,d3 + b4)\; q1 \wedge q2 \wedge q3 + (a2 - b3)\; q1 \wedge q2 \wedge q4 + (a1\,d3 + a2\,f3 - d1\,a3 - f2\,a3 + a3\,g3 - f3\,b3 + d3\,c3)\; q1 \wedge q2 \wedge q5 + (b2 + c3)\; q1 \wedge q3 \wedge q4 - (a3\,g2 - b1\,d3 - b2\,f3 + b3\,d1 + c3\,d2)\; q1 \wedge q3 \wedge q5 - (f3 + d2)\; q1 \wedge q4 \wedge q5 + c2\; q2 \wedge q3 \wedge q4 + (a3\,g1 - b3\,f1 + c1\,d3 + c2\,f3 - c3\,f2)\; q2 \wedge q3 \wedge q5 - (f2 - g3 + 1)\; q2 \wedge q4 \wedge q5 - g2\; q3 \wedge q4 \wedge q5$$

$$ddq4 := (a1\,b4 + a2\,c4 - b1\,a4 - c2\,a4 + b3\,c4 - c3\,b4)\; q1 \wedge q2 \wedge q3 - (b4 + a5)\; q1 \wedge q2 \wedge q4 - (d1\,a4 + f2\,a4 + f3\,b4 - d3\,c4 - a4)\; q1 \wedge q2 \wedge q5 - (-c4 + b5)\; q1 \wedge q3 \wedge q4 - (g2\,a4 + d1\,b4 + g3\,b4 + d2\,c4 - b4)\; q1 \wedge q3 \wedge q5 - c5\; q2 \wedge q3 \wedge q4 + (g1\,a4 - f1\,b4 - f2\,c4 - g3\,c4 + c4)\; q2 \wedge q3 \wedge q5$$

$$ddq5 := (a1\,b5 + a2\,c5 - b1\,a5 - c2\,a5 + b3\,c5 - c3\,b5)\; q1 \wedge q2 \wedge q3 - b5\; q1 \wedge q2 \wedge q4 - (d1\,a5 + f2\,a5 + f3\,b5 - d3\,c5)\; q1 \wedge q2 \wedge q5 + c5\; q1 \wedge q3 \wedge q4 - (g2\,a5 + d1\,b5 + g3\,b5 + d2\,c5)\; q1 \wedge q3 \wedge q5 + (g1\,a5 - f1\,b5 - f2\,c5 - g3\,c5)\; q2 \wedge q3 \wedge q5 \qquad \textbf{(1.5)}$$

Now, recall that, in addition to the Jacobi identities, ad(z5) acting on the first three vectors must be traceless.

We first examine the linear parts together with the requirement that ad(z5) be traceless on the first three basis vectors.

```
> Eq1:={
  Hook(Hook(Hook(ddq1,z1),z2),z4),
  Hook(Hook(Hook(ddq1,z1),z3),z4),
  Hook(Hook(Hook(ddq1,z1),z4),z5),
  Hook(Hook(Hook(ddq1,z2),z4),z5),

  Hook(Hook(Hook(ddq2,z1),z2),z4),
  Hook(Hook(Hook(ddq2,z1),z3),z4),
  Hook(Hook(Hook(ddq2,z1),z4),z5),
  Hook(Hook(Hook(ddq2,z2),z4),z5),

  Hook(Hook(Hook(ddq3,z1),z2),z4),
  Hook(Hook(Hook(ddq3,z1),z3),z4),
  Hook(Hook(Hook(ddq3,z1),z4),z5),
  Hook(Hook(Hook(ddq3,z2),z3),z4),
```

```
Hook(Hook(Hook(ddq3,z2),z4),z5),
Hook(Hook(Hook(ddq3,z3),z4),z5),

Hook(Hook(Hook(ddq4,z1),z2),z4),
Hook(Hook(Hook(ddq4,z1),z3),z4),
Hook(Hook(Hook(ddq4,z2),z3),z4),

Hook(Hook(Hook(ddq5,z1),z2),z4),
Hook(Hook(Hook(ddq5,z1),z3),z4),

LinearAlgebra:-Trace(eval(Adjoint(z5,[z1,z2,z3])))
};
```

$$Eq1 := \{c1, c2, c5, g1, -b1, -b5, -c5, -f1, -g2, -a1 - b2, a2 - b3, -b1 + c2, b2 + c3, -b4 - a5, c4 - b5, f1 + g2, -f3 - d2, -d1 - f2 - g3, d1 + 1 - f2, -f2 + g3 - 1\} \quad (1.6)$$

```
> Eq2:=solve(Eq1,{b1,b2,b3,b4,b5,c1,c2,c3,c4,c5,d1,f1,f2,f3,g1,g2,g3});
```

$$Eq2 := \{b1 = 0, b2 = -a1, b3 = a2, b4 = -a5, b5 = 0, c1 = 0, c2 = 0, c3 = a1, c4 = 0, c5 = 0, d1 = -1, f1 = 0, f2 = 0, f3 = -d2, g1 = 0, g2 = 0, g3 = 1\} \quad (1.7)$$

Evaluating the Jacobi identity with this solution simlifies the quadratic parts. We extract these equations and solve them.

```
> ddq1_s:=Tools:-DGsimplify(eval(ddq1,Eq2));
ddq2_s:=Tools:-DGsimplify(eval(ddq2,Eq2));
ddq3_s:=Tools:-DGsimplify(eval(ddq3,Eq2));
ddq4_s:=Tools:-DGsimplify(eval(ddq4,Eq2));
ddq5_s:=Tools:-DGsimplify(eval(ddq5,Eq2));
```

$$ddq1\_s := 0\; q1 \wedge q2 \wedge q3$$

$$ddq2\_s := a2\; q1 \wedge q2 \wedge q5$$

$$ddq3\_s := (2\; a1\; a2 - 2\; a5)\; q1 \wedge q2 \wedge q3 + (2\; a1\; d3 + 2\; a3)\; q1 \wedge q2 \wedge q5 + a2\; q1 \wedge q3 \wedge q5$$

$$ddq4\_s := (-d2\; a5 + 2\; a4)\; q1 \wedge q2 \wedge q5 - a5\; q1 \wedge q3 \wedge q5$$

$$ddq5\_s := a5\; q1 \wedge q2 \wedge q5 \quad (1.8)$$

```
> Eq3:={
Hook(Hook(Hook(ddq2_s,z1),z2),z5),
Hook(Hook(Hook(ddq3_s,z1),z2),z3),
Hook(Hook(Hook(ddq3_s,z1),z2),z5),
Hook(Hook(Hook(ddq4_s,z1),z2),z5),
Hook(Hook(Hook(ddq4_s,z1),z3),z5),
Hook(Hook(Hook(ddq5_s,z1),z2),z5)
};
```

$$Eq3 := \{a2, a5, -a5, 2\; a1\; a2 - 2\; a5, 2\; a1\; d3 + 2\; a3, -d2\; a5 + 2\; a4\} \quad (1.9)$$

```
> Eq4:=solve(Eq3);
```

$$Eq4 := \{a1 = a1, a2 = 0, a3 = -a1\; d3, a4 = 0, a5 = 0, d2 = d2, d3 = d3\} \quad (1.10)$$

We now combine the equations and confirm that the Jacobi identities are fully satisfied.

```
> Eq5:={op(eval(Eq2,Eq4)),op(Eq4)};
```

$$Eq5 := \{a1 = a1, a2 = 0, a3 = -a1\; d3, a4 = 0, a5 = 0, b1 = 0, b2 = -a1, b3 = 0, b4 = 0, b5 = 0, c1 = 0, c2 = 0, c3 = a1, c4 = 0, c5 = 0, d1 = -1, d2 = d2, d3 = d3, f1 = 0, f2 = 0, f3 = -d2, g1 = 0, g2 = 0, g3 = 1\} \quad (1.11)$$

```
> Tools:-DGsimplify(eval(ddq1,Eq5));
Tools:-DGsimplify(eval(ddq2,Eq5));
Tools:-DGsimplify(eval(ddq3,Eq5));
Tools:-DGsimplify(eval(ddq4,Eq5));
Tools:-DGsimplify(eval(ddq5,Eq5));
```

$$0\ q1 \wedge q2 \wedge q3$$
$$0\ q1 \wedge q2 \wedge q3$$
$$0\ q1 \wedge q2 \wedge q3$$
$$0\ q1 \wedge q2 \wedge q3$$
$$0\ q1 \wedge q2 \wedge q3 \quad \textbf{(1.12)}$$

Thus, the structure equations are:

```
> LD_Null2:=eval(LieAlgebraData([z1,z2,z3,z4,z5],alg_null2),Eq5);
```

$LD\_Null2 := [e1, e2] = a1\ e1 - a1\ d3\ e3, [e1, e3] = -a1\ e2, [e1, e4] = -e2, [e1, e5] = -e1 + d2\ e2 + d3\ e3, [e2, e3] = a1\ e3, [e2, e4] = e3, [e2, e5] = -d2\ e3, [e3, e4] = 0, [e3, e5] = e3, [e4, e5] = e4$ **(1.13)**

We now initialize the algebra and calculate the derived series for a1 nonzero and a1 zero.

```
> DGsetup(LD_Null2,[w],[r]);
```

Lie algebra: alg_null2 **(1.14)**

```
> LD_Null2_0:=eval(LieAlgebraData([w1,w2,w3,w4,w5],alg_null2_0),a1=0):
  DGsetup(LD_Null2_0,[v],[r]):
> Tools:-DGsimplify(Series(alg_null2,"Derived")[3]);
  Tools:-DGsimplify(Series(alg_null2_0,"Derived")[3]);
```

$$[-a1^3\ w1,\ -a1^3\ w2,\ -a1^3\ w3]$$
$$[-v3, v2 + d2\ v3] \quad \textbf{(1.15)}$$

The second derived algebra is either two-dimensional (a1=0) or three-dimensional (a1 nonzero).

If a1 is nonzero, then we have sl(2,R)+s_2,1
with isotropy spanned by e3+e4 and e2-2*e5.
Note that e3+e4 is null and e2-2*e5 has
eigenvalues 0, 1, and -1; thus
e5 is a boost and the isotropy is of type F8.

```
> LieAlgebraData(
  [
  2*w1-d3*w3,
  2/a1*w2,
  1/a1^2*w3,
  -1/a1^2*w3+1/a1*w4,
  1/a1*w2+d2*w4+w5
  ]);
```

$[e1, e2] = 2\ e1, [e1, e3] = -e2, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = 2\ e3, [e2, e4] = 0, [e2, e5] = 0, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = e4$ **(1.16)**

If a1 is zero, then we have s_5,35 with a=-1
and isotropy spanned by e4 and e5.
Note that e5 has eigenvalues 0, 1, and -1; thus
e5 is a boost and the isotropy is of type F8.

```
> eval(LieAlgebraData([w3,w2,-w1+d3/2*w3,w4,-d2*w4-w5]),a1=0);
```

$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -e1, [e2, e3] = 0, [e2, e4] = e1, [e2, e5] = 0, [e3, e4] = e2, [e3, e5] = e3, [e4, e5] = -e4$ **(1.17)**

B.2. F11 Maple Worksheet

Maple Worksheet
Type F11 Isotropy:

This worksheet steps through Chapter 4, serving to help validate the results. This chapters cover the isotropies of type F11, the loxodromes.

We begin by loading the packages needed.

```
> with(DifferentialGeometry):
  with(LieAlgebras):
```

We now initialize the most generic five-dimensional algebra with the subalgebra spanned by e5 of type F11. Note that theta is between 0 and pi/2 (not inclusive).

```
> LD_Loxodrome:=LieAlgebraData([
  '[e1,e2]=a1*e1+a2*e2+a3*e3+a4*e4+a5*e5',
  '[e1,e3]=b1*e1+b2*e2+b3*e3+b4*e4+b5*e5',
  '[e1,e4]=c1*e1+c2*e2+c3*e3+c4*e4+c5*e5',
  '[e2,e3]=d1*e1+d2*e2+d3*e3+d4*e4+d5*e5',
  '[e2,e4]=f1*e1+f2*e2+f3*e3+f4*e4+f5*e5',
  '[e3,e4]=g1*e1+g2*e2+g3*e3+g4*e4+g5*e5',
  '[e5,e1]=-cos(theta)*e2',
  '[e5,e2]=cos(theta)*e1',
  '[e5,e3]=-sin(theta)*e4',
  '[e5,e4]=-sin(theta)*e3'
  ],[e1,e2,e3,e4,e5],alg_lox);
```

$$LD\_Loxodrome := [e1, e2] = a1\ e1 + a2\ e2 + a3\ e3 + a4\ e4 + a5\ e5, [e1, e3] = b1\ e1 + b2\ e2 + b3\ e3 + b4\ e4 + b5\ e5, [e1, e4] = c1\ e1 + c2\ e2 + c3\ e3 + c4\ e4 + c5\ e5, [e1, e5] = \cos(\theta)\ e2, [e2, e3] = d1\ e1 + d2\ e2 + d3\ e3 + d4\ e4 + d5\ e5, [e2, e4] = f1\ e1 + f2\ e2 + f3\ e3 + f4\ e4 + f5\ e5, [e2, e5] = -\cos(\theta)\ e1, [e3, e4] = g1\ e1 + g2\ e2 + g3\ e3 + g4\ e4 + g5\ e5, [e3, e5] = \sin(\theta)\ e4, [e4, e5] = \sin(\theta)\ e3 \quad (1.1)$$

```
> DGsetup(LD_Loxodrome);
```

Lie algebra: alg_lox **(1.2)**

We give the Jacobi identity with e1, e2, and e5, and also with e3, e4, and e5:

```
> 0*e1=evalDG(
  LieBracket(e2,LieBracket(e5,e1))+
  LieBracket(e1,LieBracket(e2,e5))+
  LieBracket(e5,LieBracket(e1,e2)));

  0*e1=evalDG(
  LieBracket(e3,LieBracket(e4,e5))+
  LieBracket(e5,LieBracket(e3,e4))+
  LieBracket(e4,LieBracket(e5,e3)));
```

$$0 = a2\cos(\theta)\ e1 - a1\cos(\theta)\ e2 - a4\sin(\theta)\ e3 - a3\sin(\theta)\ e4$$

$$0 = g2\cos(\theta)\ e1 - g1\cos(\theta)\ e2 - g4\sin(\theta)\ e3 - g3\sin(\theta)\ e4 \quad (1.3)$$

Since cos(theta) and sin(theta) are nonzero, these imply the following:

```
> Eq1:=[a1=0,a2=0,a3=0,a4=0,g1=0,g2=0,g3=0,g4=0];
```

$$Eq1 := [a1 = 0, a2 = 0, a3 = 0, a4 = 0, g1 = 0, g2 = 0, g3 = 0, g4 = 0] \quad (1.4)$$

Now consider the remaining Jacobi identities

on the basis vectors involving e5. The items in
the lists below are all zero.

```
> Id1:=GetComponents(eval(
  LieBracket(e5,LieBracket(e1,e3))+
  LieBracket(e3,LieBracket(e5,e1))+
  LieBracket(e1,LieBracket(e3,e5)),
  Eq1),[e1,e2,e3,e4,e5]);

  Id2:=GetComponents(eval(
  LieBracket(e5,LieBracket(e2,e3))+
  LieBracket(e3,LieBracket(e5,e2))+
  LieBracket(e2,LieBracket(e3,e5)),
  Eq1),[e1,e2,e3,e4,e5]);

  Id3:=GetComponents(eval(
  LieBracket(e5,LieBracket(e2,e4))+
  LieBracket(e4,LieBracket(e5,e2))+
  LieBracket(e2,LieBracket(e4,e5)),
  Eq1),[e1,e2,e3,e4,e5]);

  Id4:=GetComponents(eval(
  LieBracket(e5,LieBracket(e1,e4))+
  LieBracket(e4,LieBracket(e5,e1))+
  LieBracket(e1,LieBracket(e4,e5)),
  Eq1),[e1,e2,e3,e4,e5]);
```

$$Id1 := [\sin(\theta)\ c1 + b2\cos(\theta) + \cos(\theta)\ d1, \sin(\theta)\ c2 - b1\cos(\theta) + \cos(\theta)\ d2, -b4\sin(\theta) + \cos(\theta)\ d3 + \sin(\theta)\ c3, -b3\sin(\theta) + \cos(\theta)\ d4 + \sin(\theta)\ c4, \cos(\theta)\ d5 + \sin(\theta)\ c5]$$

$$Id2 := [\sin(\theta)\ f1 - b1\cos(\theta) + \cos(\theta)\ d2, \sin(\theta)\ f2 - b2\cos(\theta) - \cos(\theta)\ d1, -d4\sin(\theta) - \cos(\theta)\ b3 + \sin(\theta)\ f3, -d3\sin(\theta) - \cos(\theta)\ b4 + \sin(\theta)\ f4, -\cos(\theta)\ b5 + \sin(\theta)\ f5]$$

$$Id3 := [\sin(\theta)\ d1 - \cos(\theta)\ c1 + f2\cos(\theta), \sin(\theta)\ d2 - \cos(\theta)\ c2 - f1\cos(\theta), -\sin(\theta)\ f4 - \cos(\theta)\ c3 + d3\sin(\theta), -\sin(\theta)\ f3 - \cos(\theta)\ c4 + d4\sin(\theta), -\cos(\theta)\ c5 + \sin(\theta)\ d5]$$

$$Id4 := [\cos(\theta)\ c2 + f1\cos(\theta) + \sin(\theta)\ b1, -\cos(\theta)\ c1 + f2\cos(\theta) + \sin(\theta)\ b2, -\sin(\theta)\ c4 + \cos(\theta)\ f3 + b3\sin(\theta), -\sin(\theta)\ c3 + \cos(\theta)\ f4 + b4\sin(\theta), \cos(\theta)\ f5 + \sin(\theta)\ b5] \tag{1.5}$$

The following linear combinations yield b5=c5=d5=f5=0.

```
> simplify(Id1[5]*sin(theta)-Id3[5]*cos(theta));
  simplify(Id1[5]*cos(theta)+Id3[5]*sin(theta));
  simplify(Id2[5]*sin(theta)+Id4[5]*cos(theta));
  simplify(-Id2[5]*cos(theta)+Id4[5]*sin(theta));
```

$$c5$$
$$d5$$
$$f5$$
$$b5 \tag{1.6}$$

By forming linear combinations of the above, we find that
f2=-c1, f1=c2, d1=b2, and d2=-b2:

```
> simplify(Id1[1]+Id2[2]);
  simplify(Id1[2]-Id2[1]);
  simplify(-Id3[1]+Id4[2]);
  simplify(Id3[2]+Id4[1]);
```

$$\sin(\theta)\ (c1 + f2)$$
$$\sin(\theta)\ (c2 - f1)$$
$$\sin(\theta)\ (b2 - d1)$$
$$\sin(\theta)\ (b1 + d2) \tag{1.7}$$

Using another linear combination and the information above, we find b2=0 and then b1=b2=c1=c2=d1=d2=f1=f2=0.

```
> simplify(eval(
  Id1[1]*2*cos(theta)+Id3[1]*sin(theta),
  {f2=-c1,f1=c2,d1=b2,d2=-b1}));

  eval(Id1[1],{b2=0,d1=0});
  eval(Id2[2],{b2=0,d1=0});
```

$$b2\left(3\cos(\theta)^2+1\right)$$

$$\sin(\theta)\ c1$$

$$\sin(\theta)\ f2 \quad (1.8)$$

Continuing in the same vein, b4=-c3, b3=-c4, f4=-d3, f3=-d4:

```
> simplify(Id2[4]+Id3[3]);
  simplify(Id1[4]+Id4[3]);
  simplify(Id2[3]+Id3[4]);
  simplify(Id1[3]+Id4[4]);
```

$$-\cos(\theta)\ (b4+c3)$$

$$\cos(\theta)\ (d4+f3)$$

$$-\cos(\theta)\ (b3+c4)$$

$$\cos(\theta)\ (d3+f4) \quad (1.9)$$

We also find b3=b4=c3=c4=d3=d4=f3=f4=0

```
> simplify(eval(
  Id1[3]*2*sin(theta)+Id2[4]*cos(theta),
  {b4=-c3, b3=-c4, f4=-d3, f3=-d4}));

  simplify(eval(
  Id1[4]*2*sin(theta)-Id3[4]*cos(theta),
  {b4=-c3, b3=-c4, f4=-d3, f3=-d4}));

  eval(Id1[3],{c3=0,b4=0,c4=0,b3=0});
  eval(Id1[4],{c3=0,b4=0,c4=0,b3=0});
```

$$-c3\left(3\cos(\theta)^2-4\right)$$

$$-c4\left(3\cos(\theta)^2-4\right)$$

$$\cos(\theta)\ d3$$

$$\cos(\theta)\ d4 \quad (1.10)$$

Thus far, we have that all parameters except a5 and g5 are zero.

```
> Eq2:={
  a1=0,a2=0,a3=0,a4=0,
  b1=0,b2=0,b3=0,b4=0,b5=0,
  c1=0,c2=0,c3=0,c4=0,c5=0,
  d1=0,d2=0,d3=0,d4=0,d5=0,
  f1=0,f2=0,f3=0,f4=0,f5=0,
  g1=0,g2=0,g3=0,g4=0
  }:
```

We now show that a5 and g5 are also zero.

```
> Id5:=GetComponents(eval(
  LieBracket(e3,LieBracket(e1,e4))+
  LieBracket(e4,LieBracket(e3,e1))+
  LieBracket(e1,LieBracket(e4,e3)),
  Eq2),[e1,e2,e3,e4,e5]);

  Id6:=GetComponents(eval(
```

```
LieBracket(e1,LieBracket(e2,e4))+
LieBracket(e4,LieBracket(e1,e2))+
LieBracket(e2,LieBracket(e4,e1)),
Eq2),[e1,e2,e3,e4,e5]);
```

$$Id5 := [0, -g5\cos(\theta), 0, 0, 0]$$

$$Id6 := [0, 0, a5\sin(\theta), 0, 0] \quad (1.11)$$

```
> Eq3:={op(Eq2),a5=0,g5=0}:
```

Therefore, the structure equations are given by
the following:

```
> LD2:=eval(LieAlgebraData([e1,e2,e3,e4,e5],alg_lox2),Eq3);
```

$$LD2 := [e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = \cos(\theta)\ e2, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -\cos(\theta)\ e1, [e3, e4] = 0, [e3, e5] = \sin(\theta)\ e4, [e4, e5] = \sin(\theta)\ e3 \quad (1.12)$$

```
> DGsetup(LD2,[x],[o]);
```

Lie algebra: alg_lox2 **(1.13)**

This change of basis gives the algebra in standard form as s_5,11
with alpha = -tan(theta), beta = tan(theta), and gamma = 0.
The isotropy is still spanned by e5.

```
> LieAlgebraData([x3+x4,x3-x4,x1,x2,sec(theta)*x5]);
```

$$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = \frac{\sin(\theta)}{\cos(\theta)}\ e1, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -\frac{\sin(\theta)}{\cos(\theta)}\ e2, [e3, e4] = 0, [e3, e5] = e4, [e4, e5] = -e3 \quad (1.14)$$

B.3. F12 Maple Worksheet

Maple Worksheet
Isotropy Type F12: Rotations

This worksheet steps through Chapter 5, serving to help validate the results. This chapters cover the isotropies of type F12, the rotations.

We begin by loading the packages needed.

```
> with(DifferentialGeometry):
  with(LieAlgebras):
```

We now initialize the most generic five-dimensional algebra with the subalgebra spanned by e5 of type F12.

```
> LD_Rotation:=LieAlgebraData([
  '[e1,e2]=a3*e3+a4*e4+a5*e5',
  '[e1,e3]=b1*e1+b2*e2+b3*e3+b4*e4+b5*e5',
  '[e2,e3]=c1*e1+c2*e2+c3*e3+c4*e4+c5*e5',
  '[e1,e4]=d1*e1+d2*e2+d3*e3+d4*e4+d5*e5',
  '[e2,e4]=g1*e1+g2*e2+g3*e3+g4*e4+g5*e5',
  '[e3,e4]=h1*e1+h2*e2+h3*e3+h4*e4+h5*e5',
  '[e5,e1]=-e2',
  '[e5,e2]=e1'
  ],[e1,e2,e3,e4,e5],alg_rot);
```

$$LD\_Rotation := [e1, e2] = a3\,e3 + a4\,e4 + a5\,e5, [e1, e3] = b1\,e1 + b2\,e2 + b3\,e3 + b4\,e4 + b5\,e5, [e1, e4] = d1\,e1 + d2\,e2 + d3\,e3 + d4\,e4 + d5\,e5, [e1, e5] = e2, [e2, e3] = c1\,e1 + c2\,e2 + c3\,e3 + c4\,e4 + c5\,e5, [e2, e4] = g1\,e1 + g2\,e2 + g3\,e3 + g4\,e4 + g5\,e5, [e2, e5] = -e1, [e3, e4] = h1\,e1 + h2\,e2 + h3\,e3 + h4\,e4 + h5\,e5, [e3, e5] = 0, [e4, e5] = 0 \quad (1.1)$$

```
> DGsetup(LD_Rotation,[e],[theta]);
```

Lie algebra: alg_rot **(1.2)**

Now, we examine the Jacobi identities.

```
> ddtheta1:=ExteriorDerivative(ExteriorDerivative(theta1));
  ddtheta2:=ExteriorDerivative(ExteriorDerivative(theta2));
  ddtheta3:=ExteriorDerivative(ExteriorDerivative(theta3));
  ddtheta4:=ExteriorDerivative(ExteriorDerivative(theta4));
  ddtheta5:=ExteriorDerivative(ExteriorDerivative(theta5));
```

$$ddtheta1 := -(a4\,h1 + c3\,b1 - b3\,c1 - b4\,g1 + c4\,d1 + b5)\,\theta1 \wedge \theta2 \wedge \theta3 + (a3\,h1 - g3\,b1 + d3\,c1 - g4\,d1 + d4\,g1 - d5)\,\theta1 \wedge \theta2 \wedge \theta4 - (h3\,b1 - b2\,g1 - b3\,h1 + d2\,c1 + h4\,d1 - d4\,h1)\,\theta1 \wedge \theta3 \wedge \theta4 - (b2 + c1)\,\theta1 \wedge \theta3 \wedge \theta5 - (d2 + g1)\,\theta1 \wedge \theta4 \wedge \theta5 - (g1\,b1 - c1\,d1 + g2\,c1 + h3\,c1 - c2\,g1 - c3\,h1 + h4\,g1 - g4\,h1 - h5)\,\theta2 \wedge \theta3 \wedge \theta4 + (b1 - c2)\,\theta2 \wedge \theta3 \wedge \theta5 + (d1 - g2)\,\theta2 \wedge \theta4 \wedge \theta5 - h2\,\theta3 \wedge \theta4 \wedge \theta5$$

$$ddtheta2 := -(a4\,h2 + c3\,b2 - b3\,c2 - b4\,g2 + c4\,d2 + c5)\,\theta1 \wedge \theta2 \wedge \theta3 + (a3\,h2 - g3\,b2 + d3\,c2 - g4\,d2 + d4\,g2 - g5)\,\theta1 \wedge \theta2 \wedge \theta4 + (b1\,d2 - d1\,b2 + b2\,g2 - h3\,b2 + b3\,h2 - d2\,c2 - h4\,d2 + d4\,h2 - h5)\,\theta1 \wedge \theta3 \wedge \theta4 + (b1 - c2)\,\theta1 \wedge \theta3 \wedge \theta5 + (d1 - g2)\,\theta1 \wedge \theta4 \wedge \theta5 - (b2\,g1 - d2\,c1 + h3\,c2 - c3\,h2 + h4\,g2 - g4\,h2)\,\theta2 \wedge \theta3 \wedge \theta4 + (b2 + c1)\,\theta2 \wedge \theta3 \wedge \theta5 + (d2 + g1)\,\theta2 \wedge \theta4 \wedge \theta5 + h1\,\theta3 \wedge \theta4 \wedge \theta5$$

$$ddtheta3 := -(b1\,a3 + c2\,a3 + a4\,h3 - b4\,g3 + c4\,d3)\,\theta1 \wedge \theta2 \wedge \theta3 - (d1\,a3 + g2\,a3 - a3\,h3 + g3\,b3 - d3\,c3 + g4\,d3 - d4\,g3)\,\theta1 \wedge \theta2 \wedge \theta4 - (a3\,h2 - b1\,d3 - g3\,b2 + d1\,b3$$

$+ d2\, c3 + h4\, d3 - d4\, h3)\ \theta1 \wedge \theta3 \wedge \theta4 - c3\, \theta1 \wedge \theta3 \wedge \theta5 - g3\, \theta1 \wedge \theta4 \wedge \theta5 + (a3\, h1 - g1\, b3 + d3\, c1 + c2\, g3 - g2\, c3 - h4\, g3 + g4\, h3)\ \theta2 \wedge \theta3 \wedge \theta4 + b3\, \theta2 \wedge \theta3 \wedge \theta5 + d3\, \theta2 \wedge \theta4 \wedge \theta5$

$ddtheta4 := -(b1\, a4 + c2\, a4 + a4\, h4 - b3\, c4 + c3\, b4 - b4\, g4 + c4\, d4)\ \theta1 \wedge \theta2 \wedge \theta3 + (a3\, h4 - d1\, a4 - g2\, a4 - b4\, g3 + c4\, d3)\ \theta1 \wedge \theta2 \wedge \theta4 - (a4\, h2 - b1\, d4 - b2\, g4 - b3\, h4 + d1\, b4 + h3\, b4 + c4\, d2)\ \theta1 \wedge \theta3 \wedge \theta4 - c4\, \theta1 \wedge \theta3 \wedge \theta5 - g4\, \theta1 \wedge \theta4 \wedge \theta5 + (a4\, h1 - b4\, g1 + c1\, d4 + c2\, g4 + c3\, h4 - g2\, c4 - h3\, c4)\ \theta2 \wedge \theta3 \wedge \theta4 + b4\, \theta2 \wedge \theta3 \wedge \theta5 + d4\, \theta2 \wedge \theta4 \wedge \theta5$

$ddtheta5 := -(a4\, h5 + b1\, a5 + c2\, a5 - b3\, c5 - b4\, g5 + c3\, b5 + c4\, d5)\ \theta1 \wedge \theta2 \wedge \theta3 + (a3\, h5 - d1\, a5 - g2\, a5 - g3\, b5 + d3\, c5 + d4\, g5 - g4\, d5)\ \theta1 \wedge \theta2 \wedge \theta4 - (h2\, a5 - b1\, d5 - b2\, g5 - b3\, h5 + d1\, b5 + h3\, b5 + d2\, c5 - d4\, h5 + h4\, d5)\ \theta1 \wedge \theta3 \wedge \theta4 - c5\, \theta1 \wedge \theta3 \wedge \theta5 - g5\, \theta1 \wedge \theta4 \wedge \theta5 + (h1\, a5 - g1\, b5 + c1\, d5 + c2\, g5 + c3\, h5 - g2\, c5 - h3\, c5 + g4\, h5 - h4\, g5)\ \theta2 \wedge \theta3 \wedge \theta4 + b5\, \theta2 \wedge \theta3 \wedge \theta5 + d5\, \theta2 \wedge \theta4 \wedge \theta5$ **(1.3)**

We now examine the linear parts of the equations given by the Jacobi identities:

```
> Eq1:={
  Hook(Hook(Hook(ddtheta1,e1),e3),e5),
  Hook(Hook(Hook(ddtheta1,e1),e4),e5),
  Hook(Hook(Hook(ddtheta1,e2),e3),e5),
  Hook(Hook(Hook(ddtheta1,e2),e4),e5),
  Hook(Hook(Hook(ddtheta1,e3),e4),e5),

  Hook(Hook(Hook(ddtheta2,e1),e3),e5),
  Hook(Hook(Hook(ddtheta2,e1),e4),e5),
  Hook(Hook(Hook(ddtheta2,e2),e3),e5),
  Hook(Hook(Hook(ddtheta2,e2),e4),e5),
  Hook(Hook(Hook(ddtheta2,e3),e4),e5),

  Hook(Hook(Hook(ddtheta3,e1),e3),e5),
  Hook(Hook(Hook(ddtheta3,e1),e4),e5),
  Hook(Hook(Hook(ddtheta3,e2),e3),e5),
  Hook(Hook(Hook(ddtheta3,e2),e4),e5),

  Hook(Hook(Hook(ddtheta4,e1),e3),e5),
  Hook(Hook(Hook(ddtheta4,e1),e4),e5),
  Hook(Hook(Hook(ddtheta4,e2),e3),e5),
  Hook(Hook(Hook(ddtheta4,e2),e4),e5),

  Hook(Hook(Hook(ddtheta5,e1),e3),e5),
  Hook(Hook(Hook(ddtheta5,e1),e4),e5),
  Hook(Hook(Hook(ddtheta5,e2),e3),e5),
  Hook(Hook(Hook(ddtheta5,e2),e4),e5)
  } ;
```

$Eq1 := \{b3, b4, b5, d3, d4, d5, h1, -c3, -c4, -c5, -g3, -g4, -g5, -h2, b1 - c2, -b2 - c1, b2 + c1, d1 - g2, -d2 - g1, d2 + g1\}$ **(1.4)**

```
> Eq2:=solve(Eq1,{b3,b4,b5,c1,c2,c3,c4,c5,d3,d4,d5,g1,g2,g3,g4,g5,h1,h2});
```

$Eq2 := \{b3 = 0, b4 = 0, b5 = 0, c1 = -b2, c2 = b1, c3 = 0, c4 = 0, c5 = 0, d3 = 0, d4 = 0, d5 = 0, g1 = -d2, g2 = d1, g3 = 0, g4 = 0, g5 = 0, h1 = 0, h2 = 0\}$ **(1.5)**

Evaluating the Jacobi identity with this solution simlifies the quadratic parts. We extract these equations and solve them.

```
> ddtheta1_s:=Tools:-DGsimplify(eval(ddtheta1,Eq2));
  ddtheta2_s:=Tools:-DGsimplify(eval(ddtheta2,Eq2));
  ddtheta3_s:=Tools:-DGsimplify(eval(ddtheta3,Eq2));
  ddtheta4_s:=Tools:-DGsimplify(eval(ddtheta4,Eq2));
  ddtheta5_s:=Tools:-DGsimplify(eval(ddtheta5,Eq2));
```

$$ddtheta1\_s := -\left(h3\,b1 + h4\,d1\right)\,\theta 1 \wedge \theta 3 \wedge \theta 4 + \left(h3\,b2 + h4\,d2 + h5\right)\,\theta 2 \wedge \theta 3 \wedge \theta 4$$

$$ddtheta2\_s := -\left(h3\,b2 + h4\,d2 + h5\right)\,\theta 1 \wedge \theta 3 \wedge \theta 4 - \left(h3\,b1 + h4\,d1\right)\,\theta 2 \wedge \theta 3 \wedge \theta 4$$

$$ddtheta3\_s := -\left(2\,b1\,a3 + a4\,h3\right)\,\theta 1 \wedge \theta 2 \wedge \theta 3 - \left(2\,d1\,a3 - a3\,h3\right)\,\theta 1 \wedge \theta 2 \wedge \theta 4$$

$$ddtheta4\_s := -\left(2\,b1\,a4 + a4\,h4\right)\,\theta 1 \wedge \theta 2 \wedge \theta 3 + \left(a3\,h4 - 2\,d1\,a4\right)\,\theta 1 \wedge \theta 2 \wedge \theta 4$$

$$ddtheta5\_s := -\left(a4\,h5 + 2\,b1\,a5\right)\,\theta 1 \wedge \theta 2 \wedge \theta 3 + \left(a3\,h5 - 2\,d1\,a5\right)\,\theta 1 \wedge \theta 2 \wedge \theta 4 \quad \textbf{(1.6)}$$

Applying the change of basis below further eliminates without changing the isotropy. We also relable constants for convenience; the combination b2*h3+d2*h4 plays the role of h5 and is therefore zero by the Jacobi identities, and a3*b2+a4*d2+a5 plays the role of a5, so we relable it as such. Using b2*h3+d2*h4+h5=0 from ddtheta1_s, we find we may take h5=0 in this basis. This change of basis also gives b2=d2=0.

```
> LD_R2:=
  eval(eval(LieAlgebraData([e1,e2,e3-b2*e5,e4-d2*e5,e5],alg_R2),Eq2),
  {a3*b2+a4*d2+a5=a5,b2*h3+d2*h4+h5=0});
```

$$LD\_R2 := \left[e1, e2\right] = a3\,e3 + a4\,e4 + a5\,e5, \left[e1, e3\right] = b1\,e1, \left[e1, e4\right] = d1\,e1, \left[e1, e5\right] = e2, \left[e2, e3\right] = b1\,e2, \left[e2, e4\right] = d1\,e2, \left[e2, e5\right] = -e1, \left[e3, e4\right] = h3\,e3 + h4\,e4, \left[e3, e5\right] = 0, \left[e4, e5\right] = 0 \quad \textbf{(1.7)}$$

We initialize the simplified algebra and examine the Jacobi identities again:

```
> DGsetup(LD_R2,[e],[theta]);
```

Lie algebra: alg_R2 **(1.8)**

```
> ddtheta1:=ExteriorDerivative(ExteriorDerivative(theta1));
  ddtheta2:=ExteriorDerivative(ExteriorDerivative(theta2));
  ddtheta3:=ExteriorDerivative(ExteriorDerivative(theta3));
  ddtheta4:=ExteriorDerivative(ExteriorDerivative(theta4));
  ddtheta5:=ExteriorDerivative(ExteriorDerivative(theta5));
```

$$ddtheta1 := -\left(h3\,b1 + h4\,d1\right)\,\theta 1 \wedge \theta 3 \wedge \theta 4$$

$$ddtheta2 := -\left(h3\,b1 + h4\,d1\right)\,\theta 2 \wedge \theta 3 \wedge \theta 4$$

$$ddtheta3 := -\left(2\,b1\,a3 + a4\,h3\right)\,\theta 1 \wedge \theta 2 \wedge \theta 3 - \left(2\,d1\,a3 - a3\,h3\right)\,\theta 1 \wedge \theta 2 \wedge \theta 4$$

$$ddtheta4 := -\left(2\,b1\,a4 + a4\,h4\right)\,\theta 1 \wedge \theta 2 \wedge \theta 3 + \left(a3\,h4 - 2\,d1\,a4\right)\,\theta 1 \wedge \theta 2 \wedge \theta 4$$

$$ddtheta5 := -2\,b1\,a5\,\theta 1 \wedge \theta 2 \wedge \theta 3 - 2\,d1\,a5\,\theta 1 \wedge \theta 2 \wedge \theta 4 \quad \textbf{(1.9)}$$

For reference, here are the structure equations again:

```
> LieAlgebraData([e1,e2,e3,e4,e5]);
```

$$\left[e1, e2\right] = a3\,e3 + a4\,e4 + a5\,e5, \left[e1, e3\right] = b1\,e1, \left[e1, e4\right] = d1\,e1, \left[e1, e5\right] = e2, \left[e2, e3\right] = b1\,e2, \left[e2, e4\right] = d1\,e2, \left[e2, e5\right] = -e1, \left[e3, e4\right] = h3\,e3 + h4\,e4, \left[e3, e5\right] = 0, \left[e4, e5\right] = 0 \quad \textbf{(1.10)}$$

We consider two cases by constructing the largest subalgebra, i, of the center, n, of the isotropy that is also an ideal in the algebra.

Any vector in n has the form of y below and any vector in the span of e1 and e2 has the form of x below:

```
> y:=evalDG(mu*e3+lambda*e4+nu*e5);
  x:=evalDG(alpha*e1+beta*e2);
```

$$y := \mu\, e3 + \lambda\, e4 + \nu\, e5$$

$$x := \alpha\, e1 + \beta\, e2 \quad \textbf{(1.11)}$$

The Lie bracket is:

```
> LieBracket(x,y);
```

$$(\alpha\,\mu\, b1 + \alpha\,\lambda\, d1 - \beta\,\nu)\; e1 + (\beta\,\mu\, b1 + \beta\,\lambda\, d1 + \alpha\,\nu)\; e2 \quad \textbf{(1.12)}$$

If y is in i, then [x,y] is in n, so

$\alpha\mu b1 + \alpha\lambda d1 - \beta\nu = 0$

$\beta\mu b1 + \beta\lambda d1 + \alpha\nu = 0$

for all α and β.

This requires $\nu = 0$ and so i takes the form $i=\{\mu e3+\lambda e4 \mid \mu b1=-\lambda d1\}$.
Note that i is two-dimensional if and only if b1=d1=0. We case split on this.

Section 5.1: i is Two-Dimensional

In this case, b1=d1=0. We initialize this algebra.

```
> LD_R51:=eval(
  LieAlgebraData(
  [e1,e2,e3,e4,e5],
  alg_R51),
  {b1=0,d1=0});
```

$$LD\_R51 := \big[e1, e2\big] = a3\, e3 + a4\, e4 + a5\, e5,\ [e1, e3] = 0,\ [e1, e4] = 0,\ [e1, e5] = e2, \quad \textbf{(1.1.1)}$$

$$[e2, e3] = 0,\ [e2, e4] = 0,\ [e2, e5] = -e1,\ \big[e3, e4\big] = h3\, e3 + h4\, e4,\ [e3, e5] = 0,$$

$$[e4, e5] = 0$$

```
> DGsetup(LD_R51,['x'],[o]);
```

Lie algebra: alg_R51 **(1.1.2)**

We examine the Jacobi identities in this algebra.

```
> ExteriorDerivative(ExteriorDerivative(o1));
  ExteriorDerivative(ExteriorDerivative(o2));
  ExteriorDerivative(ExteriorDerivative(o3));
  ExteriorDerivative(ExteriorDerivative(o4));
  ExteriorDerivative(ExteriorDerivative(o5));
```

$$0\ o1 \wedge o2 \wedge o3$$

$$0\ o1 \wedge o2 \wedge o3$$

$$-a4\, h3\, o1 \wedge o2 \wedge o3 + a3\, h3\, o1 \wedge o2 \wedge o4$$

$$-a4\, h4\, o1 \wedge o2 \wedge o3 + a3\, h4\, o1 \wedge o2 \wedge o4$$

$$0\ o1 \wedge o2 \wedge o3 \quad \textbf{(1.1.3)}$$

We see that a4*h3=a3*h3=a4*h4=a4*h3=0, so either a3=a4=0 or h3=h4=0.

To distinguish these, we consider the center. The center is trivial if and only if one or both of h3 and h4 is nonzero, in which case a3=a4=0. Otherwise, h3=h4=0 and the center contains e3 and e4.

5.1.1: The Center is Trivial

In this case, a3=a4=0. The subalgebra spanned by e3 and e4 decomposes and is nonabelian. Thus there is a basis in which h3=1 and h4=0. We initialize the algebra in this basis:

```
> LD_R511:=eval(
  LieAlgebraData(
  [x1,x2,x3,x4,x5],
  alg_R511),
  {a3=0,a4=0,h3=1,h4=0});
```

$$LD\_R511 := [e1, e2] = a5\, e5, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = e2, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e1, [e3, e4] = e3, [e3, e5] = 0, [e4, e5] = 0 \quad (1.1.1.1)$$

```
> DGsetup(LD_R511,['y'],[p]);
```

Lie algebra: alg_R511 **(1.1.1.2)**

Consider the Killing form:

```
> Killing();
```

$$\begin{bmatrix} 2\, a5 & 0 & 0 & 0 & 0 \\ 0 & 2\, a5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} \quad (1.1.1.3)$$

The sign of a5 determines the signature of the Killing form by its sign.

If a5 is not zero, the following scales a5 to +/-1. We may thus take a5 in {-1,0,1}:

```
> LieAlgebraData([
  y1/sqrt(abs(a5)),
  y2/sqrt(abs(a5)),
  y3,
  y4,
  y5
  ]);
```

$$[e1, e2] = \frac{a5}{|a5|}\, e5, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = e2, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e1, [e3, e4] = e3, [e3, e5] = 0, [e4, e5] = 0 \quad (1.1.1.4)$$

If a5 = 0, the change of basis below gives s_3,3+s_2,1 with a=0 and isotropy e3

```
> eval(LieAlgebraData([-y1,-y2,y5,y3,-y4]),a5=0);
```

$$[e1, e2] = 0, [e1, e3] = e2, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = -e1, [e2, e4] = 0, [e2, e5] = 0, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = -e4 \quad (1.1.1.5)$$

If a5 = 1, the change of basis below gives sl(2,F)+s_2,1 with isotropy e1-e3

```
> eval(LieAlgebraData([
  -y2+y5,
  2*y1,
  -y2-y5,
  y3,
  y4
  ]),a5=1);
```

$$[e1, e2] = 2\, e1, [e1, e3] = -e2, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = 2\, e3, [e2, e4] = 0, [e2, e5] = 0, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = e4 \quad (1.1.1.6)$$

If a5 = -1, the change of basis below gives so(3,R)+s_2,1 with isotropy e1

```
> eval(LieAlgebraData([y5,y2,y1,y3,y4]),a5=-1);
```

$[e1, e2] = e3, [e1, e3] = -e2, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = 0, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = e4$ **(1.1.1.7)**

5.1.2: The Center is Two-Dimensional

If the center is two-dimensional, then h3=h4=0. We initialize this algebra:

```
> LD_R512:=eval(
  LieAlgebraData([x1,x2,x3,x4,x5],alg_R512),
  {h3=0,h4=0});
```

$LD\_R512 := [e1, e2] = a3\, e3 + a4\, e4 + a5\, e5, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = e2, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e1, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0$ **(1.1.2.1)**

```
> DGsetup(LD_R512,['y'],[p]);
```

Lie algebra: alg_R512 **(1.1.2.2)**

Now consider the Killing form.

```
> Killing();
```

$$\begin{bmatrix} 2\,a5 & 0 & 0 & 0 & 0 \\ 0 & 2\,a5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} \quad \textbf{(1.1.2.3)}$$

The sign of a5 determines the signature of the Killing form by its sign.

If a5 is not zero, the following scales a5 to +/-1. We may thus take a5 in {-1,0,1}:

```
> LieAlgebraData([
  y1/sqrt(abs(a5)),
  y2/sqrt(abs(a5)),
  y3,
  y4,
  y5
  ]);
```

$\left[e1, e2\right] = \frac{a3}{|a5|}\, e3 + \frac{a4}{|a5|}\, e4 + \frac{a5}{|a5|}\, e5, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = e2, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e1, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0$ **(1.1.2.4)**

Consider also the derived algebra:

```
> Series(alg_R512,"Derived")[2];
```

$[a3\, y3 + a4\, y4 + a5\, y5, y2, -y1]$ **(1.1.2.5)**

First, suppose a5=0. Then the of the derived algebra is two if a3=a4=0 and three otherwise.

If a5 is nonzero, then the isotropy is in the derived algebra if and only if a3=a4=0.

If a3 or a4 is nonzero, we may take a3=1 and a4=0 via one of the following changes of basis (whichever is nondegenerate):

```
> LieAlgebraData([y1,y2,a3*y3+a4*y4,y3,y5]);
```

```
LieAlgebraData([y1,y2,a3*y3+a4*y4,y4,y5]);
```

$[e1, e2] = e3 + a5\, e5, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = e2, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e1, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0$

$[e1, e2] = e3 + a5\, e5, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = e2, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e1, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0$ **(1.1.2.6)**

We thus have six possibilities:

a5=1, a3=0:
The change of basis below gives
sl(2,F)+2n_1,1 with isotropy e1-e3

```
> eval(LieAlgebraData([
  -y2+y5,
  2*y1,
  -y2-y5,
  y3,
  y4
  ]),{a5=1,a3=0,a4=0});
```

$[e1, e2] = 2\, e1, [e1, e3] = -e2, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = 2\, e3, [e2, e4] = 0, [e2, e5] = 0, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0$ **(1.1.2.7)**

a5=1, a3=1:
The change of basis below gives
sl(2,F)+2n_1,1 with isotropy e1-e3-2e4

```
> eval(LieAlgebraData([
  -y2+y3+y5,
  2*y1,
  -y2-y3-y5,
  y3,
  y4
  ]),{a5=1,a3=1,a4=0});
```

$[e1, e2] = 2\, e1, [e1, e3] = -e2, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = 2\, e3, [e2, e4] = 0, [e2, e5] = 0, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0$ **(1.1.2.8)**

a5=-1, a3=0:
The change of basis below gives
so(3,R)+2n_1,1 with isotropy e1

```
> eval(LieAlgebraData([
  y5,
  y2,
  y1,
  y3,
  y4
  ]),{a5=-1,a3=0,a4=0});
```

$[e1, e2] = e3, [e1, e3] = -e2, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = 0, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0$ **(1.1.2.9)**

a5=-1, a3=1:
The change of basis below gives
so(3,R)+2n_1,1 with isotropy e1-e4

```
> eval(LieAlgebraData([
  y3-y5,
  -y2,
  y1,
  y3,
  y4
  ]),{a5=-1,a3=1,a4=0});
```

$[e1, e2] = e3, [e1, e3] = -e2, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = 0, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0$ **(1.1.2.10)**

a5=0, a3=0:
The change of basis below gives
s_3,3+2n_1,1 with a=0 and isotropy e3

```
> eval(LieAlgebraData([
  -y1,
  -y2,
  y5,
  y3,
  y4
  ]),{a5=0,a3=0,a4=0});
```

$$[e1, e2] = 0, [e1, e3] = e2, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = -e1, [e2, e4] = 0, [e2, e5] = 0, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0 \quad \textbf{(1.1.2.11)}$$

a5=0, a3=1:
The change of basis below gives
s_4,7+n_1,1 with isotropy e4

```
> eval(LieAlgebraData([
  -y3,
  -y1,
  y2,
  -y5,
  y4
  ]),{a5=0,a3=1,a4=0});
```

$$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = e1, [e2, e4] = e3, [e2, e5] = 0, [e3, e4] = -e2, [e3, e5] = 0, [e4, e5] = 0 \quad \textbf{(1.1.2.12)}$$

Section 5.2: The Dimension of i is Less Than Two

In this case, at least either or both of b1 and d1 is nonzero.

Recall the Jacobi identities:

```
> ddtheta1;
  ddtheta2;
  ddtheta3;
  ddtheta4;
  ddtheta5;
```

$$-(b1\ h3 + d1\ h4)\ \theta 1 \wedge \theta 3 \wedge \theta 4$$

$$-(b1\ h3 + d1\ h4)\ \theta 2 \wedge \theta 3 \wedge \theta 4$$

$$-(2\ a3\ b1 + a4\ h3)\ \theta 1 \wedge \theta 2 \wedge \theta 3 - (2\ a3\ d1 - a3\ h3)\ \theta 1 \wedge \theta 2 \wedge \theta 4$$

$$-(2\ a4\ b1 + a4\ h4)\ \theta 1 \wedge \theta 2 \wedge \theta 3 + (a3\ h4 - 2\ a4\ d1)\ \theta 1 \wedge \theta 2 \wedge \theta 4$$

$$-2\ b1\ a5\ \theta 1 \wedge \theta 2 \wedge \theta 3 - 2\ d1\ a5\ \theta 1 \wedge \theta 2 \wedge \theta 4 \quad \textbf{(1.2.1)}$$

If b1 is nonzero, we apply the following change of basis and relabel constants, using b1*h3+d1*h4=0 from ddtheta1=0.

```
> LD_R52:=eval(LieAlgebraData([
  e1,e2,1/b1*e3,-d1/b1*e3+e4,e5
  ],alg_R52),
  {b1*h3+d1*h4=0,a3*b1+a4*d1=a3,h4/b1=h4});
```

$$LD\_R52 := [e1, e2] = a3\ e3 + a4\ e4 + a5\ e5, [e1, e3] = e1, [e1, e4] = 0, [e1, e5] = e2, [e2, e3] = e2, [e2, e4] = 0, [e2, e5] = -e1, [e3, e4] = h4\ e4, [e3, e5] = 0, [e4, e5] = 0 \quad \textbf{(1.2.2)}$$

If d1 is nonzero, we apply the following change of basis and relabel constants, using b1*h3+d1*h4=0 from ddtheta1=0.

```
> LD_R52:=eval(LieAlgebraData([
```

```
  e1,e2,1/d1*e4,-b1/d1*e4+e3,e5
  ],alg_R52),
  {b1*h3+d1*h4=0,a3=a4,a3*b1+a4*d1=a3,h3/d1=-h4});
```

$LD\_R52 := [e1, e2] = a3\, e3 + a4\, e4 + a5\, e5, [e1, e3] = e1, [e1, e4] = 0, [e1, e5] = e2, [e2, e3] = e2, [e2, e4] = 0, [e2, e5] = -e1, [e3, e4] = h4\, e4, [e3, e5] = 0, [e4, e5] = 0$ **(1.2.3)**

In either case, we can take b1=1, d1=0, and h3=0. We now initialize the algebra and impose the Jacobi identities.

```
> DGsetup(LD_R52,['x'],[o]);
```

Lie algebra: alg_R52 **(1.2.4)**

```
> ExteriorDerivative(ExteriorDerivative(o1));
  ExteriorDerivative(ExteriorDerivative(o2));
  ExteriorDerivative(ExteriorDerivative(o3));
  ExteriorDerivative(ExteriorDerivative(o4));
  ExteriorDerivative(ExteriorDerivative(o5));
```

$$0\; o1 \wedge o2 \wedge o3$$

$$0\; o1 \wedge o2 \wedge o3$$

$$-2\, a3\; o1 \wedge o2 \wedge o3$$

$$-(a4\, h4 + 2\, a4)\; o1 \wedge o2 \wedge o3 + a3\, h4\; o1 \wedge o2 \wedge o4$$

$$-2\, a5\; o1 \wedge o2 \wedge o3$$ **(1.2.5)**

We thus find a3=a5=0, so we initialize this algebra. Note that we also will require either a4 = 0 or h4 = -2 to completely satisfy the Jacobi identities.

```
> DGsetup(eval(LD_R52,{a3=0,a5=0}),['y'],[p]);
```

Lie algebra: alg_R52 **(1.2.6)**

The structure equations are as follows:

```
> LieAlgebraData([y1,y2,y3,y4,y5]);
```

$[e1, e2] = a4\, e4, [e1, e3] = e1, [e1, e4] = 0, [e1, e5] = e2, [e2, e3] = e2, [e2, e4] = 0, [e2, e5] = -e1, [e3, e4] = h4\, e4, [e3, e5] = 0, [e4, e5] = 0$ **(1.2.7)**

Consider the derived series. It is spanned by {a4*y4, y1, y2, h4*y4}.

The structure Lie brackets given by these vectors are:

```
> LieDerivative(
  [a4*y4, y1, y2, h4*y4],
  [a4*y4, y1, y2, h4*y4]);
```

$[[0\, y1, 0\, y1, 0\, y1, 0\, y1], [0\, y1, 0\, y1, a4\, y4, 0\, y1], [0\, y1, -a4\, y4, 0\, y1, 0\, y1], [0\, y1, 0\, y1, 0\, y1, 0\, y1]]$ **(1.2.8)**

Thus, a4 is nonzero if and only if the second derived algebra is one-dimensional, in which case h4 = -2 to satisfy the Jacobi identities and the following change of basis gives s_5,45 with isotropy spanned by e5.

```
> eval(LieAlgebraData([-a4*y4,y1,-y2,-y3-y4,y5]),h4=-2);
```

$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = -2\, e1, [e1, e5] = 0, [e2, e3] = e1, [e2, e4] = -e2, [e2, e5] = -e3, [e3, e4] = -e3, [e3, e5] = e2, [e4, e5] = 0$ **(1.2.9)**

Given a4 = 0, the derived algebra is two-dimensional for h4 = 0, in which case the following change of basis gives s_4,12+n_1,1 with isotropy spanned by e4.

```
> eval(LieAlgebraData([y1,y2,-y3,-y5,y4]),{h4=0,a4=0});
```

(1.2.10)

$[e1, e2] = 0, [e1, e3] = -e1, [e1, e4] = -e2, [e1, e5] = 0, [e2, e3] = -e2, [e2, e4] = e1, [e2, e5] = 0, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0$ **(1.2.10)**

Given a4 = 0, the derived algebra is three-dimensional for h4 nonzero,in which case the following change of basis gives s_5,43 with isotropy spanned by e5. We identify our parameter h4 with the parameter β in the algebra classification tables.

```
> eval(
  LieAlgebraData([y4,-y1-y2,-y1+y2,-y3,-y5]),
  {h4=beta,a4=0});
```

$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = \beta\, e1, [e1, e5] = 0, [e2, e3] = 0, [e2, e4] = -e2, [e2, e5] = e3, [e3, e4] = -e3, [e3, e5] = -e2, [e4, e5] = 0$ **(1.2.11)**

B.4. F13 Maple Worksheet

Maple Worksheet
Isotropy Type F13: Boosts

This worksheet steps through Chapter 6, serving to help validate the results. This chapters cover the isotropies of type F13, the boosts.

We begin by loading the packages needed.

```
> with(DifferentialGeometry):
  with(LieAlgebras):
```

We now initialize the most generic five-dimensional algebra with the subalgebra spanned by e5 of type F13.

```
> LD_Boost:=LieAlgebraData([
  '[e1,e2]=a3*e3+a4*e4+a5*e5',
  '[e1,e3]=b1*e1+b2*e2+b3*e3+b4*e4+b5*e5',
  '[e2,e3]=c1*e1+c2*e2+c3*e3+c4*e4+c5*e5',
  '[e1,e4]=d1*e1+d2*e2+d3*e3+d4*e4+d5*e5',
  '[e2,e4]=g1*e1+g2*e2+g3*e3+g4*e4+g5*e5',
  '[e3,e4]=h1*e1+h2*e2+h3*e3+h4*e4+h5*e5',
  '[e5,e1]=e2',
  '[e5,e2]=e1'
  ],[e1,e2,e3,e4,e5],alg_bst);
```

$$LD\_Boost := [e1, e2] = a3\, e3 + a4\, e4 + a5\, e5, [e1, e3] = b1\, e1 + b2\, e2 + b3\, e3 + b4\, e4 + b5\, e5, [e1, e4] = d1\, e1 + d2\, e2 + d3\, e3 + d4\, e4 + d5\, e5, [e1, e5] = -e2, [e2, e3] = c1\, e1 + c2\, e2 + c3\, e3 + c4\, e4 + c5\, e5, [e2, e4] = g1\, e1 + g2\, e2 + g3\, e3 + g4\, e4 + g5\, e5, [e2, e5] = -e1, [e3, e4] = h1\, e1 + h2\, e2 + h3\, e3 + h4\, e4 + h5\, e5, [e3, e5] = 0, [e4, e5] = 0 \quad \textbf{(1.1)}$$

```
> DGsetup(LD_Boost,[e],[theta]);
```

Lie algebra: alg_bst **(1.2)**

Now, we examine the Jacobi identities.

```
> ddtheta1:=ExteriorDerivative(ExteriorDerivative(theta1));
  ddtheta2:=ExteriorDerivative(ExteriorDerivative(theta2));
  ddtheta3:=ExteriorDerivative(ExteriorDerivative(theta3));
  ddtheta4:=ExteriorDerivative(ExteriorDerivative(theta4));
  ddtheta5:=ExteriorDerivative(ExteriorDerivative(theta5));
```

$$ddtheta1 := -(a4\, h1 + c3\, b1 - b3\, c1 - b4\, g1 + c4\, d1 + b5)\, \theta1 \wedge \theta2 \wedge \theta3 + (a3\, h1 - g3\, b1 + d3\, c1 - g4\, d1 + d4\, g1 - d5)\, \theta1 \wedge \theta2 \wedge \theta4 - (h3\, b1 - b2\, g1 - b3\, h1 + d2\, c1 + h4\, d1 - d4\, h1)\, \theta1 \wedge \theta3 \wedge \theta4 - (b2 - c1)\, \theta1 \wedge \theta3 \wedge \theta5 - (d2 - g1)\, \theta1 \wedge \theta4 \wedge \theta5 - (g1\, b1 - c1\, d1 + g2\, c1 + h3\, c1 - c2\, g1 - c3\, h1 + h4\, g1 - g4\, h1 - h5)\, \theta2 \wedge \theta3 \wedge \theta4 + (b1 - c2)\, \theta2 \wedge \theta3 \wedge \theta5 + (d1 - g2)\, \theta2 \wedge \theta4 \wedge \theta5 - h2\, \theta3 \wedge \theta4 \wedge \theta5$$

$$ddtheta2 := -(a4\, h2 + c3\, b2 - b3\, c2 - b4\, g2 + c4\, d2 - c5)\, \theta1 \wedge \theta2 \wedge \theta3 + (a3\, h2 - g3\, b2 + d3\, c2 - g4\, d2 + d4\, g2 + g5)\, \theta1 \wedge \theta2 \wedge \theta4 + (b1\, d2 - d1\, b2 + b2\, g2 - h3\, b2 + b3\, h2 - d2\, c2 - h4\, d2 + d4\, h2 + h5)\, \theta1 \wedge \theta3 \wedge \theta4 - (b1 - c2)\, \theta1 \wedge \theta3 \wedge \theta5 - (d1 - g2)\, \theta1 \wedge \theta4 \wedge \theta5 - (b2\, g1 - d2\, c1 + h3\, c2 - c3\, h2 + h4\, g2 - g4\, h2)\, \theta2 \wedge \theta3 \wedge \theta4 + (b2 - c1)\, \theta2 \wedge \theta3 \wedge \theta5 + (d2 - g1)\, \theta2 \wedge \theta4 \wedge \theta5 - h1\, \theta3 \wedge \theta4 \wedge \theta5$$

$$ddtheta3 := -(b1\, a3 + c2\, a3 + a4\, h3 - b4\, g3 + c4\, d3)\, \theta1 \wedge \theta2 \wedge \theta3 - (d1\, a3 + g2\, a3 - a3\, h3 + g3\, b3 - d3\, c3 + g4\, d3 - d4\, g3)\, \theta1 \wedge \theta2 \wedge \theta4 - (a3\, h2 - b1\, d3 - g3\, b2 + d1\, b3$$

$+ d2\, c3 + h4\, d3 - d4\, h3)\ \theta 1 \wedge \theta 3 \wedge \theta 4 + c3\, \theta 1 \wedge \theta 3 \wedge \theta 5 + g3\, \theta 1 \wedge \theta 4 \wedge \theta 5 + (a3\, h1$
$- g1\, b3 + d3\, c1 + c2\, g3 - g2\, c3 - h4\, g3 + g4\, h3)\ \theta 2 \wedge \theta 3 \wedge \theta 4 + b3\, \theta 2 \wedge \theta 3 \wedge \theta 5 + d3\, \theta 2$
$\wedge \theta 4 \wedge \theta 5$

$ddtheta4 := -(b1\, a4 + c2\, a4 + a4\, h4 - b3\, c4 + c3\, b4 - b4\, g4 + c4\, d4)\ \theta 1 \wedge \theta 2 \wedge \theta 3 + (a3\, h4$
$- d1\, a4 - g2\, a4 - b4\, g3 + c4\, d3)\ \theta 1 \wedge \theta 2 \wedge \theta 4 - (a4\, h2 - b1\, d4 - b2\, g4 - b3\, h4$
$+ d1\, b4 + h3\, b4 + c4\, d2)\ \theta 1 \wedge \theta 3 \wedge \theta 4 + c4\, \theta 1 \wedge \theta 3 \wedge \theta 5 + g4\, \theta 1 \wedge \theta 4 \wedge \theta 5 + (a4\, h1$
$- b4\, g1 + c1\, d4 + c2\, g4 + c3\, h4 - g2\, c4 - h3\, c4)\ \theta 2 \wedge \theta 3 \wedge \theta 4 + b4\, \theta 2 \wedge \theta 3 \wedge \theta 5 + d4\, \theta 2$
$\wedge \theta 4 \wedge \theta 5$

$ddtheta5 := -(a4\, h5 + b1\, a5 + c2\, a5 - b3\, c5 - b4\, g5 + c3\, b5 + c4\, d5)\ \theta 1 \wedge \theta 2 \wedge \theta 3 + (a3\, h5$ **(1.3)**
$- d1\, a5 - g2\, a5 - g3\, b5 + d3\, c5 + d4\, g5 - g4\, d5)\ \theta 1 \wedge \theta 2 \wedge \theta 4 - (h2\, a5 - b1\, d5$
$- b2\, g5 - b3\, h5 + d1\, b5 + h3\, b5 + d2\, c5 - d4\, h5 + h4\, d5)\ \theta 1 \wedge \theta 3 \wedge \theta 4 + c5\, \theta 1 \wedge \theta 3 \wedge \theta 5$
$+ g5\, \theta 1 \wedge \theta 4 \wedge \theta 5 + (h1\, a5 - g1\, b5 + c1\, d5 + c2\, g5 + c3\, h5 - g2\, c5 - h3\, c5 + g4\, h5$
$- h4\, g5)\ \theta 2 \wedge \theta 3 \wedge \theta 4 + b5\, \theta 2 \wedge \theta 3 \wedge \theta 5 + d5\, \theta 2 \wedge \theta 4 \wedge \theta 5$

We now examine the linear parts of the equations
given by the Jacobi identities:

```
> Eq1:={
  Hook(Hook(Hook(ddtheta1,e1),e3),e5),
  Hook(Hook(Hook(ddtheta1,e1),e4),e5),
  Hook(Hook(Hook(ddtheta1,e2),e3),e5),
  Hook(Hook(Hook(ddtheta1,e2),e4),e5),
  Hook(Hook(Hook(ddtheta1,e3),e4),e5),

  Hook(Hook(Hook(ddtheta2,e1),e3),e5),
  Hook(Hook(Hook(ddtheta2,e1),e4),e5),
  Hook(Hook(Hook(ddtheta2,e2),e3),e5),
  Hook(Hook(Hook(ddtheta2,e2),e4),e5),
  Hook(Hook(Hook(ddtheta2,e3),e4),e5),

  Hook(Hook(Hook(ddtheta3,e1),e3),e5),
  Hook(Hook(Hook(ddtheta3,e1),e4),e5),
  Hook(Hook(Hook(ddtheta3,e2),e3),e5),
  Hook(Hook(Hook(ddtheta3,e2),e4),e5),

  Hook(Hook(Hook(ddtheta4,e1),e3),e5),
  Hook(Hook(Hook(ddtheta4,e1),e4),e5),
  Hook(Hook(Hook(ddtheta4,e2),e3),e5),
  Hook(Hook(Hook(ddtheta4,e2),e4),e5),

  Hook(Hook(Hook(ddtheta5,e1),e3),e5),
  Hook(Hook(Hook(ddtheta5,e1),e4),e5),
  Hook(Hook(Hook(ddtheta5,e2),e3),e5),
  Hook(Hook(Hook(ddtheta5,e2),e4),e5)
  };
```

$Eq1 := \{b3, b4, b5, c3, c4, c5, d3, d4, d5, g3, g4, g5, -h1, -h2, -b1 + c2, b1 - c2, -b2 + c1, b2$ **(1.4)**
$- c1, -d1 + g2, d1 - g2, -d2 + g1, d2 - g1\}$

```
> Eq2:=solve(Eq1,{b3,b4,b5,c1,c2,c3,c4,c5,d3,d4,d5,g1,g2,g3,g4,g5,h1,h2});
```

$Eq2 := \{b3 = 0, b4 = 0, b5 = 0, c1 = b2, c2 = b1, c3 = 0, c4 = 0, c5 = 0, d3 = 0, d4 = 0, d5 = 0,$ **(1.5)**
$g1 = d2, g2 = d1, g3 = 0, g4 = 0, g5 = 0, h1 = 0, h2 = 0\}$

Evaluating the Jacobi identity with this solution
simlifies the quadratic parts. We extract these
equations and solve them.

```
> ddtheta1_s:=Tools:-DGsimplify(eval(ddtheta1,Eq2));
  ddtheta2_s:=Tools:-DGsimplify(eval(ddtheta2,Eq2));
  ddtheta3_s:=Tools:-DGsimplify(eval(ddtheta3,Eq2));
  ddtheta4_s:=Tools:-DGsimplify(eval(ddtheta4,Eq2));
  ddtheta5_s:=Tools:-DGsimplify(eval(ddtheta5,Eq2));
```

$$ddtheta1\_s := -\left(h3\,b1 + h4\,d1\right)\,\theta1 \wedge \theta3 \wedge \theta4 - \left(h3\,b2 + h4\,d2 - h5\right)\,\theta2 \wedge \theta3 \wedge \theta4$$

$$ddtheta2\_s := -\left(h3\,b2 + h4\,d2 - h5\right)\,\theta1 \wedge \theta3 \wedge \theta4 - \left(h3\,b1 + h4\,d1\right)\,\theta2 \wedge \theta3 \wedge \theta4$$

$$ddtheta3\_s := -\left(2\,b1\,a3 + a4\,h3\right)\,\theta1 \wedge \theta2 \wedge \theta3 - \left(2\,d1\,a3 - a3\,h3\right)\,\theta1 \wedge \theta2 \wedge \theta4$$

$$ddtheta4\_s := -\left(2\,b1\,a4 + a4\,h4\right)\,\theta1 \wedge \theta2 \wedge \theta3 + \left(a3\,h4 - 2\,d1\,a4\right)\,\theta1 \wedge \theta2 \wedge \theta4$$

$$ddtheta5\_s := -\left(a4\,h5 + 2\,b1\,a5\right)\,\theta1 \wedge \theta2 \wedge \theta3 + \left(a3\,h5 - 2\,d1\,a5\right)\,\theta1 \wedge \theta2 \wedge \theta4 \quad \textbf{(1.6)}$$

Applying the change of basis below further eliminates without changing the isotropy. We also relable constants for convenience; the combination b2*h3+d2*h4 plays the role of h5 and is therefore zero by the Jacobi identities, and -a3*b2-a4*d2+a5 plays the role of a5, so we relable it as such. Using b2*h3+d2*h4-h5=0 from ddtheta1_s, we find we may take h5=0 in this basis. This change of basis also gives b2=d2=0.

```
> LD_B2:=
  eval(eval(LieAlgebraData([e1,e2,e3+b2*e5,e4+d2*e5,e5],alg_B2),Eq2),
  {-b2*h3-d2*h4+h5=0,-a3*b2-a4*d2+a5=a5});
```

$$LD\_B2 := \left[e1, e2\right] = a3\,e3 + a4\,e4 + a5\,e5, \left[e1, e3\right] = b1\,e1, \left[e1, e4\right] = d1\,e1, \left[e1, e5\right] = -e2, \left[e2, e3\right] = b1\,e2, \left[e2, e4\right] = d1\,e2, \left[e2, e5\right] = -e1, \left[e3, e4\right] = h3\,e3 + h4\,e4, \left[e3, e5\right] = 0, \left[e4, e5\right] = 0 \quad \textbf{(1.7)}$$

We initialize the simplified algebra and examine the Jacobi identities again:

```
> DGsetup(LD_B2,[e],[theta]);
```

Lie algebra: alg_B2 **(1.8)**

```
> ddtheta1:=ExteriorDerivative(ExteriorDerivative(theta1));
  ddtheta2:=ExteriorDerivative(ExteriorDerivative(theta2));
  ddtheta3:=ExteriorDerivative(ExteriorDerivative(theta3));
  ddtheta4:=ExteriorDerivative(ExteriorDerivative(theta4));
  ddtheta5:=ExteriorDerivative(ExteriorDerivative(theta5));
```

$$ddtheta1 := -\left(h3\,b1 + h4\,d1\right)\,\theta1 \wedge \theta3 \wedge \theta4$$

$$ddtheta2 := -\left(h3\,b1 + h4\,d1\right)\,\theta2 \wedge \theta3 \wedge \theta4$$

$$ddtheta3 := -\left(2\,b1\,a3 + a4\,h3\right)\,\theta1 \wedge \theta2 \wedge \theta3 - \left(2\,d1\,a3 - a3\,h3\right)\,\theta1 \wedge \theta2 \wedge \theta4$$

$$ddtheta4 := -\left(2\,b1\,a4 + a4\,h4\right)\,\theta1 \wedge \theta2 \wedge \theta3 + \left(a3\,h4 - 2\,d1\,a4\right)\,\theta1 \wedge \theta2 \wedge \theta4$$

$$ddtheta5 := -2\,b1\,a5\,\theta1 \wedge \theta2 \wedge \theta3 - 2\,d1\,a5\,\theta1 \wedge \theta2 \wedge \theta4 \quad \textbf{(1.9)}$$

For reference, here are the structure equations again:

```
> LieAlgebraData([e1,e2,e3,e4,e5]);
```

$$\left[e1, e2\right] = a3\,e3 + a4\,e4 + a5\,e5, \left[e1, e3\right] = b1\,e1, \left[e1, e4\right] = d1\,e1, \left[e1, e5\right] = -e2, \left[e2, e3\right] = b1\,e2, \left[e2, e4\right] = d1\,e2, \left[e2, e5\right] = -e1, \left[e3, e4\right] = h3\,e3 + h4\,e4, \left[e3, e5\right] = 0, \left[e4, e5\right] = 0 \quad \textbf{(1.10)}$$

We consider two cases by constructing the largest subalgebra, i, of the center, n, of the isotropy that is also an ideal in the algebra.

Any vector in n has the form of y below and any vector in the span of e1 and e2 has the form of x below:

```
> y:=evalDG(mu*e3+lambda*e4+nu*e5);
  x:=evalDG(alpha*e1+beta*e2);
```

$$y := \mu\, e3 + \lambda\, e4 + \nu\, e5$$

$$x := \alpha\, e1 + \beta\, e2 \tag{1.11}$$

The Lie bracket is:

```
> LieBracket(x,y);
```

$$(\alpha\,\mu\, b1 + \alpha\,\lambda\, d1 - \beta\,\nu)\, e1 - (-\beta\,\mu\, b1 - \beta\,\lambda\, d1 + \alpha\,\nu)\, e2 \tag{1.12}$$

If y is in i, then [x,y] is in n, so
$\alpha\mu b1 + \alpha\lambda d1 - \beta\nu = 0$
$\beta\mu b1 + \beta\lambda d1 + \alpha\nu = 0$
for all α and β.

This requires $\nu = 0$ and so i takes the form i=$\{\mu e3+\lambda e4 \mid \mu b1=-\lambda d1\}$.
Note that i is two-dimensional if and only if b1=d1=0. We case split on this.

Section 6.1: i is Two-Dimensional

In this case, b1=d1=0. We initialize this algebra.

```
> LD_B61:=eval(
  LieAlgebraData(
  [e1,e2,e3,e4,e5],
  alg_B61),
  {b1=0,d1=0});
```

$$LD\_B61 := [e1, e2] = a3\, e3 + a4\, e4 + a5\, e5, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -e2, \tag{1.1.1}$$
$$[e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e1, [e3, e4] = h3\, e3 + h4\, e4, [e3, e5] = 0,$$
$$[e4, e5] = 0$$

```
> DGsetup(LD_B61,['x'],[o]);
```

Lie algebra: alg_B61 **(1.1.2)**

We examine the Jacobi identities in this algebra.

```
> ExteriorDerivative(ExteriorDerivative(o1));
  ExteriorDerivative(ExteriorDerivative(o2));
  ExteriorDerivative(ExteriorDerivative(o3));
  ExteriorDerivative(ExteriorDerivative(o4));
  ExteriorDerivative(ExteriorDerivative(o5));
```

$$0\; o1 \wedge o2 \wedge o3$$
$$0\; o1 \wedge o2 \wedge o3$$
$$-a4\, h3\, o1 \wedge o2 \wedge o3 + a3\, h3\, o1 \wedge o2 \wedge o4$$
$$-a4\, h4\, o1 \wedge o2 \wedge o3 + a3\, h4\, o1 \wedge o2 \wedge o4$$
$$0\; o1 \wedge o2 \wedge o3 \tag{1.1.3}$$

We see that a4*h3=a3*h3=a4*h4=a4*h3=0, so either a3=a4=0 or h3=h4=0.

To distinguish these, we consider the center. The center is trivial if and only if one or both of h3 and h4 is nonzero, in which case a3=a4=0. Otherwise, h3=h4=0 and the center contains e3 and e4.

6.1.1: The Center is Trivial

In this case, a3=a4=0. The subalgebra spanned by e3 and e4 decomposes and is nonabelian. Thus there is a basis in which h3=1 and h4=0. We initialize the algebra in this basis:

```
> LD_B611:=eval(
  LieAlgebraData(
  [x1,x2,x3,x4,x5],
  alg_B611),
  {a3=0,a4=0,h3=1,h4=0});
```

$$LD\_B611 := [e1, e2] = a5\, e5, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -e2, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e1, [e3, e4] = e3, [e3, e5] = 0, [e4, e5] = 0 \quad (1.1.1.1)$$

```
> DGsetup(LD_B611,['y'],[p]);
```

Lie algebra: alg_B611 (1.1.1.2)

Consider the derived algebra:

```
> Series(alg_B611,"Derived")[2];
```

$$[a5\, y5, -y2, -y1, y3] \quad (1.1.1.3)$$

The dimension of the derived algebra is determined by whether or not a5=0. If a5=0, the drived algebra is three-dimensional, and it is four-dimensional otherwise.

If a5 is positive, the following scales a5 to 1.

```
> LieAlgebraData([
  y1/sqrt(a5),
  y2/sqrt(a5),
  y3,
  y4,
  y5
  ]);
```

$$[e1, e2] = e5, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -e2, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e1, [e3, e4] = e3, [e3, e5] = 0, [e4, e5] = 0 \quad (1.1.1.4)$$

If a5 is negative, the following scales a5 to 1.

```
> LieAlgebraData([
  y2/sqrt(-a5),
  y1/sqrt(-a5),
  y3,
  y4,
  y5
  ]);
```

$$[e1, e2] = e5, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -e2, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e1, [e3, e4] = e3, [e3, e5] = 0, [e4, e5] = 0 \quad (1.1.1.5)$$

We may thus take a5 to be zero or one.

If a5 = 0, the change of basis below gives s_3,1+s_2,1 with a=-1 and isotropy e3

```
> eval(LieAlgebraData([-y1+y2,y1+y2,y5,y3,y4]),a5=0);
```

$$[e1, e2] = 0, [e1, e3] = e1, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = -e2, [e2, e4] = 0, [e2, e5] = 0, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = e4 \quad (1.1.1.6)$$

If a5 = 1, the change of basis below gives sl(2,F)+s_2,1 with isotropy e2

```
> eval(LieAlgebraData([
  y1-y2,
  2*y5,
  -y1-y2,
  y3,
  y4
  ]),a5=1);
```

$[e1, e2] = 2\ e1, [e1, e3] = -e2, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = 2\ e3, [e2, e4] = 0, [e2, e5] = 0, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = e4$ **(1.1.1.7)**

6.1.2: The Center is Two-Dimensional

If the center is two-dimensional, then h3=h4=0. We initialize this algebra:

```
> LD_B612:=eval(
  LieAlgebraData([x1,x2,x3,x4,x5],alg_B612),
  {h3=0,h4=0});
```

$LD\_B612 := [e1, e2] = a3\ e3 + a4\ e4 + a5\ e5, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -e2, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e1, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0$ **(1.1.2.1)**

```
> DGsetup(LD_B612,['y'],[p]);
```

Lie algebra: alg_B612 **(1.1.2.2)**

Now consider the derived series.

```
> Series(alg_B612,"Derived");
```

$[[y1, y2, y3, y4, y5], [a3\ y3 + a4\ y4 + a5\ y5, -y2, -y1], [-a5\ y1, -a5\ y2, -a3\ y3 - a4\ y4 - a5\ y5]]$ **(1.1.2.3)**

If a5 is zero, the last term above is spanned by a3*y3+a4*y4
and the derived series necessarily terminates in the zero algebra.
Thus a5 determines whether or not the algebra is solvable.

If a5 is positive, the following scales a5 to 1.

```
> LieAlgebraData([
  y1/sqrt(a5),
  y2/sqrt(a5),
  y3,
  y4,
  y5
  ]);
```

$[e1, e2] = \frac{a3}{a5}\ e3 + \frac{a4}{a5}\ e4 + e5, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -e2, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e1, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0$ **(1.1.2.4)**

If a5 is negative, the following scales a5 to 1.

```
> LieAlgebraData([
  y2/sqrt(-a5),
  y1/sqrt(-a5),
  y3,
  y4,
  y5
  ]);
```

$[e1, e2] = \frac{a3}{a5}\ e3 + \frac{a4}{a5}\ e4 + e5, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -e2, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e1, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0$ **(1.1.2.5)**

We may thus take a5 in {0,1}.

Recall the derived series:

```
> Series(alg_B612,"Derived");
```

$[[y1, y2, y3, y4, y5], [a3\ y3 + a4\ y4 + a5\ y5, -y2, -y1], [-a5\ y1, -a5\ y2, -a3\ y3 - a4\ y4 - a5\ y5]]$ **(1.1.2.6)**

In the case that a5 = 0, the second derived algebra is trivial if and only if a3=a4=0. As an alternate invariant, a3=a4=0 if and only if the isotropy is in the derived algebra.

If a3 or a4 is nonzero, we may take a3=1 and a4=0 via one of the following changes of basis (whichever is nondegenerate):

```
> LieAlgebraData([y1,y2,a3*y3+a4*y4,y3,y5]);
  LieAlgebraData([y1,y2,a3*y3+a4*y4,y4,y5]);
```

$[e1, e2] = e3 + a5\, e5, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -e2, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e1, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0$

$[e1, e2] = e3 + a5\, e5, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -e2, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e1, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0$ **(1.1.2.7)**

We thus have four possibilities:

a5=0, a3=0:
The change of basis below gives
s_3,1+2n_1,1 with a=-1 and isotropy e3

```
> eval(LieAlgebraData([
  -y1+y2,
  -y1-y2,
  y5,
  y3,
  y4
  ]),{a5=0,a3=0,a4=0});
```

$[e1, e2] = 0, [e1, e3] = e1, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = -e2, [e2, e4] = 0, [e2, e5] = 0, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0$ **(1.1.2.8)**

a5=0, a3=1:
The change of basis below gives
s_4,6+n_1,1 with isotropy e4

```
> eval(LieAlgebraData([
  -y3,
  y1-y2,
  1/2*y1+1/2*y2,
  -y5,
  y4
  ]),{a5=0,a3=1,a4=0});
```

$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = -e1, [e2, e4] = -e2, [e2, e5] = 0, [e3, e4] = e3, [e3, e5] = 0, [e4, e5] = 0$ **(1.1.2.9)**

a5=1, a3=0:
The change of basis below gives
sl(2,F)+2n_1,1 with isotropy e2

```
> eval(LieAlgebraData([
  y1-y2,
  2*y5,
  -y1-y2,
  y3,
  y4
  ]),{a5=1,a3=0,a4=0});
```

$[e1, e2] = 2\, e1, [e1, e3] = -e2, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = 2\, e3, [e2, e4] = 0, [e2, e5] = 0, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0$ **(1.1.2.10)**

a5=1, a3=1:
The change of basis below gives
sl(2,F)+2n_1,1 with isotropy e2-2e4

```
> eval(LieAlgebraData([
  y1-y2,
```

```
2*y3+2*y5,
-y1-y2,
y3,
y4
]),{a5=1,a3=1,a4=0});
```

$[e1, e2] = 2\ e1, [e1, e3] = -e2, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = 2\ e3, [e2, e4] = 0, [e2, e5] = 0, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0$ **(1.1.2.11)**

Section 6.2: The Dimension of i is Less Than Two

In this case, at least either or both of b1 and d1 is nonzero.

Recall the Jacobi identities:

```
> ddtheta1;
  ddtheta2;
  ddtheta3;
  ddtheta4;
  ddtheta5;
```

$$-(b1\ h3 + d1\ h4)\ \theta1 \wedge \theta3 \wedge \theta4$$
$$-(b1\ h3 + d1\ h4)\ \theta2 \wedge \theta3 \wedge \theta4$$
$$-(2\ a3\ b1 + a4\ h3)\ \theta1 \wedge \theta2 \wedge \theta3 - (2\ a3\ d1 - a3\ h3)\ \theta1 \wedge \theta2 \wedge \theta4$$
$$-(2\ a4\ b1 + a4\ h4)\ \theta1 \wedge \theta2 \wedge \theta3 + (a3\ h4 - 2\ a4\ d1)\ \theta1 \wedge \theta2 \wedge \theta4$$
$$-2\ b1\ a5\ \theta1 \wedge \theta2 \wedge \theta3 - 2\ d1\ a5\ \theta1 \wedge \theta2 \wedge \theta4 \qquad \textbf{(1.2.1)}$$

If b1 is nonzero, we apply the following change of basis and relabel constants, using b1*h3+d1*h4=0 from ddtheta1=0.

```
> LD_B62:=eval(LieAlgebraData([
  e1,e2,1/b1*e3,-d1/b1*e3+e4,e5
  ],alg_B62),
  {b1*h3+d1*h4=0,a3*b1+a4*d1=a3,h4/b1=h4});
```

$LD\_B62 := [e1, e2] = a3\ e3 + a4\ e4 + a5\ e5, [e1, e3] = e1, [e1, e4] = 0, [e1, e5] = -e2, [e2, e3] = e2, [e2, e4] = 0, [e2, e5] = -e1, [e3, e4] = h4\ e4, [e3, e5] = 0, [e4, e5] = 0$ **(1.2.2)**

If d1 is nonzero, we apply the following change of basis and relabel constants, using b1*h3+d1*h4=0 from ddtheta1=0.

```
> LD_B62:=eval(LieAlgebraData([
  e1,e2,1/d1*e4,-b1/d1*e4+e3,e5
  ],alg_B62),
  {b1*h3+d1*h4=0,a3=a4,a3*b1+a4*d1=a3,h3/d1=-h4});
```

$LD\_B62 := [e1, e2] = a3\ e3 + a4\ e4 + a5\ e5, [e1, e3] = e1, [e1, e4] = 0, [e1, e5] = -e2, [e2, e3] = e2, [e2, e4] = 0, [e2, e5] = -e1, [e3, e4] = h4\ e4, [e3, e5] = 0, [e4, e5] = 0$ **(1.2.3)**

In either case, we can take b1=1, d1=0, and h3=0. We now initialize the algebra and impose the Jacobi identities.

```
> DGsetup(LD_B62,['x'],[o]);
```

Lie algebra: alg_B62 **(1.2.4)**

```
> ExteriorDerivative(ExteriorDerivative(o1));
  ExteriorDerivative(ExteriorDerivative(o2));
  ExteriorDerivative(ExteriorDerivative(o3));
  ExteriorDerivative(ExteriorDerivative(o4));
```

```
ExteriorDerivative(ExteriorDerivative(o5));
```

$$0\ o1 \wedge o2 \wedge o3$$

$$0\ o1 \wedge o2 \wedge o3$$

$$-2\ a3\ o1 \wedge o2 \wedge o3$$

$$-(a4\ h4 + 2\ a4)\ o1 \wedge o2 \wedge o3 + a3\ h4\ o1 \wedge o2 \wedge o4$$

$$-2\ a5\ o1 \wedge o2 \wedge o3 \quad \textbf{(1.2.5)}$$

We thus find a3=a5=0, so we initialize this algebra. Note that we also will require either a4 = 0 or h4 = -2 to completely satisfy the Jacobi identities.

```
> DGsetup(eval(LD_B62,{a3=0,a5=0}),['y'],[p]);
```

Lie algebra: alg_B62 **(1.2.6)**

The structure equations are as follows:

```
> LieAlgebraData([y1,y2,y3,y4,y5]);
```

$$[e1, e2] = a4\ e4,\ [e1, e3] = e1,\ [e1, e4] = 0,\ [e1, e5] = -e2,\ [e2, e3] = e2,\ [e2, e4] = 0,\ [e2, e5] = -e1,\ [e3, e4] = h4\ e4,\ [e3, e5] = 0,\ [e4, e5] = 0 \quad \textbf{(1.2.7)}$$

Consider the derived series. It is spanned by {a4*y4, y1, y2, h4*y4}.

The structure Lie brackets given by these vectors are:

```
> LieDerivative(
  [a4*y4, y1, y2, h4*y4],
  [a4*y4, y1, y2, h4*y4]);
```

$$[[0\ y1, 0\ y1, 0\ y1, 0\ y1], [0\ y1, 0\ y1, a4\ y4, 0\ y1], [0\ y1, -a4\ y4, 0\ y1, 0\ y1], [0\ y1, 0\ y1, 0\ y1, 0\ y1]] \quad \textbf{(1.2.8)}$$

Thus, a4 is nonzero if and only if the second derived algebra is one-dimensional, in which case h4 = -2 to satisfy the Jacobi identities and the following change of basis gives s_5,44 with isotropy spanned by e5.

```
> eval(
  LieAlgebraData([-2*a4*y4,y1+y2,y1-y2,-1/2*y3-1/2*y5,-y5]),
  {h4=-2});
```

$$[e1, e2] = 0,\ [e1, e3] = 0,\ [e1, e4] = -e1,\ [e1, e5] = 0,\ [e2, e3] = e1,\ [e2, e4] = 0,\ [e2, e5] = e2,\ [e3, e4] = -e3,\ [e3, e5] = -e3,\ [e4, e5] = 0 \quad \textbf{(1.2.9)}$$

Given a4 = 0, the derived algebra is two-dimensional for h4 = 0, in which case the following change of basis gives 2s_2,1+n_1,1 with isotropy spanned by e2-e4.

```
> eval(
  LieAlgebraData([2*y1-2*y2,1/2*y3+1/2*y5,2*y1+2*y2,1/2*y3-1/2*y5,y4]),
  {a4=0,h4=0});
```

$$[e1, e2] = e1,\ [e1, e3] = 0,\ [e1, e4] = 0,\ [e1, e5] = 0,\ [e2, e3] = 0,\ [e2, e4] = 0,\ [e2, e5] = 0,\ [e3, e4] = e3,\ [e3, e5] = 0,\ [e4, e5] = 0 \quad \textbf{(1.2.10)}$$

Given a4 = 0, the derived algebra is three-dimensional for h4 nonzero,in which case the following change of basis gives s_5,41 with α=β isotropy spanned by e5. We identify our parameter h4 with the parameter α in the algebra classification tables via h4=-2α.

```
> eval(
  LieAlgebraData([y1+y2,y1-y2,-2*y4,-1/2*y3-1/2*y5,-1/2*y3+1/2*y5]),
  {a4=0,h4=-2*alpha});
```

$$[e1, e2] = 0,\ [e1, e3] = 0,\ [e1, e4] = 0,\ [e1, e5] = -e1,\ [e2, e3] = 0,\ [e2, e4] = -e2,\ [e2, e5] = 0,\ [e3, e4] = -\alpha\ e3,\ [e3, e5] = -\alpha\ e3,\ [e4, e5] = 0 \quad \textbf{(1.2.11)}$$

B.5. F14 Maple Worksheet

Maple Worksheet
Isotropy Type F14: Null Rotations

This worksheet steps through Chapter 7, serving to help validate the results. This chapters cover the isotropies of type F14, the null rotations.

We begin by loading the packages needed.

```
> with(DifferentialGeometry):
  with(LieAlgebras):
```

We now initialize the most generic five-dimensional algebra with the subalgebra spanned by e5 of type F14.

```
> LD_Null:=LieAlgebraData([
  '[e1,e2]=a1*e1+a2*e2+a3*e3+a4*e4+a5*e5',
  '[e1,e3]=b1*e1+b2*e2+b3*e3+b4*e4+b5*e5',
  '[e2,e3]=c1*e1+c2*e2+c3*e3+c4*e4+c5*e5',
  '[e1,e4]=d1*e1+d2*e2+d3*e3+d4*e4+d5*e5',
  '[e2,e4]=g1*e1+g2*e2+g3*e3+g4*e4+g5*e5',
  '[e3,e4]=h1*e1+h2*e2+h3*e3+h4*e4+h5*e5',
  '[e5,e1]=-e2',
  '[e5,e2]=e3'
  ],[e1,e2,e3,e4,e5],alg_nul);
```

$$LD\_Null := [e1, e2] = a1\, e1 + a2\, e2 + a3\, e3 + a4\, e4 + a5\, e5, [e1, e3] = b1\, e1 + b2\, e2 + b3\, e3 + b4\, e4 + b5\, e5, [e1, e4] = d1\, e1 + d2\, e2 + d3\, e3 + d4\, e4 + d5\, e5, [e1, e5] = e2, [e2, e3] = c1\, e1 + c2\, e2 + c3\, e3 + c4\, e4 + c5\, e5, [e2, e4] = g1\, e1 + g2\, e2 + g3\, e3 + g4\, e4 + g5\, e5, [e2, e5] = -e3, [e3, e4] = h1\, e1 + h2\, e2 + h3\, e3 + h4\, e4 + h5\, e5, [e3, e5] = 0, [e4, e5] = 0 \quad (1.1)$$

```
> DGsetup(LD_Null,[e],[theta]);
```

Lie algebra: alg_nul **(1.2)**

Note that the isotropy, spanned by e5, centralizes a three-dimensional space spanned by {e3,e4,e5}.:

```
> LieDerivative([e5],[e1,e2,e3,e4,e5]);
```

$$[[-e2, e3, 0\, e1, 0\, e1, 0\, e1]] \quad (1.3)$$

Consider span{e3,e4,e5} and the following Jacobi identities:

```
> 0=evalDG(
  LieBracket(e3,LieBracket(e4,e5))+
  LieBracket(e5,LieBracket(e3,e4))+
  LieBracket(e4,LieBracket(e5,e3))
  ) ;

  0=evalDG(
  LieBracket(e2,LieBracket(e4,e5))+
  LieBracket(e5,LieBracket(e2,e4))+
  LieBracket(e4,LieBracket(e5,e2))
  ) ;
```

$$0 = -h1\, e2 + h2\, e3$$

$$0 = -h1\, e1 - (g1 + h2)\, e2 + (g2 - h3)\, e3 - h4\, e4 - h5\, e5 \quad (1.4)$$

Thus h1=h2=h4=h5=0 (g1=0 also, but this is unimportant for now). We see that [e3,e4] = h3*e3 and that span{e3,e4,e5} forms a Lie algebra that is either abelian if h3=0, or nonabelian if h3 is nonzero.

Section 7.1: The Centralizer of the Isotropy is Non-Abelian

In this case, h3 is nonzero. There is a basis in which [e3,e4]=e3 and the adjoint of e5 is unchanged, so we take h3=1 and examine the Jacobi identities.

First recall that h1=h2=h4=h5=g1=0 from above.

```
> Eq1:={h1=0,h2=0,h3=1,h4=0,h5=0,g1=0};
```

$$Eq1 := \{g1 = 0, h1 = 0, h2 = 0, h3 = 1, h4 = 0, h5 = 0\} \quad \textbf{(1.1.1)}$$

```
> ddtheta1:=ExteriorDerivative(ExteriorDerivative(theta1)):
  ddtheta2:=ExteriorDerivative(ExteriorDerivative(theta2)):
  ddtheta3:=ExteriorDerivative(ExteriorDerivative(theta3)):
  ddtheta4:=ExteriorDerivative(ExteriorDerivative(theta4)):
  ddtheta5:=ExteriorDerivative(ExteriorDerivative(theta5)):
```

```
> ddtheta1:=Tools:-DGsimplify(eval(ddtheta1,Eq1));
  ddtheta2:=Tools:-DGsimplify(eval(ddtheta2,Eq1));
  ddtheta3:=Tools:-DGsimplify(eval(ddtheta3,Eq1));
  ddtheta4:=Tools:-DGsimplify(eval(ddtheta4,Eq1));
  ddtheta5:=Tools:-DGsimplify(eval(ddtheta5,Eq1));
```

$$ddtheta1 := -(c2\,a1 - a2\,c1 + c3\,b1 - b3\,c1 + c4\,d1)\,\theta1\wedge\theta2\wedge\theta3 - (g2\,a1 + g3\,b1 - d3\,c1 + g4\,d1)\,\theta1\wedge\theta2\wedge\theta4 + b1\,\theta1\wedge\theta2\wedge\theta5 - (d2\,c1 + b1)\,\theta1\wedge\theta3\wedge\theta4 - c1\,\theta1\wedge\theta3\wedge\theta5 + (c1\,d1 - g2\,c1 - c1)\,\theta2\wedge\theta3\wedge\theta4$$

$$ddtheta2 := (a1\,b2 - b1\,a2 - c3\,b2 + b3\,c2 + b4\,g2 - c4\,d2 - c5)\,\theta1\wedge\theta2\wedge\theta3 + (a1\,d2 - d1\,a2 - g3\,b2 + d3\,c2 - g4\,d2 + d4\,g2 - g5)\,\theta1\wedge\theta2\wedge\theta4 + (a1 + b2)\,\theta1\wedge\theta2\wedge\theta5 + (b1\,d2 - d1\,b2 + b2\,g2 - d2\,c2 - b2)\,\theta1\wedge\theta3\wedge\theta4 + (b1 - c2)\,\theta1\wedge\theta3\wedge\theta5 + (d1 - g2)\,\theta1\wedge\theta4\wedge\theta5 + (d2\,c1 - c2)\,\theta2\wedge\theta3\wedge\theta4 + c1\,\theta2\wedge\theta3\wedge\theta5$$

$$ddtheta3 := (a1\,b3 + a2\,c3 - b1\,a3 - c2\,a3 + b4\,g3 - c4\,d3 - a4 - b5)\,\theta1\wedge\theta2\wedge\theta3 + (a1\,d3 + a2\,g3 - d1\,a3 - g2\,a3 - g3\,b3 + d3\,c3 - g4\,d3 + d4\,g3 + a3 - d5)\,\theta1\wedge\theta2\wedge\theta4 - (a2 - b3)\,\theta1\wedge\theta2\wedge\theta5 + (b1\,d3 + g3\,b2 - d1\,b3 - d2\,c3 + d4)\,\theta1\wedge\theta3\wedge\theta4 - (b2 + c3)\,\theta1\wedge\theta3\wedge\theta5 - (d2 + g3)\,\theta1\wedge\theta4\wedge\theta5 + (d3\,c1 + c2\,g3 - g2\,c3 + g4)\,\theta2\wedge\theta3\wedge\theta4 - c2\,\theta2\wedge\theta3\wedge\theta5 - (g2 - 1)\,\theta2\wedge\theta4\wedge\theta5$$

$$ddtheta4 := (a1\,b4 + a2\,c4 - b1\,a4 - c2\,a4 + b3\,c4 - c3\,b4 + b4\,g4 - c4\,d4)\,\theta1\wedge\theta2\wedge\theta3 + (a1\,d4 + a2\,g4 - d1\,a4 - g2\,a4 - b4\,g3 + c4\,d3)\,\theta1\wedge\theta2\wedge\theta4 + b4\,\theta1\wedge\theta2\wedge\theta5 + (b1\,d4 + b2\,g4 - d1\,b4 - c4\,d2 - b4)\,\theta1\wedge\theta3\wedge\theta4 - c4\,\theta1\wedge\theta3\wedge\theta5 - g4\,\theta1\wedge\theta4\wedge\theta5 + (c1\,d4 + c2\,g4 - g2\,c4 - c4)\,\theta2\wedge\theta3\wedge\theta4$$

$$ddtheta5 := (a1\,b5 + a2\,c5 - b1\,a5 - c2\,a5 + b3\,c5 + b4\,g5 - c3\,b5 - c4\,d5)\,\theta1\wedge\theta2\wedge\theta3 + (a1\,d5 + a2\,g5 - d1\,a5 - g2\,a5 - g3\,b5 + d3\,c5 + d4\,g5 - g4\,d5)\,\theta1\wedge\theta2\wedge\theta4 + b5\,\theta1\wedge\theta2\wedge\theta5 + (b1\,d5 + b2\,g5 - d1\,b5 - d2\,c5 - b5)\,\theta1\wedge\theta3\wedge\theta4 - c5\,\theta1\wedge\theta3\wedge\theta5 - g5\,\theta1\wedge\theta4\wedge\theta5 + (c1\,d5 + c2\,g5 - g2\,c5 - c5)\,\theta2\wedge\theta3\wedge\theta4 \quad \textbf{(1.1.2)}$$

We consider the linear terms:

```
> Eq2:={
  Hook(Hook(Hook(ddtheta1,e1),e2),e5),
  Hook(Hook(Hook(ddtheta1,e1),e3),e5),

  Hook(Hook(Hook(ddtheta2,e1),e2),e5),
  Hook(Hook(Hook(ddtheta2,e1),e3),e5),
  Hook(Hook(Hook(ddtheta2,e1),e4),e5),
  Hook(Hook(Hook(ddtheta2,e2),e3),e5),
```

```
Hook(Hook(Hook(ddtheta3,e1),e2),e5),
Hook(Hook(Hook(ddtheta3,e1),e3),e5),
Hook(Hook(Hook(ddtheta3,e1),e4),e5),
Hook(Hook(Hook(ddtheta3,e2),e3),e5),
Hook(Hook(Hook(ddtheta3,e2),e4),e5),

Hook(Hook(Hook(ddtheta4,e1),e2),e5),
Hook(Hook(Hook(ddtheta4,e1),e3),e5),
Hook(Hook(Hook(ddtheta4,e1),e4),e5),

Hook(Hook(Hook(ddtheta5,e1),e2),e5),
Hook(Hook(Hook(ddtheta5,e1),e4),e5)
} ;
```

$$Eq2 := \{b1, b4, b5, c1, -c1, -c2, -c4, -g4, -g5, a1 + b2, -a2 + b3, b1 - c2, -b2 - c3, d1 - g2, -d2 - g3, -g2 + 1\} \quad \textbf{(1.1.3)}$$

```
> Eq3:=solve(Eq2,{b1,b2,b3,b4,b5,c1,c2,c3,c4,d1,g2,g3,g4,g5});
```

$$Eq3 := \{b1 = 0, b2 = -a1, b3 = a2, b4 = 0, b5 = 0, c1 = 0, c2 = 0, c3 = a1, c4 = 0, d1 = 1, g2 = 1, g3 = -d2, g4 = 0, g5 = 0\} \quad \textbf{(1.1.4)}$$

Now we reexamine the Jacobi identities using this partial solution:

```
> ddtheta1_s:=Tools:-DGsimplify(eval(ddtheta1,Eq3));
ddtheta2_s:=Tools:-DGsimplify(eval(ddtheta2,Eq3));
ddtheta3_s:=Tools:-DGsimplify(eval(ddtheta3,Eq3));
ddtheta4_s:=Tools:-DGsimplify(eval(ddtheta4,Eq3));
ddtheta5_s:=Tools:-DGsimplify(eval(ddtheta5,Eq3));
```

$$ddtheta1\_s := -a1\ \theta1 \wedge \theta2 \wedge \theta4$$

$$ddtheta2\_s := -c5\,\theta1 \wedge \theta2 \wedge \theta3 - (a2 - d4)\ \theta1 \wedge \theta2 \wedge \theta4 + a1\ \theta1 \wedge \theta3 \wedge \theta4$$

$$ddtheta3\_s := (2\ a1\ a2 - a4)\ \theta1 \wedge \theta2 \wedge \theta3 + (2\ a1\ d3 - d4\ d2 - a3 - d5)\ \theta1 \wedge \theta2 \wedge \theta4 - (a2 - d4)\ \theta1 \wedge \theta3 \wedge \theta4 - a1\ \theta2 \wedge \theta3 \wedge \theta4$$

$$ddtheta4\_s := (a1\ d4 - 2\ a4)\ \theta1 \wedge \theta2 \wedge \theta4$$

$$ddtheta5\_s := 2\ a2\ c5\,\theta1 \wedge \theta2 \wedge \theta3 + (a1\ d5 + c5\,d3 - 2\ a5)\ \theta1 \wedge \theta2 \wedge \theta4 - c5\,d2\,\theta1 \wedge \theta3 \wedge \theta4 - c5\,\theta1 \wedge \theta3 \wedge \theta5 - 2\ c5\,\theta2 \wedge \theta3 \wedge \theta4 \quad \textbf{(1.1.5)}$$

We again examine the linear parts:

```
> Eq4:={
Hook(Hook(Hook(ddtheta1_s,e1),e2),e4),

Hook(Hook(Hook(ddtheta2_s,e1),e2),e3),
Hook(Hook(Hook(ddtheta2_s,e1),e2),e4),
Hook(Hook(Hook(ddtheta2_s,e1),e3),e4),

Hook(Hook(Hook(ddtheta3_s,e1),e3),e4),
Hook(Hook(Hook(ddtheta3_s,e2),e3),e4),

Hook(Hook(Hook(ddtheta5_s,e1),e3),e5),
Hook(Hook(Hook(ddtheta5_s,e2),e3),e4)
} ;
```

$$Eq4 := \{a1, -a1, -2\ c5, -c5, -a2 + d4\} \quad \textbf{(1.1.6)}$$

```
> Eq5:=solve(Eq4,{a1,d4,c5});
```

$$Eq5 := \{a1 = 0, c5 = 0, d4 = a2\} \quad \textbf{(1.1.7)}$$

We examine the Jacobi identites one more time:

```
> ddtheta1_s2:=Tools:-DGsimplify(eval(ddtheta1_s,Eq5));
ddtheta2_s2:=Tools:-DGsimplify(eval(ddtheta2_s,Eq5));
ddtheta3_s2:=Tools:-DGsimplify(eval(ddtheta3_s,Eq5));
ddtheta4_s2:=Tools:-DGsimplify(eval(ddtheta4_s,Eq5));
```

```
ddtheta5_s2:=Tools:-DGsimplify(eval(ddtheta5_s,Eq5));
```

$$ddtheta1\_s2 := 0\ \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta2\_s2 := 0\ \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta3\_s2 := -a4\ \theta1 \wedge \theta2 \wedge \theta3 - (a2\ d2 + a3 + d5)\ \theta1 \wedge \theta2 \wedge \theta4$$

$$ddtheta4\_s2 := -2\ a4\ \theta1 \wedge \theta2 \wedge \theta4$$

$$ddtheta5\_s2 := -2\ a5\ \theta1 \wedge \theta2 \wedge \theta4 \quad \textbf{(1.1.8)}$$

We now have the following restrictions on the structure constants:

```
> Eq6:={
  op(Eq1),
  eval(op(Eq3),Eq5),
  op(Eq5),

  op(solve({
  Hook(Hook(Hook(ddtheta3_s2,e1),e2),e4),
  Hook(Hook(Hook(ddtheta4_s2,e1),e2),e4),
  Hook(Hook(Hook(ddtheta5_s2,e1),e2),e4)
  },{d5,a4,a5}))

  };
```

$$Eq6 := \{a1 = 0, a4 = 0, a5 = 0, b1 = 0, b2 = 0, b3 = a2, b4 = 0, b5 = 0, c1 = 0, c2 = 0, c3 = 0, c4 = 0, c5 = 0, d1 = 1, d4 = a2, d5 = -a2\ d2 - a3, g1 = 0, g2 = 1, g3 = -d2, g4 = 0, g5 = 0, h1 = 0, h2 = 0, h3 = 1, h4 = 0, h5 = 0\} \quad \textbf{(1.1.9)}$$

The structure equations thus take the following form:

```
> eval(LieAlgebraData([e1,e2,e3,e4,e5],alg_N1),Eq6);
```

$$[e1, e2] = a2\ e2 + a3\ e3, [e1, e3] = a2\ e3, [e1, e4] = e1 + d2\ e2 + d3\ e3 + a2\ e4 - (a2\ d2 + a3)\ e5, [e1, e5] = e2, [e2, e3] = 0, [e2, e4] = e2 - d2\ e3, [e2, e5] = -e3, [e3, e4] = e3, [e3, e5] = 0, [e4, e5] = 0 \quad \textbf{(1.1.10)}$$

The following change of basis simplifies the algebra:

```
> eval(LieAlgebraData(
  [e1-a3*e5,e2,e3,e4-d2*e5,e5],
  alg_N1),Eq6);
```

$$[e1, e2] = a2\ e2, [e1, e3] = a2\ e3, [e1, e4] = e1 + d3\ e3 + a2\ e4, [e1, e5] = e2, [e2, e3] = 0, [e2, e4] = e2, [e2, e5] = -e3, [e3, e4] = e3, [e3, e5] = 0, [e4, e5] = 0 \quad \textbf{(1.1.11)}$$

Suppose a2 is nonzero. Then the following change of basis and relabeling of constants gives:

```
> eval(eval(LieAlgebraData([
  (1/a2)*(e1-a3*e5)+e4-d2*e5,
  e2,
  a2*e3,
  e4-d2*e5,
  a2*e5
  ],alg_N1),Eq6),d3/a2^2=d3);
```

$$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = e1 + d3\ e3, [e1, e5] = e2, [e2, e3] = 0, [e2, e4] = e2, [e2, e5] = -e3, [e3, e4] = e3, [e3, e5] = 0, [e4, e5] = 0 \quad \textbf{(1.1.12)}$$

Thus we may take a2=0 without loss of generality:

```
> Eq7:={op(eval(Eq6,{a2=0})),a2=0};
```

$$Eq7 := \{a1 = 0, a2 = 0, a4 = 0, a5 = 0, b1 = 0, b2 = 0, b3 = 0, b4 = 0, b5 = 0, c1 = 0, c2 = 0, c3 = 0, c4 = 0, c5 = 0, d1 = 1, d4 = 0, d5 = -a3, g1 = 0, g2 = 1, g3 = -d2, g4 = 0, g5 = 0, h1 = 0, h2 = 0, h3 = 1, h4 = 0, h5 = 0\} \quad \textbf{(1.1.13)}$$

We initialize this algebra now:

```
> LD_N1:=eval(LieAlgebraData(
  [e1-a3*e5,e2,e3,e4-d2*e5,e5],
  alg_N1),Eq7);
```

$$LD\_N1 := [e1, e2] = 0, [e1, e3] = 0, [e1, e4] = e1 + d3\, e3, [e1, e5] = e2, [e2, e3] = 0, [e2, e4] = e2, [e2, e5] = -e3, [e3, e4] = e3, [e3, e5] = 0, [e4, e5] = 0 \quad (1.1.14)$$

```
> DGsetup(LD_N1,[x],[o]);
```

Lie algebra: alg_N1 **(1.1.15)**

We find the nilradical:

```
> Nilradical();
```

$$[x1, x2, x3, x5] \quad (1.1.16)$$

Consider an arbitrary vector not in the nilradical (i.e., any vector with an x4 component). By requiring that it also be in the centralizer of the isotropy, we may take the x1 and x2 components to be zero.

```
> X:=evalDG(alpha*x4+beta*x3+gamma*x5);
```

$$X := \beta\, x3 + \alpha\, x4 + \gamma\, x5 \quad (1.1.17)$$

We find its adjoint restricted to the nilradical:

```
> AX:=Adjoint(X,[x1,x2,x3,x5]);
```

$$AX := \begin{bmatrix} -\alpha & 0 & 0 & 0 \\ -\gamma & -\alpha & 0 & 0 \\ -\alpha\, d3 & \gamma & -\alpha & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (1.1.18)$$

The matrix AX is diagonalizable if and only if d3=γ=0. Thus, within the centalizer of the isotropy, there is a diagonalizable complement to the nilradical if and only if d3=0.

In the d3=0 case, the following change of basis gives the algebra s_5,37 with isotropy spanned by e4:

```
> eval(LieAlgebraData([-x3,x2,x1,x5,x4]),d3=0);
```

$$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = e1, [e2, e3] = 0, [e2, e4] = e1, [e2, e5] = e2, [e3, e4] = e2, [e3, e5] = e3, [e4, e5] = 0 \quad (1.1.19)$$

In the d3 nonzero case, the following change of basis gives the algebra s_5,38 with isotropy spanned by e4:

```
> eval(LieAlgebraData([
  -abs(d3)*x3,
  sqrt(abs(d3))*x2,
  x1,
  sqrt(abs(d3))*x5,
  x4
  ]),d3/abs(d3)=epsilon);
```

$$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = e1, [e2, e3] = 0, [e2, e4] = e1, [e2, e5] = e2, [e3, e4] = e2, [e3, e5] = -\varepsilon\, e1 + e3, [e4, e5] = 0 \quad (1.1.20)$$

Section 7.2: The Centralizer of the Isotropy is Abelian

In this case, h3 is zero. We examine the Jacobi identities.

First recall that given h3=0, h1=h2=h4=h5=g1=g2=0 from the following:

```
> 0=evalDG(
```

```
LieBracket(e2,LieBracket(e4,e5))+
LieBracket(e5,LieBracket(e2,e4))+
LieBracket(e4,LieBracket(e5,e2))
) ;
```

$$0 = -h1\,e1 - (g1 + h2)\,e2 + (g2 - h3)\,e3 - h4\,e4 - h5\,e5 \quad \textbf{(1.2.1)}$$

```
> Eq1:={h1=0,h2=0,h3=0,h4=0,h5=0,g1=0,g2=0};
```

$$Eq1 := \{g1 = 0, g2 = 0, h1 = 0, h2 = 0, h3 = 0, h4 = 0, h5 = 0\} \quad \textbf{(1.2.2)}$$

```
> ddtheta1:=ExteriorDerivative(ExteriorDerivative(theta1)):
ddtheta2:=ExteriorDerivative(ExteriorDerivative(theta2)):
ddtheta3:=ExteriorDerivative(ExteriorDerivative(theta3)):
ddtheta4:=ExteriorDerivative(ExteriorDerivative(theta4)):
ddtheta5:=ExteriorDerivative(ExteriorDerivative(theta5)):
```

```
> ddtheta1:=Tools:-DGsimplify(eval(ddtheta1,Eq1));
ddtheta2:=Tools:-DGsimplify(eval(ddtheta2,Eq1));
ddtheta3:=Tools:-DGsimplify(eval(ddtheta3,Eq1));
ddtheta4:=Tools:-DGsimplify(eval(ddtheta4,Eq1));
ddtheta5:=Tools:-DGsimplify(eval(ddtheta5,Eq1));
```

$$ddtheta1 := -(c2\,a1 - a2\,c1 + c3\,b1 - b3\,c1 + c4\,d1)\,\theta1 \wedge \theta2 \wedge \theta3 - (g3\,b1 - d3\,c1 + g4\,d1)\,\theta1 \wedge \theta2 \wedge \theta4 + b1\,\theta1 \wedge \theta2 \wedge \theta5 - d2\,c1\,\theta1 \wedge \theta3 \wedge \theta4 - c1\,\theta1 \wedge \theta3 \wedge \theta5 + c1\,d1\,\theta2 \wedge \theta3 \wedge \theta4$$

$$ddtheta2 := (a1\,b2 - b1\,a2 - c3\,b2 + b3\,c2 - c4\,d2 - c5)\,\theta1 \wedge \theta2 \wedge \theta3 + (a1\,d2 - d1\,a2 - g3\,b2 + d3\,c2 - g4\,d2 - g5)\,\theta1 \wedge \theta2 \wedge \theta4 + (a1 + b2)\,\theta1 \wedge \theta2 \wedge \theta5 + (b1\,d2 - d1\,b2 - d2\,c2)\,\theta1 \wedge \theta3 \wedge \theta4 + (b1 - c2)\,\theta1 \wedge \theta3 \wedge \theta5 + d1\,\theta1 \wedge \theta4 \wedge \theta5 + d2\,c1\,\theta2 \wedge \theta3 \wedge \theta4 + c1\,\theta2 \wedge \theta3 \wedge \theta5$$

$$ddtheta3 := (a1\,b3 + a2\,c3 - b1\,a3 - c2\,a3 + b4\,g3 - c4\,d3 - b5)\,\theta1 \wedge \theta2 \wedge \theta3 + (a1\,d3 + a2\,g3 - d1\,a3 - g3\,b3 + d3\,c3 - g4\,d3 + d4\,g3 - d5)\,\theta1 \wedge \theta2 \wedge \theta4 - (a2 - b3)\,\theta1 \wedge \theta2 \wedge \theta5 + (b1\,d3 + g3\,b2 - d1\,b3 - d2\,c3)\,\theta1 \wedge \theta3 \wedge \theta4 - (b2 + c3)\,\theta1 \wedge \theta3 \wedge \theta5 - (d2 + g3)\,\theta1 \wedge \theta4 \wedge \theta5 + (d3\,c1 + c2\,g3)\,\theta2 \wedge \theta3 \wedge \theta4 - c2\,\theta2 \wedge \theta3 \wedge \theta5$$

$$ddtheta4 := (a1\,b4 + a2\,c4 - b1\,a4 - c2\,a4 + b3\,c4 - c3\,b4 + b4\,g4 - c4\,d4)\,\theta1 \wedge \theta2 \wedge \theta3 + (a1\,d4 + a2\,g4 - d1\,a4 - b4\,g3 + c4\,d3)\,\theta1 \wedge \theta2 \wedge \theta4 + b4\,\theta1 \wedge \theta2 \wedge \theta5 + (b1\,d4 + b2\,g4 - d1\,b4 - c4\,d2)\,\theta1 \wedge \theta3 \wedge \theta4 - c4\,\theta1 \wedge \theta3 \wedge \theta5 - g4\,\theta1 \wedge \theta4 \wedge \theta5 + (c1\,d4 + c2\,g4)\,\theta2 \wedge \theta3 \wedge \theta4$$

$$ddtheta5 := (a1\,b5 + a2\,c5 - b1\,a5 - c2\,a5 + b3\,c5 + b4\,g5 - c3\,b5 - c4\,d5)\,\theta1 \wedge \theta2 \wedge \theta3 + (a1\,d5 + a2\,g5 - d1\,a5 - g3\,b5 + d3\,c5 + d4\,g5 - g4\,d5)\,\theta1 \wedge \theta2 \wedge \theta4 + b5\,\theta1 \wedge \theta2 \wedge \theta5 + (b1\,d5 + b2\,g5 - d1\,b5 - c5\,d2)\,\theta1 \wedge \theta3 \wedge \theta4 - c5\,\theta1 \wedge \theta3 \wedge \theta5 - g5\,\theta1 \wedge \theta4 \wedge \theta5 + (c1\,d5 + c2\,g5)\,\theta2 \wedge \theta3 \wedge \theta4 \quad \textbf{(1.2.3)}$$

We consider the linear terms:

```
> Eq2:={
Hook(Hook(Hook(ddtheta1,e1),e2),e5),
Hook(Hook(Hook(ddtheta1,e1),e3),e5),

Hook(Hook(Hook(ddtheta2,e1),e2),e5),
Hook(Hook(Hook(ddtheta2,e1),e3),e5),
Hook(Hook(Hook(ddtheta2,e1),e4),e5),
Hook(Hook(Hook(ddtheta2,e2),e3),e5),

Hook(Hook(Hook(ddtheta3,e1),e2),e5),
Hook(Hook(Hook(ddtheta3,e1),e3),e5),
```

```
Hook(Hook(Hook(ddtheta3,e1),e4),e5),
Hook(Hook(Hook(ddtheta3,e2),e3),e5),

Hook(Hook(Hook(ddtheta4,e1),e2),e5),
Hook(Hook(Hook(ddtheta4,e1),e3),e5),
Hook(Hook(Hook(ddtheta4,e1),e4),e5),

Hook(Hook(Hook(ddtheta5,e1),e2),e5),
Hook(Hook(Hook(ddtheta5,e1),e3),e5),
Hook(Hook(Hook(ddtheta5,e1),e4),e5)
};
```

$$Eq2 := \{b1, b4, b5, c1, d1, -c1, -c2, -c4, -c5, -g4, -g5, a1 + b2, -a2 + b3, b1 - c2, -b2 - c3, -d2 - g3\} \quad \textbf{(1.2.4)}$$

```
> Eq3:=solve(Eq2,{b1,b2,b3,b4,b5,c1,c2,c3,c4,c5,d1,g3,g4,g5});
```

$$Eq3 := \{b1 = 0, b2 = -a1, b3 = a2, b4 = 0, b5 = 0, c1 = 0, c2 = 0, c3 = a1, c4 = 0, c5 = 0, d1 = 0, g3 = -d2, g4 = 0, g5 = 0\} \quad \textbf{(1.2.5)}$$

Now we reexamine the Jacobi identities using this partial solution:

```
> ddtheta1_s:=Tools:-DGsimplify(eval(ddtheta1,Eq3));
ddtheta2_s:=Tools:-DGsimplify(eval(ddtheta2,Eq3));
ddtheta3_s:=Tools:-DGsimplify(eval(ddtheta3,Eq3));
ddtheta4_s:=Tools:-DGsimplify(eval(ddtheta4,Eq3));
ddtheta5_s:=Tools:-DGsimplify(eval(ddtheta5,Eq3));
```

$$ddtheta1\_s := 0\ \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta2\_s := 0\ \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta3\_s := 2\ a1\ a2\ \theta1 \wedge \theta2 \wedge \theta3 + (2\ a1\ d3 - d4\ d2 - d5)\ \theta1 \wedge \theta2 \wedge \theta4$$

$$ddtheta4\_s := a1\ d4\ \theta1 \wedge \theta2 \wedge \theta4$$

$$ddtheta5\_s := a1\ d5\ \theta1 \wedge \theta2 \wedge \theta4 \quad \textbf{(1.2.6)}$$

When a1 is nonzero, a2=d3=d4=d5=0 and the Jacobi identities are fully satisfied. This is the only case in which the algebra decomposes into the direct sum of a three-dimensional algebra and the two-dimensional abelian algebra, as we shall see below.

```
> eval(eval(LieAlgebraData([
e1,e2,e3,e4,e5
]),{op(Eq1),op(Eq3)}),{d3=0,d4=0,d5=0,a2=0});
```

$$[e1, e2] = a1\ e1 + a3\ e3 + a4\ e4 + a5\ e5, [e1, e3] = -a1\ e2, [e1, e4] = d2\ e2, [e1, e5] = e2, [e2, e3] = a1\ e3, [e2, e4] = -d2\ e3, [e2, e5] = -e3, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0 \quad \textbf{(1.2.7)}$$

Under the following change of basis, we find that the algebra is given by sl(2,F)+2n,1,1 with isotropy spanned by e3+e4.

```
> eval(eval(LieAlgebraData([
(e1*a1+(a3/2+(a4*d2+a5)/(2*a1))*e3+a4*e4+a5*e5)*sqrt(2)/a1^2,
2*e2/a1,
sqrt(2)*e3/a1,
-sqrt(2)*e5-sqrt(2)*e3/a1,
e4-d2*e5
]),{op(Eq1),op(Eq3)}),{d4=0,d5=0,d3=0,a2=0,-(a1^2*a3-a1*a4*d2-a1*a5)
/a1^3=a3});
```

$$[e1, e2] = 2\ e1, [e1, e3] = -e2, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = 2\ e3, [e2, e4] = 0, [e2, e5] = 0, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0 \quad \textbf{(1.2.8)}$$

Otherwise, a1=0 and the Jacobi identities are fully satisfied if and only if d5=-d2*d4, as follows:

```
> eval(eval(LieAlgebraData([e1,e2,e3,e4,e5]),
{op(Eq1),op(Eq3)}),{a1=0,d5=-d2*d4});
```

$[e1, e2] = a2\, e2 + a3\, e3 + a4\, e4 + a5\, e5, [e1, e3] = a2\, e3, [e1, e4] = d2\, e2 + d3\, e3 + d4\, e4 - d2\, d4\, e5, [e1, e5] = e2, [e2, e3] = 0, [e2, e4] = -d2\, e3, [e2, e5] = -e3, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0$ **(1.2.9)**

The following change of basis and relabeling of a5 simplifies the algebra:

```
> LD_N1:=eval(eval(eval(
  LieAlgebraData([e1-a3*e5,e2,e3,e4-d2*e5,e5],alg_N1),
  {op(Eq1),op(Eq3)}),{a1=0,d5=-d2*d4}),a4*d2+a5=a5);
```

$LD\_N1 := [e1, e2] = a2\, e2 + a4\, e4 + a5\, e5, [e1, e3] = a2\, e3, [e1, e4] = d3\, e3 + d4\, e4, [e1, e5] = e2, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e3, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0$ **(1.2.10)**

We initialize this algebra:

```
> DGsetup(LD_N1,[x],[o]);
```

Lie algebra: alg_N1 **(1.2.11)**

To identify the unique algebras in this family, we consider the derived algebra. The following Lie brackets demonstrate that the derived algebra necessarily contains x2 and x3:

```
> LieBracket(x1,x5);
  LieBracket(x5,x2);
```

$x2$

$x3$ **(1.2.12)**

The derived algebra is therefore entirely determined by the following vectors:

```
> evalDG(LieBracket(x1,x2)-a2*x2);
  evalDG(LieBracket(x1,x4)-d3*x3);
```

$a4\, x4 + a5\, x5$

$d4\, x4$ **(1.2.13)**

Specifically, the dimension of the derived algebra is two plus the rank of the following matrix, A:

```
> A:=Matrix([[a4,a5],[d4,0]]);
```

$$A := \begin{bmatrix} a4 & a5 \\ d4 & 0 \end{bmatrix}$$ **(1.2.14)**

Section 7.2.1: The Derived Algebra is Two-Dimensional

In this case, the matrix A from above is the zero matrix, i.e. a4=a5=d4=0. We initialize the algebra under this assumption:

```
> LD_N21:=eval(
  LieAlgebraData([x1,x2,x3,x4,x5],alg_N21),
  {a4=0,a5=0,d4=0});
```

$LD\_N21 := [e1, e2] = a2\, e2, [e1, e3] = a2\, e3, [e1, e4] = d3\, e3, [e1, e5] = e2, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e3, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0$ **(1.2.1.1)**

```
> DGsetup(LD_N21,[y],[p]);
```

Lie algebra: alg_N21 **(1.2.1.2)**

We begin by considering the center. The following Lie brackets demonstrate that the center cannot contain y1, y2, or y5:

```
> LieBracket(y1,y5);
  LieBracket(y2,y5);
```

$y2$

$-y3$ **(1.2.1.3)**

We consider the adjoints of y3 and y4.

```
> Adjoint(y3);
```

```
Adjoint(y4);
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -a2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -d3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \textbf{(1.2.1.4)}$$

If the center is one-dimensional, then at least one of a2 and d3 is nonzero. By considering the lower central series, we may determine whether or not a2 = 0.

```
> Series("Lower")[2];
```

$$[a2\,y2, a2\,y3] \quad \textbf{(1.2.1.5)}$$

If a2 is non-zero (i.e., the second term in the lower central series is two-dimensional), then the following change of basis gives n_5,4 with isotropy spanned by e2+e3.

```
> LieAlgebraData([
  -1/a2*y3,
  1/a2*y2,
  -1/a2*y2+y5,
  -1/a2*y1,
  d3/a2*y3-y4]);
```

$$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = e1, [e1, e5] = 0, [e2, e3] = e1, [e2, e4] = e2, [e2, e5] = 0, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0 \quad \textbf{(1.2.1.6)}$$

If a2 is zero, then d3 is non-zero, and the following change of basis gives s_4,11+n_1,1 with isotropy spanned by e5.

```
> eval(LieAlgebraData([
  -y3,
  y2+y3,
  1/d3*y4,
  y1-y2,
  y5]),a2=0);
```

$$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = e1, [e3, e4] = e1, [e3, e5] = 0, [e4, e5] = e2 \quad \textbf{(1.2.1.7)}$$

The center is two-dimensional if a2=d3=0, in which case the following change of basis gives s_4,1+n_1,1 with isotropy spanned by e5.

```
> eval(LieAlgebraData([
  -y3,
  y2+y3,
  y1-y2,
  y4,
  y5]),{a2=0,d3=0});
```

$$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = e1, [e3, e4] = 0, [e3, e5] = e2, [e4, e5] = 0 \quad \textbf{(1.2.1.8)}$$

Section 7.2.2: The Derived Algebra is Three-Dimensional

Recall A from above; its rank is one in this case:

```
> A;
```

$$\begin{bmatrix} a4 & a5 \\ d4 & 0 \end{bmatrix} \quad (1.2.2.1)$$

A is rank one if A is not the zero matrix and either a5=0 or d4 = 0.

Recall also the structure equations below:

```
> LieAlgebraData([x1,x2,x3,x4,x5]);
```

$$[e1, e2] = a2\, e2 + a4\, e4 + a5\, e5, [e1, e3] = a2\, e3, [e1, e4] = d3\, e3 + d4\, e4, [e1, e5] = e2, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e3, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0 \quad (1.2.2.2)$$

As shown above, the derived algebra contains the following vectors:

```
> Der1:=[x2,x3,a4*x4+a5*x5,d4*x4];
```

$$Der1 := [x2, x3, a4\, x4 + a5\, x5, d4\, x4] \quad (1.2.2.3)$$

Consider the Lie brackets of these:

```
> LieDerivative(Der1,Der1);
```

$$[[0\, x1, 0\, x1, -a5\, x3, 0\, x1], [0\, x1, 0\, x1, 0\, x1, 0\, x1], [a5\, x3, 0\, x1, 0\, x1, 0\, x1], [0\, x1, 0\, x1, 0\, x1, 0\, x1]] \quad (1.2.2.4)$$

Notice that the derived algebra is abelian if and only if a5=0.

Section 7.2.2.1: The Derived Algebra is Abelian

In this case, a5=0 and a4, d4, or both are nonzero.

The structure equations are as follows:

```
> eval(LieAlgebraData([x1,x2,x3,x4,x5]),a5=0);
```

$$[e1, e2] = a2\, e2 + a4\, e4, [e1, e3] = a2\, e3, [e1, e4] = d3\, e3 + d4\, e4, [e1, e5] = e2, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e3, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0 \quad (1.2.2.1.1)$$

Consider the center by examining the adjoint of an arbitrary vector:

```
> Adjoint(R*x1+S*x2+T*x3+U*x4+V*x5);
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -S\, a2 - V & R\, a2 & 0 & 0 & R \\ -T\, a2 - U\, d3 & V & R\, a2 & R\, d3 & -S \\ -S\, a4 - U\, d4 & R\, a4 & 0 & R\, d4 & 0 \\ -S\, a5 & R\, a5 & 0 & 0 & 0 \end{bmatrix} \quad (1.2.2.1.2)$$

For this matrix to be the zero matrix, R=S=V=0, so any vector in the center is necessarily a linear combination of x3 and x4:

```
> Adjoint(T*x3+U*x4);
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -T\, a2 - U\, d3 & 0 & 0 & 0 & 0 \\ -U\, d4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (1.2.2.1.3)$$

The center is two-dimensional if and only if a2=d3=d4=0, and since A is nonzero, a4 is nonzero. In this case, the following change of basis gives n_5,2 with isotropy spanned by e5.

```
> eval(
  LieAlgebraData([x3,-a4*x4,-x2,-x1,x5]),
  {a2=0,a5=0,d3=0,d4=0});
```

$$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = 0, [e3, e4] = e2, [e3, e5] = e1, [e4, e5] = e3 \quad \textbf{(1.2.2.1.4)}$$

If the center is one-dimensional, then either d4=0 and a2 and a4 are nonzero, or a2=0 and at least one of d3 and d4 is nonzero.
We consider the lower central series.
Begin by recalling the structure equations.

```
> eval(LieAlgebraData([x1,x2,x3,x4,x5]),a5=0);
```

$$[e1, e2] = a2\, e2 + a4\, e4, [e1, e3] = a2\, e3, [e1, e4] = d3\, e3 + d4\, e4, [e1, e5] = e2, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e3, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0 \quad \textbf{(1.2.2.1.5)}$$

The derived algebra is spanned by:

```
> Der2:=[x2,x3,x4];
```

$$Der2 := [x2, x3, x4] \quad \textbf{(1.2.2.1.6)}$$

The Lie bracket of this span with the original algebra gives the second algebra in the lower central series:

```
> LCS2a:=eval(LieDerivative([x1,x2,x3,x4,x5],Der2),a5=0);
```

$$LCS2a := [[a2\, x2 + a4\, x4 + 0\, x5, a2\, x3, d3\, x3 + d4\, x4], [0\, x1, 0\, x1, 0\, x1], [0\, x1, 0\, x1, 0\, x1], [0\, x1, 0\, x1, 0\, x1], [x3, 0\, x1, 0\, x1]] \quad \textbf{(1.2.2.1.7)}$$

We consider the span of these vectors by constructing a matrix with rows corresponding to vectors and columns corresponding to basis vectors. We delete the zero rows manually.

```
> LCSMatZ:=
  Matrix([seq(seq(
  GetComponents(LCS2a[i][j],[x1,x2,x3,x4,x5]),
  i=1..5),j=1..3)]):
> LCSMat:=LinearAlgebra:-DeleteRow(LCSMatZ,
  [2..4,7..10,12..15]);
```

$$LCSMat := \begin{bmatrix} 0 & a2 & 0 & a4 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & a2 & 0 & 0 \\ 0 & 0 & d3 & d4 & 0 \end{bmatrix} \quad \textbf{(1.2.2.1.8)}$$

From this matrix, we see that the lower central series' second term is spanned by:

```
> LCS2b:=evalDG([a2*x2+a4*x4,x3,d4*x4]);
```

$$LCS2b := [a2\, x2 + a4\, x4, x3, d4\, x4] \quad \textbf{(1.2.2.1.9)}$$

Now, we build the next term in the lower central series.

```
> LCS3a:=eval(LieDerivative([x1,x2,x3,x4,x5],LCS2b),a5=0);
```

$$LCS3a := [[a2^2\, x2 + a4\, d3\, x3 + (a2\, a4 + a4\, d4)\, x4 + 0\, x5, a2\, x3, d4\, d3\, x3 + d4^2\, x4], [0\, x1, 0\, x1, 0\, x1], [0\, x1, 0\, x1, 0\, x1], [0\, x1, 0\, x1, 0\, x1], [a2\, x3, 0\, x1, 0\, x1]] \quad \textbf{(1.2.2.1.10)}$$

```
> LCSMatZ2:=
  Matrix([seq(seq(
  GetComponents(LCS3a[i][j],[x1,x2,x3,x4,x5]),
  i=1..5),j=1..3)]):
```

```
> LCSMat2:=LinearAlgebra:-DeleteRow(LCSMatZ2,
  [2..4,5,7..10,12..15]);
```

$$LCSMat2 := \begin{bmatrix} 0 & a2^2 & a4\,d3 & a2\,a4 + a4\,d4 & 0 \\ 0 & 0 & a2 & 0 & 0 \\ 0 & 0 & d4\,d3 & d4^2 & 0 \end{bmatrix} \quad \textbf{(1.2.2.1.11)}$$

If a2 is nonzero, then d4=0 and a4 is nonzero. The third algebra in the lower central series is then two-dimensional and equal to the second algebra; both are spanned by x3 and a2/a4*x3+x4.

Suppose a2=0. Then since a4 and d4 are not both zero, and the case in which a2=d3=d4=0 has been done, the third algebra in the lower central series is one-dimensional and spanned by d3*x3+d4*x4. In this case, the second algebra in the lower central series is spanned by x3 and x4.

An alternative way to distinguish these cases is by whether or not the second algebra in the lower central series commutes with the isotropy:

```
> LieDerivative(LCS2b,[x5]);
```

$$[[-a2\,x3], [0\,x1], [0\,x1]] \quad \textbf{(1.2.2.1.12)}$$

Consider the a2 nonzero case first, in which d4=0 and a4 is nonzero. The following change of basis gives s_5,20 with isotropy spanned by e1-e2-e3

```
> eval(LieAlgebraData([
  -1/a2*x3,
  1/a2*x2-a4*d3/a2^3*x3+a4/a2^2*x4,
  -1/a2*x2+(a4*d3/a2^3-1/a2)*x3-a4/a2^2*x4+x5,
  -a4*d3/a2^3*x3+x4*a4/a2^2,
  1/a2*x1-1/a2*x2-a4*d3/a2^2*x5
  ]),{a5=0,d4=0});
```

$$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -e1, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = -e2, [e3, e4] = 0, [e3, e5] = e4, [e4, e5] = 0 \quad \textbf{(1.2.2.1.13)}$$

Now consider the a2=0 case. The first terms in the lower central series are spanned by:

```
> [x2,x3,x4];
  [x3,x4];
  [d3*x3+d4*x4];
```

$$[x2, x3, x4]$$
$$[x3, x4]$$
$$[d3\,x3 + d4\,x4] \quad \textbf{(1.2.2.1.14)}$$

We calculate the next term:

```
> eval(LieDerivative([d3*x3+d4*x4],[x1,x2,x3,x4,x5]),a2=0);
```

$$[[-d4\,d3\,x3 - d4^2\,x4, 0\,x1, 0\,x1, 0\,x1, 0\,x1]] \quad \textbf{(1.2.2.1.15)}$$

In this case, the lower central series terminates in the trivial algebra if and only if d4=0.

In the case where d4=0, d3 is nonzero (since a2=d3=d4=0 is the case of two-dimensional center). Furthermore, a4 is nonzero since we are considering the case of three-dimensional derived algebra. The following change of basis gives n_5,6 with isotropy spanned by e4 for a4*d3 > 0

```
> eval(LieAlgebraData([
  1/sqrt(a4*d3)*x3,
  -1/d3*x4,
  -1/sqrt(a4*d3)*x2,
```

```
x5,
1/sqrt(a4*d3)*x1
]),{d4=0,a5=0,a2=0});
```

$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = e1, [e3, e4] = e1, [e3, e5] = -e2, [e4, e5] = e3$ **(1.2.2.1.16)**

If a4*d3 < 0, the following change of basis gives the same algebra-subalgebra pair.

```
> eval(LieAlgebraData([
1/sqrt(-a4*d3)*x3,
-1/d3*x4,
1/sqrt(-a4*d3)*x2,
-x5,
1/sqrt(-a4*d3)*x1
]),{d4=0,a5=0,a2=0});
```

$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = e1, [e3, e4] = e1, [e3, e5] = -e2, [e4, e5] = e3$ **(1.2.2.1.17)**

In the case where d4 is nonzero, we must consider whether or not a4 is zero. Consider the center:

```
> eval(Adjoint(T*x3+U*x4),a2=0);
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -U\,d3 & 0 & 0 & 0 & 0 \\ -U\,d4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \textbf{(1.2.2.1.18)}$$

With d4 nonzero, the center is spanned by x3. Consider the structure equations:

```
> eval(LieAlgebraData([x1,x2,x3,x4,x5]),{a2=0,a5=0});
```

$[e1, e2] = a4\,e4, [e1, e3] = 0, [e1, e4] = d3\,e3 + d4\,e4, [e1, e5] = e2, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e3, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0$ **(1.2.2.1.19)**

We consider the upper central series. We consider the adjoint of an aribtrary vector and ignore the third row, which corresponds to the center:

```
> LinearAlgebra:-DeleteRow(eval(
Adjoint(R*x1+S*x2+T*x3+U*x4+V*x5),
{a2=0,a5=0}),[3]);
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -V & 0 & 0 & 0 & R \\ -S\,a4 - U\,d4 & R\,a4 & 0 & R\,d4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \textbf{(1.2.2.1.20)}$$

We require R=V=0.

```
> LinearAlgebra:-DeleteRow(eval(
Adjoint(S*x2+T*x3+U*x4),
{a2=0,a5=0}),[3]);
```

(1.2.2.1.21)

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -S\,a4 - U\,d4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \textbf{(1.2.2.1.21)}$$

Since d4 is nonzero, the second term in the upper central series is spanned by

```
> UCS2:=evalDG([x3,x2-a4/d4*x4]);
```

$$UCS2 := \left[x3, x2 - \frac{a4}{d4}\,x4\right] \quad \textbf{(1.2.2.1.22)}$$

We build the next term in the series by consider the adjoint of an arbitrary vector in a basis adapted to UCS2.

```
> LinearAlgebra:-DeleteRow(eval(
  Adjoint(R*x1+S*(x2-a4/d4*x4)+T*x3+U*x4+V*x5,
  [x1,x2-a4/d4*x4,x3,x4,x5]),
  {a2=0,a5=0}),[2,3]);
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -\frac{U\,d4^2 + V\,a4}{d4} & 0 & 0 & R\,d4 & \frac{R\,a4}{d4} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \textbf{(1.2.2.1.23)}$$

Since d4 is nonzero, we require R=0. If a4=0, we also require U=0, giving the algebra spanned by UCS2 together with x5. If a4 is nonzero, we require V = -U*d4/a4 and he algebra is spanned by UCS2 together with x4-d4/a4*x5. Thus a4 determines whether or not the isotropy is in the third upper central series algebra (which happens to be the terminal algebra in either case).

If a4 is nonzero, the following change of basis gives s_5,14 with isotropy spanned by e2+e3+e4:

```
> simplify(eval(LieAlgebraData([
  x3/d4,
  x2/d4-x3*d3*a4/d4^3-a4/d4^2*x4,
  -x2/d4+x5,
  x3*a4*d3/d4^3+a4/d4^2*x4,
  x1/d4+x5*a4*d3/d4^2
  ]),{a2=0,a5=0}));
```

$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = -e1, [e2, e4] = 0, [e2, e5] = 0, [e3, e4] = 0, [e3, e5] = -e2, [e4, e5] = -e4$ **(1.2.2.1.24)**

If a4 = 0, then the following change of basis gives s_5,14 with isotropy spanned by e1-e3:

```
> simplify(eval(LieAlgebraData([
  x3/d4,
  x2/d4,
  x3/d4+x5,
  d3/d4^2*x3+1/d4*x4,
  x1/d4
  ]),{a2=0,a4=0,a5=0}));
```

$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = -e1, [e2, e4] = 0, [e2, e5] = 0, [e3, e4] = 0, [e3, e5] = -e2, [e4, e5] = -e4$ **(1.2.2.1.25)**

We now consider the case in which the center is trivial, i.e., a5=0, but a2 and d4 are nonzero. The structure equations, again, are:

```
> LD_N221:=eval(LieAlgebraData([x1,x2,x3,x4,x5],alg_N221),a5=0);
```

(1.2.2.1.26)

$LD\_N221 := [e1, e2] = a2\ e2 + a4\ e4, [e1, e3] = a2\ e3, [e1, e4] = d3\ e3 + d4\ e4, [e1, e5] = e2, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e3, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0$ **(1.2.2.1.26)**

We initialize this and consider the nilradical:

```
> DGsetup(LD_N221,[y],[p]);
```

Lie algebra: alg_N221 **(1.2.2.1.27)**

```
> Nilradical();
```

$[y2, y3, y4, y5]$ **(1.2.2.1.28)**

The only nonzero Lie bracket of these basis vectors for the nilradical is

```
> LieBracket(y5,y2);
```

$y3$ **(1.2.2.1.29)**

Consider an arbitrary vector non in the nilradical, and its adjoint:

```
> X:=evalDG(alpha*y1+beta*y2+gamma*y3+delta*y4+epsilon*y5);
```

$X := \alpha\ y1 + \beta\ y2 + \gamma\ y3 + \delta\ y4 + \varepsilon\ y5$ **(1.2.2.1.30)**

```
> AX:=Adjoint(X,[X,y2,y3,y4,y5]);
```

$$AX := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha\ a2 & 0 & 0 & \alpha \\ 0 & \varepsilon & \alpha\ a2 & \alpha\ d3 & -\beta \\ 0 & \alpha\ a4 & 0 & \alpha\ d4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ **(1.2.2.1.31)**

We find the eigenvalues of this matrix (recall that a2 and d4 are nonzero):

```
> factor(LinearAlgebra:-CharacteristicPolynomial(AX,lambda));
```

$-\lambda^2 (\alpha\ a2 - \lambda)^2 (\alpha\ d4 - \lambda)$ **(1.2.2.1.32)**

Thus if a2 is not equal to d4, the eigenvalues are zero and 1 with algebraic multiplicity two and d4/a2 with algebraic multiplicty one. Otherwise, 1 is an eigenvalue of algebraic multiplicity three.

If there are three distinct eigenvalues, then the following change of basis gives s_5,30 with a not equal to one and isotropy e2+e3+e4 for a4 nonzero:

```
> chi4:=(a4*d3/(a2-d4)^2/d4)*y3-a4/d4/(a2-d4)*y4:
  eval(LieAlgebraData([
  -1/a2*y3,
  (1/a2*y2-d4/a2*chi4),
  (-1/a2*y2+(d4/a2-1)*chi4+y5),
  chi4,
  (1/a2*y1-a4*d3/(a2^2-a2*d4)*y5)
  ]),d4/a2=a);
```

$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -e1, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = -e2, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = -a\ e4$ **(1.2.2.1.33)**

If a4 = 0, the following change of basis gives the same algebra, but with isotropy e2+e3:

```
> eval(LieAlgebraData([
  -1/a2*y3,
  1/a2*y2,
  -1/a2*y2+y5,
  1/(d4*(a2-d4))*(d3/(a2-d4)*y3-y4),
  1/a2*y1+(a4*d3)/(a2*d4-a2^2)*y5
  ]),{a4=0,d4/a2=a});
```

$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -e1, [e2, e3] = e1, [e2,$ **(1.2.2.1.34)**

$e4] = 0, [e2, e5] = -e2, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = -a\, e4$

These two isotropies, h, are distinguished by whether or not there is a complement to the nilradical, X such that, such that [X,h] is an eigenvalue of ad(X).

If there are only two distinct eigenvalues (one of which is zero), then a2=d4. Consider AX–a2*αI:

```
> eval(AX,d4=a2)-a2*alpha*LinearAlgebra:-IdentityMatrix(5);
```

$$\begin{bmatrix} -\alpha\, a2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha \\ 0 & \varepsilon & 0 & \alpha\, d3 & -\beta \\ 0 & \alpha\, a4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha\, a2 \end{bmatrix} \quad \textbf{(1.2.2.1.35)}$$

When a4 is nonzero, the rank is independent of choice of X and is given by three if d3=0 and four otherwise.

Therefore, given a2=d4, a4 is zero if and only if there are two vectors X and Y not in the nilradical such that ad(X)-a2*αI and ad(Y)–a2*αI are of distinct rank. If the ranks are two and three, then d3=0. If the ranks are three and four, then d3 is nonzero.

We now find changes of basis for these cases:

If a4 and d3 are nonzero, the following change of basis gives s_5,32 with isotropy spanned by e2+e3

```
> chi3:=evalDG(
  a4*d3/a2^3*y2+
  2*d3^2*a4^2/a2^5*y3-
  a4^2*d3/a2^4*y4):

  eval(LieAlgebraData([
  a4^2*d3^2/a2^5*y3,
  -chi3+a4*d3/a2^2*y5,
  chi3,
  a4*d3/a2^3*y2-chi3,
  (1/a2*y1-a4*d3/a2^2*y5)
  ]),{d4=a2});
```

$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -e1, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = 0, [e3, e4] = 0, [e3, e5] = -e3 - e4, [e4, e5] = -e1 - e4$ **(1.2.2.1.36)**

If a4=0 and d3 is nonzero, the following change of basis gives s_5,31 with isotropy spanned by e2+e3

```
> eval(LieAlgebraData([
  -1/a2*y3,
  1/a2*y2,
  -1/a2*y2+y5,
  a2/d3*y3-1/d3*y4,
  1/a2*y1
  ]),{d4=a2,a4=0});
```

$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -e1, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = -e2, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = -e1 - e4$ **(1.2.2.1.37)**

If a4 is nonzero and d3=0, the following change of basis gives s_5,29 with isotropy spanned by e2+e3

```
> lprint(%);
_DG([["LieAlgebra", "L1", [5, table( [ ] )]], [[[1, 5, 1], -1], [
[2, 3, 1], 1], [[2, 5, 1], 0], [[2, 5, 2], -1], [[2, 5, 4], 0], [
[3, 5, 1], 0], [[3, 5, 4], 0], [[4, 5, 1], -1], [[4, 5, 4], -1]]])
```

```
> eval(LieAlgebraData([
  -1/a2*y3,
  -1/a2*y2+a4/a2^2*y4+y5,
  1/a2*y2-a4/a2^2*y4,
  a4/a2^2*y4,
  1/a2*y1
  ]),{d4=a2,d3=0});
```

$$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -e1, [e2, e3] = -e1, [e2, e4] = 0, [e2, e5] = 0, [e3, e4] = 0, [e3, e5] = -e3 - e4, [e4, e5] = -e4 \quad \textbf{(1.2.2.1.38)}$$

If a4=d3=0, the following change of basis gives
s_5,30 with a=1 and isotropy spanned by e2+e3

```
> eval(LieAlgebraData([
  -1/a2*y3,
  1/a2*y2,
  -1/a2*y2+y5,
  y4,
  1/a2*y1
  ]),{d4=a2,a4=0,d3=0});
```

$$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -e1, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = -e2, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = -e4 \quad \textbf{(1.2.2.1.39)}$$

Section 7.2.2.2: The Derived Algebra is Non-Ableian

If the derived algebra is not abelian, a5 is nonzero, d4=0, and the
structure equations are as follows:

```
> eval(LieAlgebraData([x1,x2,x3,x4,x5]),d4=0);
```

$$[e1, e2] = a2\, e2 + a4\, e4 + a5\, e5, [e1, e3] = a2\, e3, [e1, e4] = d3\, e3, [e1, e5] = e2, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e3, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0 \quad \textbf{(1.2.2.2.1)}$$

The center cannot contain x1, x2, or x5:

```
> LieBracket(x1,x5); LieBracket(x5,x2);
```

$$x2$$
$$x3 \quad \textbf{(1.2.2.2.2)}$$

Thus every vector in the center is of the form S*x3+T*x4. Examine
the adjoint of an arbitrary vector in the center:

```
> eval(Adjoint(S*x3+T*x4),d4=0);
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -S\, a2 - T\, d3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \textbf{(1.2.2.2.3)}$$

The center is thus one-dimensional or two-dimensional and is two-
dimensional if and only if a2=d3=0, in which case the following
change of basis simplifies the structure equations, which we will
initialize:

```
> LD_N222:=simplify(eval(LieAlgebraData([
```

```
1/sqrt(abs(a5))*x1,
a5/sqrt(abs(a5))*x2,
(abs(a5))^(3/2)*x3,
x4,
a4*x4+a5*x5
],alg_N222),{a2=0,d3=0,d4=0}))
assuming a5::real;
```

$LD\_N222 := \left[e1, e2\right] = \frac{a5}{|a5|}\ e5, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = e2, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e3, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0$ **(1.2.2.2.4)**

```
> DGsetup(LD_N222,[y],[p]);
```

Lie algebra: alg_N222 **(1.2.2.2.5)**

The sign of a5 is essential and determines the signature of the Killing form:

```
> Killing();
```

$$\begin{bmatrix} \frac{2\ a5}{|a5|} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ **(1.2.2.2.6)**

Further, note taht the isotropy is spanned by a4*y4-y5. This is a sub algebra of the derived algebra if and only if a4=0:

```
> DerivedAlgebra();
```

$[y2, y3, y5]$ **(1.2.2.2.7)**

If a4 is nonzero, then scaling y4 is an automorphism that allows us to take a4 = 1 (i.e., take the isotropy to be y4+y5):

```
> LieAlgebraData([y1,y2,y3,-a4*y4,y5]);
```

$\left[e1, e2\right] = \frac{a5}{|a5|}\ e5, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = e2, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e3, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0$ **(1.2.2.2.8)**

Now, if a5 > 0, the following change of basis yields s_4,6+n_1,1 with isotropy spanned by e2-2*e3 or e2-2*e3+e5.

```
> eval(LieAlgebraData([
-y3,-y2+y5,-1/2*y2-1/2*y5,-y1,y4
]),{abs(a5)=a5});
```

$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = e1, [e2, e4] = -e2, [e2, e5] = 0, [e3, e4] = e3, [e3, e5] = 0, [e4, e5] = 0$ **(1.2.2.2.9)**

If a5 < 0, the following change of basis yields s_4,7+n_1,1 with isotropy spanned by e3 or e3-e5

```
> eval(LieAlgebraData([
y3,y2,-y5,-y1,y4
]),{abs(a5)=-a5});
```

$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = e1, [e2, e4] = e3, [e2, e5] = 0, [e3, e4] = -e2, [e3, e5] = 0, [e4, e5] = 0$ **(1.2.2.2.10)**

This concludes the case of two-dimensional center. If the center is one-dimensional, that a2 and d3 cannot both be zero. Recall the structure equations:

```
> eval(LieAlgebraData([x1,x2,x3,x4,x5]),d4=0);
```

(1.2.2.2.11)

$[e1, e2] = a2\, e2 + a4\, e4 + a5\, e5, [e1, e3] = a2\, e3, [e1, e4] = d3\, e3, [e1, e5] = e2, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e3, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0$ **(1.2.2.2.11)**

Suppose a2 is zero. Then the center is spanned by x3, and the second algebra in the upper central series is spanned by x3 and x4. We show this by considering the adjoint of a generic vector and deleting the third row:

```
> LinearAlgebra:-DeleteRow(eval(
  Adjoint(R*x1+S*x2+T*x3+U*x4+V*x5),
  {d4=0,a2=0}),[3]);
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -V & 0 & 0 & 0 & R \\ -S\, a4 & R\, a4 & 0 & 0 & 0 \\ -S\, a5 & R\, a5 & 0 & 0 & 0 \end{bmatrix} \quad \textbf{(1.2.2.2.12)}$$

We thus see V=R=S=0, leaving a vector of the form T*x3+U*x4.

If, however, a2 is nonzero, the center is spanned by d3/a2*x3-x4
By the same process, we see that the upper central series terminates at the center:

```
> LinearAlgebra:-DeleteRow(eval(
  Adjoint(R*x1+S*x2+T*x3+U*x4+V*x5,[x1,x2,x3,d3/a2*x3-x4,x5]),
  {d4=0}),[4]);
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -S\, a2 - V & R\, a2 & 0 & 0 & R \\ -\frac{S\, a4\, d3 + T\, a2^2 + U\, d3\, a2}{a2} & \frac{d3\, R\, a4 + V\, a2}{a2} & R\, a2 & 0 & -S \\ -S\, a5 & R\, a5 & 0 & 0 & 0 \end{bmatrix} \quad \textbf{(1.2.2.2.13)}$$

We find that R=S=V=0 and T=-U*d3/a2, i.e., the algebra contains only the center.

Therefore, a2 is zero if and only if the upper central series has two distinct algebras. Otherwise, it terminates at the center.

Consider the a2=0 case first. We perform the following change of basis and initialize the algebra:

```
> LD_N222b:=eval(LieAlgebraData([
  1/sqrt(abs(a5))*x3,
  1/sqrt(abs(a5))*x2,
  -a4/a5*x4-x5,
  1/d3*x4,
  1/sqrt(abs(a5))*x1+a4*d3/(a5*sqrt(abs(a5)))*x2
  ],alg_N222b),{d4=0,a2=0});
```

$LD\_N222b := [e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = e1, [e2, e4] = 0, \left[e2, e5\right] = \frac{a5}{|a5|}\, e3, [e3, e4] = 0, [e3, e5] = e2, [e4, e5] = -e1$ **(1.2.2.2.14)**

```
> DGsetup(LD_N222b,[w],[q]);
```

Lie algebra: alg_N222b **(1.2.2.2.15)**

The signature of the Killing form is determined by the sign of a5:

```
> Killing();
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2\,a5}{|a5|} \end{bmatrix} \quad (1.2.2.2.16)$$

Furthermore, the isotropy is spanned by w3+d3*a4/a5*w4 and is a subalgebra of the derived algebra if and only if a4=0. If a4 is nonzero, the following automorphism allows us to take the isotropy to be spanned by w3+w4:

```
> LieAlgebraData([
  (a4*d3/a5)^2*w1,
  a4*d3/a5*w2,
  a4*d3/a5*w3,
  (a4*d3/a5)^2*w4,
  w5
  ]);
```

$$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = e1, [e2, e4] = 0, \left[e2, e5\right] = \frac{a5}{|a5|}\, e3, [e3, e4] = 0, [e3, e5] = e2, [e4, e5] = -e1 \quad (1.2.2.2.17)$$

If a5 < 0, then the algebra is already given as s_5,16 with isotropy spanned by w3 or w3+w4 depending on whether or not the isotropy is in the derived algebra:

```
> eval(LieAlgebraData([w1,w2,w3,w4,w5]),{abs(a5)=-a5});
```

$$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = -e3, [e3, e4] = 0, [e3, e5] = e2, [e4, e5] = -e1 \quad (1.2.2.2.18)$$

If a5 > 0, the following change of basis gives s_5,15 with isotropy spanned by e2-e3 or e2-e3-e4.

```
> eval(LieAlgebraData([
  -2*w1,w2+w3,w2-w3,-2*w4,w5
  ]),{abs(a5)=a5});
```

$$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = e2, [e3, e4] = 0, [e3, e5] = -e3, [e4, e5] = -e1 \quad (1.2.2.2.19)$$

Now we treat the case when the upper central series terminates at the center, i.e., when a2 is nonzero. Again, the isotropy is in the derived algebra if and only if a4=0. When a4=0, we apply the following change of basis and relabel a5/a2^2 as a5:

```
> eval(LieAlgebraData([
  1/a2*x1,1/a2*x2,1/a2*x3,d3/a2*x3-x4,x5
  ]),{d4=0,a4=0,a5/a2^2=a5});
```

$$\left[e1, e2\right] = e2 + a5\, e5, [e1, e3] = e3, [e1, e4] = 0, [e1, e5] = e2, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e3, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0 \quad (1.2.2.2.20)$$

This yields isotropy spanned by the fifth basis vector. When a4 is nonzero, the following change of basis yields the same structure equations and isotropy spanned by the sum of the fourth and fifth basis vectors:

```
> LD_N222c:=eval(LieAlgebraData([
  1/a2*x1-d3*a4/a2^2*x5,
  1/a2*x2,
  1/a2*x3,
  a4*d3/(a2*a5)*x3-a4/a5*x4,
```

```
-a4*d3/(a2*a5)*x3+a4/a5*x4+x5
],alg_N222c),{d4=0,a5/a2^2=a5});
```

$$LD\_N222c := [e1, e2] = e2 + a5\, e5, [e1, e3] = e3, [e1, e4] = 0, [e1, e5] = e2, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e3, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0 \quad (1.2.2.2.21)$$

We initialize the algebra:

```
> DGsetup(LD_N222c,[f],[r]);
```

Lie algebra: alg_N222c (1.2.2.2.22)

The nilradical is as follows:

```
> Nilradical();
```

$$[f2, f3, f4, f5] \quad (1.2.2.2.23)$$

Consider the adjoint of an arbitrary vector not in the nilradical:

```
> F:=evalDG(alpha*f1+beta*f2+gamma*f3+delta*f4+epsilon*f5);
```

$$F := \alpha\, f1 + \beta\, f2 + \gamma\, f3 + \delta\, f4 + \varepsilon\, f5 \quad (1.2.2.2.24)$$

```
> AF:=Adjoint(F,[F,f2,f3,f4,f5]);
```

$$AF := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & \alpha \\ 0 & \varepsilon & \alpha & 0 & -\beta \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha\, a5 & 0 & 0 & 0 \end{bmatrix} \quad (1.2.2.2.25)$$

We examine its eigenvalues:

```
> factor(LinearAlgebra:-
  CharacteristicPolynomial(AF,lambda));
```

$$(\alpha - \lambda)\, \lambda^2\, (\alpha^2\, a5 + \alpha\, \lambda - \lambda^2) \quad (1.2.2.2.26)$$

The quadratic factor has the following roots:

```
> solve(a5*alpha^2+alpha*lambda-lambda^2,lambda);
```

$$\left(\frac{1}{2} + \frac{\sqrt{1+4\, a5}}{2}\right)\alpha, \left(\frac{1}{2} - \frac{\sqrt{1+4\, a5}}{2}\right)\alpha \quad (1.2.2.2.27)$$

Since a5 is nonzero, neither of these roots can be α or zero. Thus, there is a nonzero eigenvalue of multiplicity two if and only if these roots are equal, i.e., a5 = -1/4. If a5 > -1/4, all eigenvalues are real, and if a5 < -1/4, there are nonreal eigenvalues. We find changes of basis for these cases:

If a5 = -1/4, then we have s_4,10+n_1,1 with isotropy e2 or e2+e5.

```
> eval(LieAlgebraData([
  -f3,
  sqrt(2)*f2-sqrt(2)/2*f5,
  sqrt(2)/2*f5,
  2*f1,
  sqrt(2)/2*f4
  ]),{a5=-1/4});
```

$$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = -2\, e1, [e1, e5] = 0, [e2, e3] = e1, [e2, e4] = -e2, [e2, e5] = 0, [e3, e4] = -e2 - e3, [e3, e5] = 0, [e4, e5] = 0 \quad (1.2.2.2.28)$$

If a5 > -1/4, then we have s_4,8+n_1,1 with isotropy e2-e3 or e2-e3+e5

```
> simplify(eval(LieAlgebraData([
  (a-1)/(1+a)*f3,f2-a/(1+a)*f5,f2-1/(1+a)*f5,(1+a)*f1,(1-a)/(1+a)*f4
  ]),{a5=-a/(1+a)^2}));
```

$$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = -(1+a)\, e1, [e1, e5] = 0, [e2, e3] = \quad (1.2.2.2.29)$$

$-e1, [e2, e4] = -e2, [e2, e5] = 0, [e3, e4] = -a\,e3, [e3, e5] = 0, [e4, e5] = 0$

If a5 < -1/4, then we have s_4,9+n_1,1 with isotropy e2 or e2+e5

```
> simplify(eval(LieAlgebraData(eval([
  -(alpha^2+1)^2/(2*alpha^3)*f3,
  -(alpha^2+1)/(2*alpha^2)*f5,
  (alpha^2+1)/alpha*f2-(alpha^2+1)/(2*alpha)*f5,
  2*alpha*f1,
  -(alpha^2+1)/(2*alpha^2)*f4
  ],{alpha=1/sqrt(-1-4*a5)})),{a5=(1/alpha^2+1)/(-4)}))
  assuming alpha::positive;
```

$$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = -2\alpha\,e1, [e1, e5] = 0, [e2, e3] = e1, [e2, e4] = -\alpha\,e2 + e3, [e2, e5] = 0, [e3, e4] = -e2 - \alpha\,e3, [e3, e5] = 0, [e4, e5] = 0 \quad \textbf{(1.2.2.2.30)}$$

Section 7.2.3: The Derived Algebra is Four-Dimensional

Recall A from above; its rank is two in this case:

```
> A;
```

$$\begin{bmatrix} a4 & a5 \\ d4 & 0 \end{bmatrix} \quad \textbf{(1.2.3.1)}$$

This requires that d4 and a5 be nonzero.

Now, recall the structure equations and consider the center of the algebra; it is either spanned by x3 or it is trivial:

```
> LieAlgebraData([x1,x2,x3,x4,x5]);
```

$$[e1, e2] = a2\,e2 + a4\,e4 + a5\,e5, [e1, e3] = a2\,e3, [e1, e4] = d3\,e3 + d4\,e4, [e1, e5] = e2, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e3, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0 \quad \textbf{(1.2.3.2)}$$

This becomes more clear upon examination of the adjoint of an arbitrary vector:

```
> Adjoint(R*x1+S*x2+T*x3+U*x4+V*x5);
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -S\,a2 - V & R\,a2 & 0 & 0 & R \\ -T\,a2 - U\,d3 & V & R\,a2 & R\,d3 & -S \\ -S\,a4 - U\,d4 & R\,a4 & 0 & R\,d4 & 0 \\ -S\,a5 & R\,a5 & 0 & 0 & 0 \end{bmatrix} \quad \textbf{(1.2.3.3)}$$

We quickly see that if this matrix is the zero matrix, then R=S=V=0, which implies (since d4 is nonzero) that U=0 as well. The matrix is the zero matrix if and only if T*a2=0. Thus, the dimension of the center is one if a2=0 and the center is trivial otherwise.

Section 7.2.3.1: The Center is One-Dimensional

Let the center be one-dimensional so that a2=0. We initialize this algebra:

```
> LD_N231:=eval(LieAlgebraData(
  [x1,x2,x3,x4,x5],alg_N231),a2=0);
```

$$LD\_N231 := [e1, e2] = a4\,e4 + a5\,e5, [e1, e3] = 0, [e1, e4] = d3\,e3 + d4\,e4, \quad \textbf{(1.2.3.1.1)}$$

$[e1, e5] = e2, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e3, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0$

```
> DGsetup(LD_N231,[y],[o]);
```

Lie algebra: alg_N231 **(1.2.3.1.2)**

We calculate the nilradical:

```
> Nilradical();
```

$[y2, y3, y4, y5]$ **(1.2.3.1.3)**

Consider a generic vector not in the nilradical and its adjoint:

```
> X:=evalDG(y1+beta*y2+gamma*y3+delta*y4+epsilon*y5);
```

$X := y1 + \beta\, y2 + \gamma\, y3 + \delta\, y4 + \varepsilon\, y5$ **(1.2.3.1.4)**

```
> AX:=Adjoint(X,[X,y2,y3,y4,y5]);
```

$$AX := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & \varepsilon & 0 & d3 & -\beta \\ 0 & a4 & 0 & d4 & 0 \\ 0 & a5 & 0 & 0 & 0 \end{bmatrix}$$ **(1.2.3.1.5)**

We examin its eigenvalues:

```
> factor(LinearAlgebra:-CharacteristicPolynomial(AX,lambda));
```

$\lambda^2 (-\lambda + d4) (-\lambda^2 + a5)$ **(1.2.3.1.6)**

Its eigenvalues are proportional to zero, d4, and ±sqrt(a5).

Thus, if there are any imaginary eigenvalues, a5 is negative. and a5 is not equal to d4^2. Furthermore, the isotropy is in the derived algebra if and only if a4=0, in which case, the following change of basis gives s_5,19 with alpha not equal to one and isotropy spanned by e3.

```
> eval(LieAlgebraData([
  -d4^4/(a5*sqrt(-a5))*y3,
  d4^2/a5*y2,
  d4^2/sqrt(-a5)*y5,
  -d3*y3-d4*y4,
  (1/sqrt(-a5)*y1)
  ]),{a4=0,d4/sqrt(-a5)=alpha});
```

$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = -e3, [e3, e4] = 0, [e3, e5] = e2, [e4, e5] = -\alpha\, e4$ **(1.2.3.1.7)**

If the isotropy is not in the derived algebra, then a4 is nonzero and the following change of basis gives the same algebra with isotropy spanned by e3-e4.

```
> chi4:=a4*d4/((d4^2-a5)*sqrt(-a5))*(-d3*y3-d4*y4):

  eval(LieAlgebraData([
  -d4^4/(a5*sqrt(-a5))*y3,
  d4^2/a5*y2-d4/sqrt(-a5)*chi4,
  chi4+d4^2/sqrt(-a5)*y5,
  chi4,
  (1/sqrt(-a5)*y1+a4*d3/(d4*sqrt(-a5))*y5)
  ]),{d4/sqrt(-a5)=alpha});
```

$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = -e3, [e3, e4] = 0, [e3, e5] = e2, [e4, e5] = -\alpha\, e4$ **(1.2.3.1.8)**

Note also by scaling e2, e3, and e5 by $\pm$, we may take $\alpha > 0$.
Furthermore, since d4^2 is not equal to a5, alpha is not one in this case.

If there are four distinct, real eigenvalues, then a5 > 0 and a5 is not equal to d4^2. As before, a4 determines whether or not the isotropy is in the derived algebra. If it is, then a4=0 and the following change of basis gives s_5,17 with a not equal to and one isotropy spanned by e2-e3.

```
> eval(LieAlgebraData([
  2*d4^4/(a5*sqrt(a5))*y3,
  d4^2/a5*y2+d4^2/sqrt(a5)*y5,
  d4^2/a5*y2-d4^2/sqrt(a5)*y5,
  -d3*y3-d4*y4,
  (1/sqrt(a5)*y1)
  ]),{a4=0,d4/sqrt(a5)=a});
```

$$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = -e2, [e3, e4] = 0, [e3, e5] = e3, [e4, e5] = -a\, e4 \tag{1.2.3.1.9}$$

If the isotropy is not in the derived algebra and there are four real and distinct eigenvalues, then the following change of basis gives the same algebra with isotropy spanned by e2-e3-e4

```
> chi4:=-2*a4/(d4*(-d4^2+a5))*(-d3*y3-d4*y4):

  simplify(eval(LieAlgebraData([
  2/sqrt(a5)*y3,
  (1/sqrt(a5)*y2+alpha*chi4+y5),
  (1/sqrt(a5)*y2+(alpha-1)*chi4-y5),
  chi4,
  1/sqrt(a5)*y1+a4*d3/(d4*sqrt(a5))*y5
  ]),{
  d4/sqrt(a5)=a,
  alpha=(-d4^2+a5)/(2*sqrt(a5)*(sqrt(a5)-d4))
  }));
```

$$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = -e2, [e3, e4] = 0, [e3, e5] = e3, [e4, e5] = -a\, e4 \tag{1.2.3.1.10}$$

Note that in the above, a is not one, since d4^2 is not equal to a5.

If there is a repeated nonzero eigenvalue, then a5=d4^2 and again the value of a4 determines whether or not the isotropy is in the derived algebra. If a4 = 0, the isotropy is in the derived algebra and the following change of basis gives s_5,17 with a=1 and isotropy spanned by e2-e3

```
> eval(LieAlgebraData([
  2*d4*y3,
  y2+d4*y5,
  y2-d4*y5,
  -d3*y3-d4*y4,
  1/d4*y1
  ]),{a4=0,a5=d4^2});
```

$$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = -e2, [e3, e4] = 0, [e3, e5] = e3, [e4, e5] = -e4 \tag{1.2.3.1.11}$$

If a5=d4^2 and the isotropy is not in the derived algebra, then a4 is not zero and the following change of basis gives s_5,18 with isotropy spanned by e2-e3-(1/2)e4

```
> eval(LieAlgebraData([
  -d4*y3,
  y2-d4*y5+a4/(2*d4^2)*(-d3*y3-d4*y4),
  y2+d4*y5,
```

```
-a4/(d4^2)*(-d3*y3-d4*y4),
1/d4*y1+a4*d3/d4^2*y5
]),{a5=d4^2});
```

$$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = 2\, e1, [e2, e4] = 0, [e2, e5] = e2, [e3, e4] = 0, [e3, e5] = -e3 - e4, [e4, e5] = -e4 \quad \textbf{(1.2.3.1.12)}$$

Section 7.2.3.2: The Center is Trivial

When the center is trivial, a2, a5, and d4 are nonzero. We may set a2=1 as follows:

```
> LD_N232:=eval(LieAlgebraData(
  [1/a2*x1,1/a2^2*x2,1/a2^3*x3,1/a2^3*x4,1/a2*x5],
  alg_N232),{a5/a2^2=a5,d3/a2=d3,d4/a2=d4});
```

$$LD\_N232 := [e1, e2] = e2 + a4\, e4 + a5\, e5, [e1, e3] = e3, [e1, e4] = d3\, e3 + d4\, e4, [e1, e5] = e2, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e3, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0 \quad \textbf{(1.2.3.2.1)}$$

```
> DGsetup(LD_N232,[y],[p]);
```

Lie algebra: alg_N232 **(1.2.3.2.2)**

We consider the nilradical:

```
> Nilradical();
```

$$[y2, y3, y4, y5] \quad \textbf{(1.2.3.2.3)}$$

We next consider an arbitrary vector not in the nilradical:

```
> X:=evalDG(lambda*y1+beta*y2+gamma*y3+delta*y4+epsilon*y5);
```

$$X := \lambda\, y1 + \beta\, y2 + \gamma\, y3 + \delta\, y4 + \varepsilon\, y5 \quad \textbf{(1.2.3.2.4)}$$

The nilradical is a four-dimensional subalgebra containing the isotropy. We shall see that when a4=0, it is the only such algebra.

We begin by noting that any other algebra with this property contains a choice of X with λ nonzero as well as the isotropy basis y5. Thus, it also contains the Lie bracket of these:

```
> W2:=LieBracket(X,y5);
```

$$W2 := \lambda\, y2 - \beta\, y3 \quad \textbf{(1.2.3.2.5)}$$

This is not spanned by X and y5 for any choice of X. We consider the Lie brackets with this vector:

```
> LieBracket(W2,y5);
  LieBracket(W2,X);
```

$$-\lambda\, y3$$

$$-\lambda^2\, y2 + (\lambda\beta - \lambda\varepsilon)\, y3 - \lambda^2\, a4\, y4 - \lambda^2\, a5\, y5 \quad \textbf{(1.2.3.2.6)}$$

If a4=0, then the vectors X, y5, W2, and x3 form an algebra. Otherwise, they do not, implying that the nilradical is the only four-dimensional subalgebra containing the isotropy. We shall use this invariant in the following discussion where necessary.

For now, we consider the adjoint of X:

```
> AX:=Adjoint(X,[y2,y3,y4,y5]);
```

$$AX := \begin{bmatrix} \lambda & 0 & 0 & \lambda \\ \varepsilon & \lambda & \lambda\, d3 & -\beta \\ \lambda\, a4 & 0 & \lambda\, d4 & 0 \\ \lambda\, a5 & 0 & 0 & 0 \end{bmatrix} \quad \textbf{(1.2.3.2.7)}$$

The following similarity transformation eliminates epsilon and beta, so when considering the eigenvectors, we may take X=lambda*x1:

```
> Q:=Matrix([
  [1,0,0,0],
  [(epsilon-a5*beta)/(lambda*a5),1,0,epsilon/(lambda*a5)],
  [0,0,1,0],
  [0,0,0,1]]);
  AX2:=simplify(Q^(-1).AX.Q);
```

$$Q := \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{-a5\,\beta+\varepsilon}{\lambda\, a5} & 1 & 0 & \frac{\varepsilon}{\lambda\, a5} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$AX2 := \begin{bmatrix} \lambda & 0 & 0 & \lambda \\ 0 & \lambda & \lambda\, d3 & 0 \\ \lambda\, a4 & 0 & \lambda\, d4 & 0 \\ \lambda\, a5 & 0 & 0 & 0 \end{bmatrix} \quad \textbf{(1.2.3.2.8)}$$

We now consider the eigenvalues of this matrix:

```
> LinearAlgebra:-Eigenvalues(AX2);
```

$$\begin{bmatrix} \lambda \\ \lambda\, d4 \\ \left(\frac{1}{2}+\frac{\sqrt{1+4\,a5}}{2}\right)\lambda \\ \left(\frac{1}{2}-\frac{\sqrt{1+4\,a5}}{2}\right)\lambda \end{bmatrix} \quad \textbf{(1.2.3.2.9)}$$

```
> simplify(eval(LinearAlgebra:-Eigenvalues(AX2),a5=d4^2-d4)) assuming
  d4>1/2;
  simplify(eval(LinearAlgebra:-Eigenvalues(AX2),a5=d4^2-d4)) assuming
  d4<1/2;
```

$$\begin{bmatrix} \lambda \\ \lambda\, d4 \\ \lambda\, d4 \\ -(-1+d4)\,\lambda \end{bmatrix}$$

$$\begin{bmatrix} \lambda \\ \lambda\, d4 \\ -(-1+d4)\,\lambda \\ \lambda\, d4 \end{bmatrix} \quad \textbf{(1.2.3.2.10)}$$

There are only nonreal eigenvalues when a5 < -1/4.

If the eigenvalues all distinct, then a5 is not -1/4, d4 is not one, and a5 is not d4^2-d4.

There are two distinct eigenvalues and two repeated when one of the following holds:
a) d4=1 and a5 is not -1/4
b) d4 is neither 1 nor 1/2 and a5 = -1/4
c) a5=d4^2-d4 is not -1/4 (i.e., d4 is not 1/2). Also, d4=1 is forbidden since it implies a5=0.

In case a), the repeated eigenvalue is λ, with eigenvector given by considering the kernel of the following:

```
> Id:=LinearAlgebra:-IdentityMatrix(4):
  eval(AX2-lambda*Id,d4=1);
```

$$\begin{bmatrix} 0 & 0 & 0 & \lambda \\ 0 & 0 & \lambda\, d3 & 0 \\ \lambda\, a4 & 0 & 0 & 0 \\ \lambda\, a5 & 0 & 0 & -\lambda \end{bmatrix} \qquad (1.2.3.2.11)$$

Thus w3 is an eigenvector, and w4 is as well when d3=0.

In case b), the repeated eigenvalue is $\lambda/2$, with eigenvector given by considering the kernel of the following:

```
> eval(AX2-lambda/2*Id,a5=-1/4);
```

$$\begin{bmatrix} \frac{\lambda}{2} & 0 & 0 & \lambda \\ 0 & \frac{\lambda}{2} & \lambda\, d3 & 0 \\ \lambda\, a4 & 0 & \lambda\, d4 - \frac{1}{2}\lambda & 0 \\ -\frac{\lambda}{4} & 0 & 0 & -\frac{\lambda}{2} \end{bmatrix} \qquad (1.2.3.2.12)$$

The rank is always three, and the eigenvector is:

```
> LinearAlgebra:-NullSpace(eval(AX2-lambda/2*Id,a5=-1/4));
```

$$\left\{ \begin{bmatrix} -2 \\ -\frac{8\, d3\, a4}{2\, d4 - 1} \\ \frac{4\, a4}{2\, d4 - 1} \\ 1 \end{bmatrix} \right\} \qquad (1.2.3.2.13)$$

Note that this eigenvector is never in the derived algebra of the nilradical, distinguishing case b) from case a).

In case c), the repeated eigenvalue is $\lambda d4$, with eigenvector given by considering the kernel of the following:

```
> eval(AX2-lambda*d4*Id,a5=d4^2-d4);
```

(1.2.3.2.14)

$$\begin{bmatrix} -\lambda\, d4+\lambda & 0 & 0 & \lambda \\ 0 & -\lambda\, d4+\lambda & \lambda\, d3 & 0 \\ \lambda\, a4 & 0 & 0 & 0 \\ \lambda\,(d4^2-d4) & 0 & 0 & -\lambda\, d4 \end{bmatrix} \quad (1.2.3.2.14)$$

Here, since d4 is not one, d3/(d4-1)w3+w4 is always an eigenvector. When a4=0, w2+(d4-1)w5 is also an eigenvalue. Never is the derived algebra of the nilradical contained in the eigenspace. When a4 is nonzero, this case may be distinguished from case b) by whether or not the eigenspace is in the center of the nilradical.

There are three repeated eigenvalues when a5 = -1/4 and d4 = 1/2.

Since a5 is nonzero, there are two sets of repeated eigevalues only when a5=-1/4 and d4=1. We examine the correpsonding eigenvectors, given by the kernel of the following matrices:

```
> eval(AX2-lambda*Id,{a5=-1/4,d4=1});
  eval(AX2-lambda/2*Id,{a5=-1/4,d4=1});
```

$$\begin{bmatrix} 0 & 0 & 0 & \lambda \\ 0 & 0 & \lambda\, d3 & 0 \\ \lambda\, a4 & 0 & 0 & 0 \\ -\frac{\lambda}{4} & 0 & 0 & -\lambda \end{bmatrix}$$

$$\begin{bmatrix} \frac{\lambda}{2} & 0 & 0 & \lambda \\ 0 & \frac{\lambda}{2} & \lambda\, d3 & 0 \\ \lambda\, a4 & 0 & \frac{\lambda}{2} & 0 \\ -\frac{\lambda}{4} & 0 & 0 & -\frac{\lambda}{2} \end{bmatrix} \quad (1.2.3.2.15)$$

As before, we may distinguish d3 = 0 from d3 nonzero by whether or not the eigenspace contains the center of the nilradical (spanned by w4). The derived algebra (spanned by w3) is always in the eigenspace.

It is not possible for all eigenvalues to be the same.

We will first consider the cases in which all eigenvalues are distinct, so that d4 is not one and a5 is not d4^2-d4. We perform the following change of basis in those cases and initialize the algebra. We may also perform this change of basis in the case b) above, wherein d4 is neither 1 nor 1/2 and a5 = -1/4.

```
> LD_Na:=LieAlgebraData([
  y1+a4*d3/(d4-1)*y5,
  y2+(a4*d3/(-d4^2+a5+d4)*y3+a4*(d4-1)/(-d4^2+a5+d4)*y4-y5),
  y3,
  -d3/((d4-1)*(-d4^2+a5+d4))*y3-1/(-d4^2+a5+d4)*y4,
  a4*d3/((d4-1)*(-d4^2+a5+d4))*y3+a4/(-d4^2+a5+d4)*y4+y5
  ],alg_Na);
```

$LD\_Na := [e1, e2] = a5\, e5, [e1, e3] = e3, [e1, e4] = d4\, e4, [e1, e5] = e2 + e5, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e3, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0$ **(1.2.3.2.16)**

```
> DGsetup(LD_Na,[w],[omega]);
```

Lie algebra: alg_Na **(1.2.3.2.17)**

The isotropy in this basis is given by a4*w4+w5, but scaling w4 is an automorphism, so we take the isotropy to be either w4+w5 or w5:

```
> LieAlgebraData([w1,w2,w3,w4*R,w5]);
```

$[e1, e2] = a5\, e5, [e1, e3] = e3, [e1, e4] = d4\, e4, [e1, e5] = e2 + e5, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e3, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0$ **(1.2.3.2.18)**

If there are nonreal eigenvalues and no repeated eigenvalues, then d4 is not one and a5 < -1/4. The following change of basis gives s_5,25 with β not equal to 2α and isotropy spanned by e2+e4 or e2, depending on the value of a4.

```
> eval(eval(LieAlgebraData([
  -2*alpha*w3,
  w5,
  -2*alpha*w2-alpha*w5,
  w4,
  2*alpha*w1
  ]),{a5=-(alpha^2+1)/(4*alpha^2)}),
  {-2*alpha*d4=-beta});
```

$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -2\,\alpha\, e1, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = -\alpha\, e2 + e3, [e3, e4] = 0, [e3, e5] = -e2 - \alpha\, e3, [e4, e5] = -\beta\, e4$ **(1.2.3.2.19)**

If there are four distinct real eigenvalues, then d4 is not one and a5 > -1/4. The following change of basis gives s_5,22 with b not equal to 1 or a+1 and isotropy spanned by e2+e3+e4 or e2+e3, depending on the value of a4.

```
> simplify(eval(simplify(eval(LieAlgebraData([
  -4*alpha*w3,
  alpha*(2*w2+w5)+w5,
  -alpha*(2*w2+w5)+w5,
  2*w4,
  2*alpha/(1+alpha)*w1
  ]),{a5=-(alpha^2-1)/(4*alpha^2)})),
  {(-1+alpha)/(1+alpha)=a,
  -2*alpha*d4/(1+alpha)=-b,
  -2*alpha/(1+alpha)=-(a+1)}));
```

$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -(1 + a)\, e1, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = -e2, [e3, e4] = 0, [e3, e5] = -a\, e3, [e4, e5] = -b\, e4$ **(1.2.3.2.20)**

In the last case for this basis, case b) from above, (i.e., a5=-1/4, but d4 is neither 1 nor 1/2) the following change of basis gives s_5,24 with a not equal to one or two and isotropy spanned by e2+e4 or e2, depending on a4.

```
> eval(LieAlgebraData([
  1/2*w3,
  1/2*w5,
  w2+1/2*w5,
  1/2*w4,
  2*w1
  ]),{a5=-1/4,-2*d4=-a});
```

(1.2.3.2.21)

$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -2\, e1, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = -e2 - e3, [e3, e4] = 0, [e3, e5] = -e3, [e4, e5] = -a\, e4$ **(1.2.3.2.21)**

We return now to the previous basis (with isotropy y5):

```
> LieAlgebraData([y1,y2,y3,y4,y5]);
```

$[e1, e2] = e2 + a4\, e4 + a5\, e5, [e1, e3] = e3, [e1, e4] = d3\, e3 + d4\, e4, [e1, e5] = e2, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e3, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0$ **(1.2.3.2.22)**

We now consider case c) from above.

When a4 is nonzero, we apply the following change of basis to eliminate parameters. We initialize the algebra:

```
> LD_Nb:=eval(LieAlgebraData([
  y1+d3*a4/(d4-1)*y5,
  y2,
  y3,
  d3*a4/(d4-1)*y3+a4*y4,
  y5
  ],alg_Nb),{a5=d4^2-d4});
```

$LD\_Nb := [e1, e2] = e2 + e4 + (d4^2 - d4)\, e5, [e1, e3] = e3, [e1, e4] = d4\, e4, [e1, e5] = e2, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e3, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0$ **(1.2.3.2.23)**

```
> DGsetup(LD_Nb,[w],[omega]);
```

Lie algebra: alg_Nb **(1.2.3.2.24)**

The following change of basis then gives s_5,23 with isotropy spanned by e2+e3.

```
> simplify(eval(simplify(eval(LieAlgebraData([
  T^3*w3,
  -T*(w2-w5)+w4+R*w5,
  P*(w2-w5)-w4+S*w5,
  T*w4/d4,
  w1/d4
  ]),
  {R=(2*d4^2-3*d4+1),S=(2*d4^2-d4),T=2*d4-1,P=2*d4-1})),
  {d4=1/(a+1)}));
```

$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -(1 + a)\, e1, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = -a\, e2, [e3, e4] = 0, [e3, e5] = -e3 - e4, [e4, e5] = -e4$ **(1.2.3.2.25)**

When a4=0 in case c) from above, we instead eliminate parameters via the following change of basis.
We initialize this algebra.

```
> LD_Nc:=eval(LieAlgebraData([
  y1,
  y2,
  y3,
  d3/(d4-1)*y3+y4,
  y5
  ],alg_Nc),{a5=d4^2-d4,a4=0});
```

$LD\_Nc := [e1, e2] = e2 + (d4^2 - d4)\, e5, [e1, e3] = e3, [e1, e4] = d4\, e4, [e1, e5] = e2, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e3, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0$ **(1.2.3.2.26)**

```
> DGsetup(LD_Nc,[w],[omega]);
```

Lie algebra: alg_Nc **(1.2.3.2.27)**

The following change of basis then gives s_5,22 with b=1
and isotropy spanned by e2+e3.

```
> simplify(eval(simplify(eval(LieAlgebraData([
  T^3*w3,
  -T*(w2-w5)+R*w5,
  P*(w2-w5)+S*w5,
  T*w4/d4,
  w1/d4
  ]),
  {R=(2*d4^2-3*d4+1),S=(2*d4^2-d4),T=2*d4-1,P=2*d4-1})),
  {d4=1/(a+1)}));
```

$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -(1+a)\ e1, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = -a\ e2, [e3, e4] = 0, [e3, e5] = -e3, [e4, e5] = -e4$ **(1.2.3.2.28)**

In the case of three repeated eigenvalues, a5 = -1/4 and d4 = 1/2.
When a4 is nonzero, the following change of basis gives s_5,21
with isotropy spanned by e2-e3+e4.

```
> chi4:=evalDG(-4*a4*d3*y3+2*a4*y4):
  eval(LieAlgebraData([
  1/2*y3,
  y2,
  y2+chi4-1/2*y5,
  chi4,
  2*y1-4*a4*d3*y5
  ]),{a5=-1/4,d4=1/2});
```

$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -2\ e1, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = -e2 - e3, [e3, e4] = 0, [e3, e5] = -e3 - e4, [e4, e5] = -e4$ **(1.2.3.2.29)**

In the case when a4=0, we instead find, via the following
change of basis, the algebra s_5,24 with a=1 and isotropy
spanned by e2-e3.

```
> chi4:=evalDG(-4*d3*y3+2*y4):
  eval(LieAlgebraData([
  1/2*y3,
  y2,
  y2-1/2*y5,
  chi4,
  2*y1
  ]),{a4=0,a5=-1/4,d4=1/2});
```

$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -2\ e1, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = -e2 - e3, [e3, e4] = 0, [e3, e5] = -e3, [e4, e5] = -e4$ **(1.2.3.2.30)**

Consider the cases in which there are repeated eigenvalues,
the derived algebra of the nilradical is in the eigenspace, and
the center of the nilradical is not in the eigenspace (d4=1, d3
nonzero). The following change of basis eliminates parameters.

```
> LD_Nd:=eval(LieAlgebraData([
  y1+a4*d3/a5*(y2-y5),
  (y2-y5),
  y3,
  1/d3*y4,
  (a4/a5*y4+y5)
  ],alg_Nd),{d4=1});
```

$LD\_Nd := [e1, e2] = a5\ e5, [e1, e3] = e3, [e1, e4] = e3 + e4, [e1, e5] = e2$ **(1.2.3.2.31)**

$+ e5, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e3, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0$

```
> DGsetup(LD_Nd,[w],[omega]);
```

Lie algebra: alg_Nd **(1.2.3.2.32)**

When a4=0, the isotropy is spanned by w5. Otherwise, the isotropy is spanned by -a4*d3/a5*w4+w5, but the following automorphism allows us to take the isotropy to be spanned by -w4+w5:

```
> LieAlgebraData([
  w1,
  w2*(a4*d3/a5),
  w3*(a4*d3/a5)^2,
  w4*(a4*d3/a5)^2,
  w5*a4*d3/a5
  ]);
```

$[e1, e2] = a5\, e5, [e1, e3] = e3, [e1, e4] = e3 + e4, [e1, e5] = e2 + e5, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e3, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0$ **(1.2.3.2.33)**

Recall that there are two real distinct eigenvalues of AX2 only if a5 > -1/4, in which case, the following change of basis gives s_5,26 with isotropy spanned by e2+e3 or e2+e3+e4, depending on whether a4 is zero or nonzero.

```
> simplify(eval(LieAlgebraData([
  (a^2-1)*w3,
  (a+1)*w2+w5,
  -(a+1)*w2-a*w5,
  (a-1)*w4,
  (a+1)*w1
  ]),{a5=-a/(a+1)^2}));
```

$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -(1 + a)\, e1, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = -e2, [e3, e4] = 0, [e3, e5] = -a\, e3, [e4, e5] = -e1 - (1 + a)\, e4$ **(1.2.3.2.34)**

If there are nonreal eigenvalues, then a5 < -1/4, and the change of basis gives s_5,28 with isotropy spanned by e3-e4 or e3, depending on the value of a4.

```
> eval(LieAlgebraData([
  2*alpha*w3,
  -2*alpha*w2-alpha*w5,
  w5,
  w4,
  2*alpha*w1
  ]),{a5=-(alpha^2+1)/(4*alpha^2)});
```

$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -2\,\alpha\, e1, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = -\alpha\, e2 - e3, [e3, e4] = 0, [e3, e5] = e2 - \alpha\, e3, [e4, e5] = -e1 - 2\,\alpha\, e4$ **(1.2.3.2.35)**

If there are two sets of repeated eigenvalues, then a5 = -1/4 and the following change of basis gives s_5,27 with isotropy spanned by e2+e3+e4 or e2+e3, depending on a4.

```
> eval(LieAlgebraData([
  2*w3,
  2*w2,
  -2*w2-w5,
  w4,
  2*w1
```

```
]),{a5=-1/4});
```

$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -2\, e1, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = -e2 - e3, [e3, e4] = 0, [e3, e5] = -e3, [e4, e5] = -e1 - 2\, e4$ **(1.2.3.2.36)**

Consider the cases in which there are repeated eigenvalues, the derived algebra of the nilradical is in the eigenspace, and the center of the nilradical is also in the eigenspace (d4=1, d3=0) The following change of basis eliminates parameters.

```
> LD_Ne:=eval(LieAlgebraData([
  y1,
  (y2-y5),
  y3,
  y4,
  (a4/a5*y4+y5)
  ],alg_Ne),{d4=1,d3=0});
```

$LD\_Ne := [e1, e2] = a5\, e5, [e1, e3] = e3, [e1, e4] = e4, [e1, e5] = e2 + e5, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e3, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0$ **(1.2.3.2.37)**

```
> DGsetup(LD_Ne,[w],[omega]);
```

Lie algebra: alg_Ne **(1.2.3.2.38)**

When a4=0, the isotropy is spanned by w5. Otherwise, the isotropy is spanned by -a4/a5*w4+w5, but the following automorphism allows us to take the isotropy to be spanned by -w4+w5:

```
> LieAlgebraData([
  w1,
  w2*(a4/a5),
  w3*(a4/a5)^2,
  w4*(a4/a5)^2,
  w5*a4/a5
  ]);
```

$[e1, e2] = a5\, e5, [e1, e3] = e3, [e1, e4] = e4, [e1, e5] = e2 + e5, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e3, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0$ **(1.2.3.2.39)**

Recall that there are two real distinct eigenvalues of AX2 only if a5 > -1/4, in which case, the following change of basis gives s_5,22 with b=a+1 and isotropy spanned by e2+e3+e4 or e2+e3, depending on whether a4 is zero or nonzero.

```
> simplify(eval(LieAlgebraData([
  (a^2-1)*w3,
  (a+1)*w2+w5,
  -(a+1)*w2-a*w5,
  (a-1)*w4,
  (a+1)*w1
  ]),{a5=-a/(a+1)^2}));
```

$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -(1 + a)\, e1, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = -e2, [e3, e4] = 0, [e3, e5] = -a\, e3, [e4, e5] = -(1 + a)\, e4$ **(1.2.3.2.40)**

If there are nonreal eigenvalues, then a5 < -1/4, and the change of basis gives s_5,25 with β=2α and isotropy spanned by e3−e4 or e3, depending on the value of a4.

```
> eval(LieAlgebraData([
  2*alpha*w3,
```

```
-2*alpha*w2-alpha*w5,
w5,
w4,
2*alpha*w1
]),{a5=-(alpha^2+1)/(4*alpha^2)});
```

$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -2\,\alpha\, e1, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = -\alpha\, e2 - e3, [e3, e4] = 0, [e3, e5] = e2 - \alpha\, e3, [e4, e5] = -2\,\alpha\, e4$ **(1.2.3.2.41)**

If there are two sets of repeated eigenvalues, then a5 = -1/4 and the following change of basis gives s_5,24 with a=2 and isotropy spanned by e2+e3+e4 or e2+e3, depending on a4.

```
> eval(LieAlgebraData([
2*w3,
2*w2,
-2*w2-w5,
w4,
2*w1
]),{a5=-1/4});
```

$[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -2\, e1, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = -e2 - e3, [e3, e4] = 0, [e3, e5] = -e3, [e4, e5] = -2\, e4$ **(1.2.3.2.42)**

B.6. Maple Worksheet for Matching Algebras with Petrov's Classification

```
> restart;
                "USU001-1114  1:14:04.55 Tue 10/27/2015"
                "Init file loaded with the following packages:"
                     "DifferentialGeometry, LieAlgebras"                    (1)
> with(DifferentialGeometry):with(LieAlgebras):with(Tensor):with(GroupActions):
> DGTable := table();
                              DGTable := table( [ ])                        (2)
> read "./5on4_5on3_Database.mm";
```

```
# Calculate isometry dimension from preloaded table of structure equations with
# isotropy.

BuildInvarianMetric := proc(indx)
local C, LD, Iso, H, M, A, S, InvG, k, g, p, V,n,CB,B0,eq;
     # Load and initialize algebra and isotropy H
     C := DGTable[indx]["StructureConstants"]:
     LD := _DG([["LieAlgebra", alg, [5, table()]], C ]):
     DGsetup(LD);
     Iso := DGTable[indx]["Isotropy"];
     V := DGinformation("FrameBaseVectors");
     H := DGzip(Iso, V, "plus");
     n := nops(H);

     # Build a generic complementary basis CB for H and determine conditions
     # needed to ensure CB is a reductive complement to the reductive isotropy H.
     CB := ComplementaryBasis(H, V, t);
     B0 := Query(H, CB, "ReductivePair")[4][1][2];

     # For most cases, setting all free parameters to zero is a suitable choice.
     # For two cases, however, this leads to difficult structure equations, so
     # we alter the parameter choice "by hand" for these.
     eq := {seq(c = 0, c = CB[2])};
     if indx = [5,F14,47] or indx = [5,F14,53] then
          eq := {t3=-1,t4=0};
     fi;
     # Now substitute the parameter values
     M := DGsimplify(subs(eq, B0));

     # Build the most general metric g on M that is invariant under H
     DGEnvironment[GSpace](M, H, P);
     S := GenerateSymmetricTensors(evalDG([seq(Theta||i, i = 1 .. 5-n)]), 2);
     InvG := GroupActions:-InvariantGeometricObjectFields([E5], S, output = "list");
     k := nops(InvG);
     g := DGzip([seq(a||i, i = 1..k)], InvG, "plus"):
end:

# Output the given index and either the isometry dimension if
# it is not five, or 'OK' if it is five.
IsometryDimension := proc(indx)
local g, LD, dim;
     g := BuildInvarianMetric(indx);
     LD := IsometryAlgebraData(g, []);
     dim :=  op(LD)[1][3][1]:
     if dim <> 5 then
          print(indx, dim)
     else
          print(indx, OK)
     fi;
```

```
> Indx := sort(map(op, [indices(DGTable)]));
```

Indx := [[5, *F11*, 0], [5, *F12*, 0], [5, *F12*, 1], [5, *F12*, 2], [5, *F12*, 3], [5, *F12*, 4], [5, *F12*, 5], [5, *F12*, 6], [5, *F12*, 7], [5, *F12*, 8], [5, *F12*, 9], [5, *F12*, 10], [5, *F12*, 11], [5, *F13*, 0], [5, *F13*, 1], [5, *F13*, 2], [5, *F13*, 3], [5, *F13*, 4], [5, *F13*, 5], [5, *F13*, 6], [5, *F13*, 7], [5, *F13*, 8], [5, *F14*, 0], [5, *F14*, 1], [5, *F14*, 2], [5, *F14*, 3], [5, *F14*, 4], [5, *F14*, 5], [5, *F14*, 6], [5, *F14*, 7], [5, *F14*, 8], [5, *F14*, 9], [5, *F14*, 10], [5, *F14*, 11], [5, *F14*, 12], [5, *F14*, 13], [5, *F14*, 14], [5, *F14*, 15], [5, *F14*, 16], [5, *F14*, 17], [5, *F14*, 18], [5, *F14*, 19], [5, *F14*, 20], [5, *F14*, 21], [5, *F14*, 22], [5, *F14*, 23], [5, *F14*, 24], [5, *F14*, 25], [5, *F14*, 26], [5, *F14*, 27], [5, *F14*, 28], [5, *F14*, 29], [5, *F14*, 30], [5, *F14*, 31], [5, *F14*, 32], [5, *F14*, 33], [5, *F14*, 34], [5, *F14*, 35], [5, *F14*, 36], [5, *F14*, 37], [5, *F14*, 38], [5, *F14*, 39], [5, *F14*, 40], [5, *F14*, 41], [5, *F14*, 42], [5, *F14*, 43], [5, *F14*, 44], [5, *F14*, 45], [5, *F14*, 46], [5, *F14*, 47], [5, *F14*, 48], [5, *F14*, 49], [5, *F14*, 50], [5, *F14*, 51], [5, *F14*, 52], [5, *F14*, 53], [5, *F14*, 54], [5, *F14*, 55], [5, *F14*, 56], [5, *F14*, 57], [5, *F14*, 58], [5, *F8*, 0], [5, *F8*, 1]] **(3)**

```
> for ind in Indx[1..10] do IsometryDimension(ind); od;
```

[5, *F11*, 0], 10
[5, *F12*, 0], 6
[5, *F12*, 1], 6
[5, *F12*, 2], 6
[5, *F12*, 3], 6
[5, *F12*, 4], *OK*
[5, *F12*, 5], 6
[5, *F12*, 6], *OK*
[5, *F12*, 7], 10
[5, *F12*, 8], *OK* **(4)**

```
> for ind in Indx[11..20] do IsometryDimension(ind); od;
```

[5, *F12*, 9], *OK*
[5, *F12*, 10], 7
[5, *F12*, 11], *OK*
[5, *F13*, 0], 6
[5, *F13*, 1], 6
[5, *F13*, 2], 10
[5, *F13*, 3], *OK*
[5, *F13*, 4], 6
[5, *F13*, 5], *OK*
[5, *F13*, 6], *OK* **(5)**

```
> for ind in Indx[21..30] do IsometryDimension(ind); od;
```

[5, *F13*, 7], 7
[5, *F13*, 8], *OK*
[5, *F14*, 0], 10
[5, *F14*, 1], *OK*
[5, *F14*, 2], *OK*
[5, *F14*, 3], 10
[5, *F14*, 4], 10
[5, *F14*, 5], 10
[5, *F14*, 6], 6
[5, *F14*, 7], 6 **(6)**

```
> for ind in Indx[31..40] do IsometryDimension(ind); od;
```

$[5, F14, 8], 6$
$[5, F14, 9], 6$
$[5, F14, 10], 6$
$[5, F14, 11], 6$
$[5, F14, 12], 6$
$[5, F14, 13], 6$
$[5, F14, 14], 10$
$[5, F14, 15], 6$
$[5, F14, 16], 10$
$[5, F14, 17], 6$ **(7)**

```
> for ind in Indx[41..50] do IsometryDimension(ind); od;
```

$[5, F14, 18], 6$
$[5, F14, 19], 6$
$[5, F14, 20], 6$
$[5, F14, 21], 6$
$[5, F14, 22], 6$
$[5, F14, 23], 6$
$[5, F14, 24], 6$
$[5, F14, 25], 6$
$[5, F14, 26], 6$
$[5, F14, 27], 6$ **(8)**

```
> for ind in Indx[51..60] do IsometryDimension(ind); od;
```

$[5, F14, 28], 6$
$[5, F14, 29], 6$
$[5, F14, 30], 6$
$[5, F14, 31], 6$
$[5, F14, 32], 6$
$[5, F14, 33], 6$
$[5, F14, 34], 6$
$[5, F14, 35], 7$
$[5, F14, 36], 7$
$[5, F14, 37], 6$ **(9)**

```
> for ind in Indx[61..70] do IsometryDimension(ind); od;
```

$[5, F14, 38], 6$
$[5, F14, 39], 6$
$[5, F14, 40], 6$
$[5, F14, 41], 6$
$[5, F14, 42], 6$
$[5, F14, 43], 6$
$[5, F14, 44], 7$
$[5, F14, 45], 6$
$[5, F14, 46], 7$
$[5, F14, 47], 6$ **(10)**

```
> for ind in Indx[71..81] do IsometryDimension(ind); od;
```

$[5, F14, 48], 6$

$$[5, F14, 49], 6$$
$$[5, F14, 50], 6$$
$$[5, F14, 51], 6$$
$$[5, F14, 52], 6$$
$$[5, F14, 53], 6$$
$$[5, F14, 54], 6$$
$$[5, F14, 55], 6$$
$$[5, F14, 56], 6$$
$$[5, F14, 57], 6$$
$$[5, F14, 58], 6 \quad \textbf{(11)}$$

These are the cases where the isometry is five-dimensional, i.e., we obtain no additional symmetries:

```
> Indx5D:=[[5,F12,4],[5,F12,6],[5,F12,8],[5,F12,9],[5,F12,11],[5,F13,3],[5,F13,5],[5,F13,6],[5,F13,8],[5,F14,1],[5,F14,2]];
```

$$Indx5D := [[5, F12, 4], [5, F12, 6], [5, F12, 8], [5, F12, 9], [5, F12, 11], [5, F13, 3], [5, F13, 5], [5, F13, 6], [5, F13, 8], [5, F14, 1], [5, F14, 2]] \quad \textbf{(12)}$$

In the following sections, we match the above with entries in Petrov's classification.
The general procedure is to . . .

1. Initialize the algebra and isotropy H.
2. Initialize the Petrov Killing vectors KV on a manifold P.
3. Calculate the isotropy IsoK at a point.
4. Find a change of basis that aligns the two algebras.
5. Check that the change of basis also aligns the isotropies (up to scaling).

(F12, 4) = (33.17) with e = -1

```
> C := DGTable[Indx5D[1]]["StructureConstants"]:
> LD := _DG([["LieAlgebra", alg, [5, table()]], C ]);
```

$$LD := [e1, e2] = 2\, e1, [e1, e3] = -2\, e2, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = 2\, e3, [e2, e4] = 0, [e2, e5] = 0, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0 \quad \textbf{(1.1)}$$

```
> DGsetup(LD):
> Iso := DGTable[Indx5D[1]]["Isotropy"]:
> V := DGinformation("FrameBaseVectors"):
> H := DGzip(Iso, V, "plus");
```

$$H := [e1 - e3 - 2\, e4] \quad \textbf{(1.2)}$$

```
> DGsetup([x1,x2,x3,x4],P);
```

Manifold: P **(1.3)**

```
> KV:=DGzip([[0, 1, 0, 0], [0, x2, 1, 0], [-exp(x3), -exp(2*x3)+x2^2, 2*x2, 0], [1, 0, 0, 0], [0, 0, 0, 1]],DGinformation("FrameBaseVectors"));
```

$$KV := \left[\partial_{x2}, x2\, \partial_{x2} + \partial_{x3}, -e^{x3}\, \partial_{x1} + \left(-e^{2\, x3} + x2^2\right) \partial_{x2} + 2\, x2\, \partial_{x3}, \partial_{x1}, \partial_{x4}\right] \quad \textbf{(1.4)}$$

```
> LDK:=LieAlgebraData(KV,algK);
```

$$LDK := [e1, e2] = e1, [e1, e3] = 2\, e2, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = e3, [e2, e4] = 0, [e2, e5] = 0, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0 \quad \textbf{(1.5)}$$

```
> DGsetup(LDK,[X],[O]);
```

Lie algebra: algK **(1.6)**

```
> IsoK:=IsotropySubalgebra(KV,[x1=0,x2=0,x3=0,x4=0],output=[algK]);
```

$$IsoK := [X1 + X3 + X4] \quad \textbf{(1.7)}$$

```
> Mat:=Matrix([[-sqrt(2), 0, 0, 0, 0], [0, 2, 0, 0, 0], [0, 0, sqrt(2), 0, 0], [0, 0, 0, 1/sqrt(2), 0], [0, 0, 0, 0, -1]]):
> COB:=evalDG(convert(Mat.Matrix([[X1],[X2],[X3],[X4],[X5]]),list));
```

$$COB := \left[-\sqrt{2}\,X1,\ 2\,X2,\ \sqrt{2}\,X3,\ \frac{\sqrt{2}}{2}\,X4,\ -X5 \right] \quad (1.8)$$

```
> DGequal(LieAlgebraData(COB),LD);
```

true **(1.9)**

```
> List:=op(GetComponents(H,DGinformation(alg,"FrameBaseVectors")));
```

$$List := [1, 0, -1, -2, 0] \quad (1.10)$$

```
> evalDG(add(List[i]*COB[i],i=1..5)*(-1/sqrt(2))-op(IsoK));
```

$$0\,X1 \quad (1.11)$$

(F12, 6) = (33.19)

```
> C := DGTable[Indx5D[2]]["StructureConstants"]:
> LD := _DG([["LieAlgebra", alg, [5, table()]], C ]);
```

$$LD := [e1, e2] = e3, [e1, e3] = -e2, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = 0, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0 \quad (2.1)$$

```
> DGsetup(LD):
> Iso := DGTable[Indx5D[2]]["Isotropy"]:
> V := DGinformation("FrameBaseVectors"):
> H := DGzip(Iso, V, "plus");
```

$$H := [e1 - e4] \quad (2.2)$$

```
> DGsetup([x1,x2,x3,x4],P);
```

Manifold: P **(2.3)**

```
> KV:=DGzip([[0, 1, 0, 0], [cos(x2), -1/sin(x1)*cos(x1)*sin(x2), 1/sin(x1)*sin(x2), 0], [-sin(x2), -1/sin(x1)*cos(x1)*cos(x2), 1/sin(x1)*cos(x2), 0], [0, 0, 1, 0], [0, 0, 0, 1]],DGinformation("FrameBaseVectors"));
```

$$KV := \left[\partial_{x2},\ \cos(x2)\,\partial_{x1} - \frac{\cos(x1)\,\sin(x2)}{\sin(x1)}\,\partial_{x2} + \frac{\sin(x2)}{\sin(x1)}\,\partial_{x3},\ -\sin(x2)\,\partial_{x1} - \frac{\cos(x1)\,\cos(x2)}{\sin(x1)}\,\partial_{x2} + \frac{\cos(x2)}{\sin(x1)}\,\partial_{x3},\ \partial_{x3},\ \partial_{x4} \right] \quad (2.4)$$

```
> LDK:=LieAlgebraData(KV,algK);
```

$$LDK := [e1, e2] = e3, [e1, e3] = -e2, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = 0, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0 \quad (2.5)$$

```
> DGsetup(LDK,[X],[O]);
```

Lie algebra: algK **(2.6)**

```
> IsoK:=IsotropySubalgebra(KV,[x1=Pi/2,x2=0,x3=0,x4=0],output=[algK]);
```

$$IsoK := [X3 - X4] \quad (2.7)$$

```
> Mat:=Matrix([[0, 0, 1, 0, 0], [-1, 0, 0, 0, 0], [0, -1, 0, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, -1]]):
> COB:=evalDG(convert(Mat.Matrix([[X1],[X2],[X3],[X4],[X5]]),list));
```

$$COB := [X3, -X1, -X2, X4, -X5] \quad (2.8)$$

```
> DGequal(LieAlgebraData(COB),LD);
```

true **(2.9)**

```
> List:=op(GetComponents(H,DGinformation(alg,"FrameBaseVectors")));
```

$$List := [1, 0, 0, -1, 0] \quad (2.10)$$

```
> evalDG(add(List[i]*COB[i],i=1..5)*(1)-op(IsoK));
```

$$0\,X1 \quad (2.11)$$

(F12, 6) = (33.20)

```
> C := DGTable[Indx5D[2]]["StructureConstants"]:
> LD := _DG([["LieAlgebra", alg, [5, table()]], C ]);
```

$$LD := [e1, e2] = e3, [e1, e3] = -e2, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = 0, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0 \quad (3.1)$$

```
> DGsetup(LD):
> Iso := DGTable[Indx5D[2]]["Isotropy"]:
> V := DGinformation("FrameBaseVectors"):
> H := DGzip(Iso, V, "plus");
```

$$H := [e1 - e4] \quad (3.2)$$

```
> DGsetup([x1,x2,x3,x4],P);
```

Manifold: P **(3.3)**

```
> KV:=DGzip([[0, 1, 0, 0], [1/cos(x3)*cos(x2), -1/cos(x3)*sin(x3)*cos(x2), sin(x2), 0], [-1/cos(x3)*sin(x2), 1/cos(x3)*sin(x3)*sin(x2), cos(x2), 0], [1, 0, 0, 0], [0, 0, 0, 1]],DGinformation("FrameBaseVectors"));
```

$$KV := \left[\partial_{x2}, \frac{\cos(x2)}{\cos(x3)}\partial_{x1} - \frac{\sin(x3)\cos(x2)}{\cos(x3)}\partial_{x2} + \sin(x2)\partial_{x3}, -\frac{\sin(x2)}{\cos(x3)}\partial_{x1} + \frac{\sin(x3)\sin(x2)}{\cos(x3)}\partial_{x2} + \cos(x2)\partial_{x3}, \partial_{x1}, \partial_{x4}\right] \quad (3.4)$$

```
> LDK:=LieAlgebraData(KV,algK);
```

$$LDK := [e1, e2] = e3, [e1, e3] = -e2, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = 0, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0 \quad (3.5)$$

```
> DGsetup(LDK,[X],[O]);
```

Lie algebra: algK **(3.6)**

```
> IsoK:=IsotropySubalgebra(KV,[x1=0,x2=0,x3=0,x4=0],output=[algK]);
```

$$IsoK := [X2 - X4] \quad (3.7)$$

```
> Mat:=Matrix([[0, 1, 0, 0, 0], [-1, 0, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]]):
> COB:=evalDG(convert(Mat.Matrix([[X1],[X2],[X3],[X4],[X5]]),list));
```

$$COB := [X2, -X1, X3, X4, X5] \quad (3.8)$$

```
> DGequal(LieAlgebraData(COB),LD);
```

true **(3.9)**

```
> List:=op(GetComponents(H,DGinformation(alg,"FrameBaseVectors")));
```

$$List := [1, 0, 0, -1, 0] \quad (3.10)$$

```
> evalDG(add(List[i]*COB[i],i=1..5)*(1)-op(IsoK));
```

$$0\ X1 \quad (3.11)$$

(F12, 8) = (33.23)

```
> C := DGTable[Indx5D[3]]["StructureConstants"]:
> LD := _DG([["LieAlgebra", alg, [5, table()]], C ]);
```

$$LD := [e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = e1, [e2, e4] = e3, [e2, e5] = 0, [e3, e4] = -e2, [e3, e5] = 0, [e4, e5] = 0 \quad (4.1)$$

```
> DGsetup(LD):
> Iso := DGTable[Indx5D[3]]["Isotropy"]:
> V := DGinformation("FrameBaseVectors"):
```

```
> H := DGzip(Iso, V, "plus");
```
$$H := [e4] \tag{4.2}$$

```
> DGsetup([x1,x2,x3,x4],P);
```
Manifold: P **(4.3)**

```
> KV:=DGzip([[0, 1, 0, 0], [0, 0, 1, 0], [-1, x3, 0, 0], [-x3, 1/2*x3^2-1/2*x1^2, x1, 0], [0, 0, 0, 1]],
  DGinformation("FrameBaseVectors"));
```
$$KV := \left[\partial_{x2}, \partial_{x3}, -\partial_{x1} + x3\,\partial_{x2}, -x3\,\partial_{x1} - \left(-\frac{x3^2}{2} + \frac{x1^2}{2}\right)\partial_{x2} + x1\,\partial_{x3}, \partial_{x4}\right] \tag{4.4}$$

```
> LDK:=LieAlgebraData(KV,algK);
```
$$LDK := [e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = e1, [e2, e4] = e3, [e2, e5] = 0, [e3, e4] = -e2, [e3, e5] = 0, [e4, e5] = 0 \tag{4.5}$$

```
> DGsetup(LDK,[X],[O]);
```
Lie algebra: algK **(4.6)**

```
> IsoK:=IsotropySubalgebra(KV,[x1=0,x2=0,x3=0,x4=0],output=[algK]);
```
$$IsoK := [X4] \tag{4.7}$$

```
> Mat:=Matrix([[-1, 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, -1, 0, 0], [0, 0, 0, -1, 0], [0, 0, 0, 0, 1]]):
> COB:=evalDG(convert(Mat.Matrix([[X1],[X2],[X3],[X4],[X5]]),list));
```
$$COB := [-X1, X2, -X3, -X4, X5] \tag{4.8}$$

```
> DGequal(LieAlgebraData(COB),LD);
```
true **(4.9)**

```
> List:=op(GetComponents(H,DGinformation(alg,"FrameBaseVectors")));
```
$$List := [0, 0, 0, 1, 0] \tag{4.10}$$

```
> evalDG(add(List[i]*COB[i],i=1..5)*(-1)-op(IsoK));
```
$$0\,X1 \tag{4.11}$$

(F12, 9) = (33.22)

```
> C := DGTable[Indx5D[4]]["StructureConstants"]:
> LD := _DG([["LieAlgebra", alg, [5, table()]], C ]);
```
$$LD := [e1, e2] = 0, [e1, e3] = 0, [e1, e4] = -2\,e1, [e1, e5] = 0, [e2, e3] = e1, [e2, e4] = -e2, [e2, e5] = -e3, [e3, e4] = -e3, [e3, e5] = e2, [e4, e5] = 0 \tag{5.1}$$

```
> DGsetup(LD):
> Iso := DGTable[Indx5D[4]]["Isotropy"]:
> V := DGinformation("FrameBaseVectors"):
> H := DGzip(Iso, V, "plus");
```
$$H := [e5] \tag{5.2}$$

```
> DGsetup([x1,x2,x3,x4],P);
```
Manifold: P **(5.3)**

```
> KV:=DGzip([[0, 1, 0, 0], [0, 0, 1, 0], [-1, x3, 0, 0], [-x3, 1/2*x3^2-1/2*x1^2, x1, 0], [x1, 2*x2,
  x3, 1]],DGinformation("FrameBaseVectors"));
```
$$KV := \left[\partial_{x2}, \partial_{x3}, -\partial_{x1} + x3\,\partial_{x2}, -x3\,\partial_{x1} - \left(-\frac{x3^2}{2} + \frac{x1^2}{2}\right)\partial_{x2} + x1\,\partial_{x3}, x1\,\partial_{x1} + 2\,x2\,\partial_{x2} + x3\,\partial_{x3} + \partial_{x4}\right] \tag{5.4}$$

```
> LDK:=LieAlgebraData(KV,algK);
```
$$LDK := [e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = 2\,e1, [e2, e3] = e1, [e2, e4] = e3, [e2, e5] = e2, [e3, e4] = -e2, [e3, e5] = e3, [e4, e5] = 0 \tag{5.5}$$

```
> DGsetup(LDK,[X],[O]);
```
(5.6)

Lie algebra: algK **(5.6)**

```
> IsoK:=IsotropySubalgebra(KV,[x1=0,x2=0,x3=0,x4=0],output=[algK]);
```

$$IsoK := [X4] \quad \textbf{(5.7)}$$

```
> Mat:=Matrix([[1, 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 0, -1], [0, 0, 0, -1, 0]]):
> COB:=evalDG(convert(Mat.Matrix([[X1],[X2],[X3],[X4],[X5]]),list));
```

$$COB := [X1, X2, X3, -X5, -X4] \quad \textbf{(5.8)}$$

```
> DGequal(LieAlgebraData(COB),LD);
```

true **(5.9)**

```
> List:=op(GetComponents(H,DGinformation(alg,"FrameBaseVectors")));
```

$$List := [0, 0, 0, 0, 1] \quad \textbf{(5.10)}$$

```
> evalDG(add(List[i]*COB[i],i=1..5)*(-1)-op(IsoK));
```

$$0\ X1 \quad \textbf{(5.11)}$$

(F12, 11) = (33.31)

```
> C := DGTable[Indx5D[5]]["StructureConstants"]:
> LD := _DG([["LieAlgebra", alg, [5, table()]], C ]);
```

$$LD := [e1, e2] = 0, [e1, e3] = 0, [e1, e4] = \beta\, e1, [e1, e5] = 0, [e2, e3] = 0, [e2, e4] = -e2, [e2, e5] = e3, [e3, e4] = -e3, [e3, e5] = -e2, [e4, e5] = 0 \quad \textbf{(6.1)}$$

```
> DGsetup(LD):
> Iso := DGTable[Indx5D[5]]["Isotropy"]:
> V := DGinformation("FrameBaseVectors"):
> H := DGzip(Iso, V, "plus");
```

$$H := [e5] \quad \textbf{(6.2)}$$

```
> DGsetup([x1,x2,x3,x4],P);
```

Manifold: P **(6.3)**

```
> KV:=DGzip([[0, 1, 0, 0], [0, 0, 1, 0], [-1, 0, 0, 0], [0, -x3, x2, 0], [x1*_l, _k*x2, x3*_k, 1]],
  DGinformation("FrameBaseVectors"));
```

$$KV := \left[\partial_{x2}, \partial_{x3}, -\partial_{x1}, -x3\,\partial_{x2} + x2\,\partial_{x3}, x1\ \_l\,\partial_{x1} + \_k\,x2\,\partial_{x2} + x3\ \_k\,\partial_{x3} + \partial_{x4}\right] \quad \textbf{(6.4)}$$

```
> LDK:=LieAlgebraData(KV,algK);
```

$$LDK := [e1, e2] = 0, [e1, e3] = 0, [e1, e4] = e2, [e1, e5] = \_k\,e1, [e2, e3] = 0, [e2, e4] = -e1, [e2, e5] = \_k\,e2, [e3, e4] = 0, [e3, e5] = \_l\,e3, [e4, e5] = 0 \quad \textbf{(6.5)}$$

```
> DGsetup(LDK,[X],[O]);
```

Lie algebra: algK **(6.6)**

```
> IsoK:=IsotropySubalgebra(KV,[x1=0,x2=0,x3=0,x4=0],output=[algK]);
```

$$IsoK := [X4] \quad \textbf{(6.7)}$$

```
> Mat:=Matrix([[0, 0, 1, 0, 0], [0, -1, 0, 0, 0], [-1, 0, 0, 0, 0], [0, 0, 0, 0, -1/_k], [0, 0, 0, -1, 0]]):
> COB:=evalDG(convert(Mat.Matrix([[X1],[X2],[X3],[X4],[X5]]),list));
```

$$COB := \left[X3, -X2, -X1, -\frac{1}{\_k}\,X5, -X4\right] \quad \textbf{(6.8)}$$

```
> DGequal(eval(LieAlgebraData(COB),-_l/_k=beta),LD);
```

true **(6.9)**

```
> List:=op(GetComponents(H,DGinformation(alg,"FrameBaseVectors")));
```

$$List := [0, 0, 0, 0, 1] \quad \textbf{(6.10)}$$

```
> evalDG(add(List[i]*COB[i],i=1..5)*(-1)-op(IsoK));
```

$$0\ X1 \quad \textbf{(6.11)}$$

(F13, 3) = (33.21) with c = 0

```
> C := DGTable[Indx5D[6]]["StructureConstants"]:
> LD := _DG([["LieAlgebra", alg, [5, table()]], C ]);
```

$LD := [e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = -e1, [e2, e4] = -e2, [e2, e5] = 0, [e3, e4] = e3, [e3, e5] = 0, [e4, e5] = 0$ **(7.1)**

```
> DGsetup(LD):
> Iso := DGTable[Indx5D[6]]["Isotropy"]:
> V := DGinformation("FrameBaseVectors"):
> H := DGzip(Iso, V, "plus");
```

$H := [e4]$ **(7.2)**

```
> DGsetup([x1,x2,x3,x4],P);
```

Manifold: P **(7.3)**

```
> KV:=DGzip([[0, 1, 0, 0], [0, 0, 1, 0], [-1, x3, 0, 0], [-x1, 0, x3, 0], [0, 0, 0, 1]],DGinformation
  ("FrameBaseVectors"));
```

$KV := [\partial_{x2}, \partial_{x3}, -\partial_{x1} + x3\,\partial_{x2}, -x1\,\partial_{x1} + x3\,\partial_{x3}, \partial_{x4}]$ **(7.4)**

```
> LDK:=LieAlgebraData(KV,algK);
```

$LDK := [e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = e1, [e2, e4] = e2, [e2, e5] = 0, [e3, e4] = -e3, [e3, e5] = 0, [e4, e5] = 0$ **(7.5)**

```
> DGsetup(LDK,[X],[O]);
```

Lie algebra: algK **(7.6)**

```
> IsoK:=IsotropySubalgebra(KV,[x1=0,x2=0,x3=0,x4=0],output=[algK]);
```

$IsoK := [X4]$ **(7.7)**

```
> Mat:=Matrix([[-1, 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, -1, 0], [0, 0, 0, 0, 1]]):
> COB:=evalDG(convert(Mat.Matrix([[X1],[X2],[X3],[X4],[X5]]),list));
```

$COB := [-X1, X2, X3, -X4, X5]$ **(7.8)**

```
> DGequal(LieAlgebraData(COB),LD);
```

true **(7.9)**

```
> List:=op(GetComponents(H,DGinformation(alg,"FrameBaseVectors")));
```

$List := [0, 0, 0, 1, 0]$ **(7.10)**

```
> evalDG(add(List[i]*COB[i],i=1..5)*(-1)-op(IsoK));
```

$0\,X1$ **(7.11)**

(F13, 5) = (33.17) with epsilon = 1

```
> C := DGTable[Indx5D[7]]["StructureConstants"]:
> LD := _DG([["LieAlgebra", alg, [5, table()]], C ]);
```

$LD := [e1, e2] = 2\,e1, [e1, e3] = e2, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = 2\,e3, [e2, e4] = 0, [e2, e5] = 0, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0$ **(8.1)**

```
> DGsetup(LD):
> Iso := DGTable[Indx5D[7]]["Isotropy"]:
> V := DGinformation("FrameBaseVectors"):
> H := DGzip(Iso, V, "plus");
```

$H := [e2 - 2\,e4]$ **(8.2)**

```
> DGsetup([x1,x2,x3,x4],P);
```

Manifold: P **(8.3)**

```
> KV:=DGzip([[0, 1, 0, 0], [0, x2, 1, 0], [-exp(x3), exp(2*x3)+x2^2, 2*x2, 0], [1, 0, 0, 0], [0, 0, 0,
  1]],DGinformation("FrameBaseVectors"));
```

$$KV := \left[\partial_{x2}, x2\,\partial_{x2} + \partial_{x3}, -e^{x3}\,\partial_{x1} + \left(e^{2\,x3} + x2^2\right)\partial_{x2} + 2\,x2\,\partial_{x3}, \partial_{x1}, \partial_{x4}\right] \quad (8.4)$$

```
> LDK:=LieAlgebraData(KV,algK);
```

$$LDK := [e1, e2] = e1, [e1, e3] = 2\,e2, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = e3, [e2, e4] = 0, [e2, e5] = 0, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0 \quad (8.5)$$

```
> DGsetup(LDK,[X],[O]);
```

Lie algebra: algK (8.6)

```
> IsoK:=IsotropySubalgebra(KV,[x1=0,x2=0,x3=0,x4=0],output=[algK]);
```

$$IsoK := [X1 - X3 - X4] \quad (8.7)$$

```
> Mat:=Matrix([[1/2, -1, 1/2, 0, 0], [1, 0, -1, 0, 0], [1/2, 1, 1/2, 0, 0], [0, 0, 0, 1/2, 0], [0, 0, 0, 0, 1]]):
> COB:=evalDG(convert(Mat.Matrix([[X1],[X2],[X3],[X4],[X5]]),list));
```

$$COB := \left[\frac{1}{2}\,X1 - X2 + \frac{1}{2}\,X3, X1 - X3, \frac{1}{2}\,X1 + X2 + \frac{1}{2}\,X3, \frac{1}{2}\,X4, X5\right] \quad (8.8)$$

```
> DGequal(LieAlgebraData(COB),LD);
```

true (8.9)

```
> List:=op(GetComponents(H,DGinformation(alg,"FrameBaseVectors")));
```

$$List := [0, 1, 0, -2, 0] \quad (8.10)$$

```
> evalDG(add(List[i]*COB[i],i=1..5)*(1)-op(IsoK));
```

$$0\,X1 \quad (8.11)$$

(F13, 6) = (33.21) with c nonzero

```
> C := DGTable[Indx5D[8]]["StructureConstants"]:
> LD := _DG([["LieAlgebra", alg, [5, table()]], C ]);
```

$$LD := [e1, e2] = 0, [e1, e3] = 0, [e1, e4] = -e1, [e1, e5] = 0, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = e2, [e3, e4] = -e3, [e3, e5] = -e3, [e4, e5] = 0 \quad (9.1)$$

```
> DGsetup(LD):
> Iso := DGTable[Indx5D[8]]["Isotropy"]:
> V := DGinformation("FrameBaseVectors"):
> H := DGzip(Iso, V, "plus");
```

$$H := [e5] \quad (9.2)$$

```
> DGsetup([x1,x2,x3,x4],P);
```

Manifold: P (9.3)

```
> KV:=DGzip([[0, 1, 0, 0], [0, 0, 1, 0], [-1, x3, 0, 0], [-x1, 0, x3, 0], [_c*x1, _c*x2, 0, 1]], DGinformation("FrameBaseVectors"));
```

$$KV := \left[\partial_{x2}, \partial_{x3}, -\partial_{x1} + x3\,\partial_{x2}, -x1\,\partial_{x1} + x3\,\partial_{x3}, \_c\,x1\,\partial_{x1} + \_c\,x2\,\partial_{x2} + \partial_{x4}\right] \quad (9.4)$$

```
> LDK:=LieAlgebraData(KV,algK);
```

$$LDK := [e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = \_c\,e1, [e2, e3] = e1, [e2, e4] = e2, [e2, e5] = 0, [e3, e4] = -e3, [e3, e5] = \_c\,e3, [e4, e5] = 0 \quad (9.5)$$

```
> DGsetup(LDK,[X],[O]);
```

Lie algebra: algK (9.6)

```
> IsoK:=IsotropySubalgebra(KV,[x1=0,x2=0,x3=0,x4=0],output=[algK]);
```

$$IsoK := [X4] \quad (9.7)$$

```
> Mat:=Matrix([[1, 0, 0, 0, 0], [0, 0, -1, 0, 0], [0, 1, 0, 0, 0], [0, 0, 0, -1, -1/_c], [0, 0, 0, -1, 0]]):
> COB:=evalDG(convert(Mat.Matrix([[X1],[X2],[X3],[X4],[X5]]),list));
```

(9.8)

$$COB := \left[X1,\ -X3, X2,\ -X4 - \frac{1}{\_c}\ X5,\ -X4\right] \quad (9.8)$$

```
> DGequal(LieAlgebraData(COB),LD);
```

true **(9.9)**

```
> List:=op(GetComponents(H,DGinformation(alg,"FrameBaseVectors")));
```

$$List := [0, 0, 0, 0, 1] \quad (9.10)$$

```
> evalDG(add(List[i]*COB[i],i=1..5)*(-1)-op(IsoK));
```

$$0\ X1 \quad (9.11)$$

(F13, 8) = (33.28) with kappa = k + epsilon nonzero

```
> C := DGTable[Indx5D[9]]["StructureConstants"]:
> LD := _DG([["LieAlgebra", alg, [5, table()]], C ]);
```

$$LD := [e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -e1, [e2, e3] = 0, [e2, e4] = -e2, [e2, e5] = 0, [e3, e4] = -a\ e3, [e3, e5] = -a\ e3, [e4, e5] = 0 \quad (10.1)$$

```
> DGsetup(LD):
> Iso := DGTable[Indx5D[9]]["Isotropy"]:
> V := DGinformation("FrameBaseVectors"):
> H := DGzip(Iso, V, "plus");
```

$$H := [e4 - e5] \quad (10.2)$$

```
> DGsetup([x1,x2,x3,x4],P);
```

Manifold: P **(10.3)**

```
> # We will recalculate the Killing vectors, since there seems to be
  # an error in Petrov's vector fields.
> g:=convert(Matrix([[_k11*exp(-2*_l*x4), 0, 0, 0], [0, 0, _k23*exp(-(_k+_epsilon)*x4), 0], [0,
  _k23*exp(-(_k+_epsilon)*x4), 0, 0], [0, 0, 0, _k44]]),DGtensor,[["cov_bas","cov_bas"],[]]);
```

$$g := \_k11\, e^{-2\,\_l\,x4}\, dx1 \otimes dx1 + \_k23\, e^{-(\_k+\_\epsilon)\,x4}\, dx2 \otimes dx3 + \_k23\, e^{-(\_k+\_\epsilon)\,x4}\, dx3 \otimes dx2 + \_k44\, dx4 \otimes dx4 \quad (10.4)$$

```
> KVG:=KillingVectors(g);
```

$$KVG := KillingVectors\left(\_k11\, e^{-2\,\_l\,x4}\, dx1 \otimes dx1 + \_k23\, e^{-(\_k+\_\epsilon)\,x4}\, dx2 \otimes dx3 + \_k23\, e^{-(\_k+\_\epsilon)\,x4}\, dx3 \otimes dx2 + \_k44\, dx4 \otimes dx4\right) \quad (10.5)$$

```
> # By scaling, rearranging, and setting _kappa = _k + _epsilon, we obtain
  # the following. We write this manually so that the ordering of vector
  # fields is always the same, regardless of Maple version, etc.
> KV:=DGzip([[0, 1, 0, 0], [0, 0, 1, 0], [-1, 0, 0, 0], [0, x2, -x3, 0], [_l*x1, 0, _kappa*x3, 1]],
  DGinformation("FrameBaseVectors"));
```

$$KV := \left[\partial_{x2}, \partial_{x3}, -\partial_{x1}, x2\,\partial_{x2} - x3\,\partial_{x3}, \_l\,x1\,\partial_{x1} + \_\kappa\,x3\,\partial_{x3} + \partial_{x4}\right] \quad (10.6)$$

```
> LDK:=LieAlgebraData(KV,algK);
```

$$LDK := [e1, e2] = 0, [e1, e3] = 0, [e1, e4] = e1, [e1, e5] = 0, [e2, e3] = 0, [e2, e4] = -e2, [e2, e5] = \_\kappa\, e2, [e3, e4] = 0, [e3, e5] = \_l\, e3, [e4, e5] = 0 \quad (10.7)$$

```
> DGsetup(LDK,[X],[O]);
```

Lie algebra: algK **(10.8)**

```
> IsoK:=IsotropySubalgebra(KV,[x1=0,x2=0,x3=0,x4=0],output=[algK]);
```

$$IsoK := [X4] \quad (10.9)$$

```
> Mat:=Matrix([[1, 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 0, -1/_kappa], [0, 0, 0, -1,
  -1/_kappa]]):
> COB:=evalDG(convert(Mat.Matrix([[X1],[X2],[X3],[X4],[X5]]),list));
```

$$COB := \left[X1, X2, X3, -\frac{1}{\_\kappa} X5, -X4 - \frac{1}{\_\kappa} X5\right] \quad (10.10)$$

```
> DGequal(eval(LieAlgebraData(COB),_l/_kappa=a),LD);
```
true **(10.11)**

```
> List:=op(GetComponents(H,DGinformation(alg,"FrameBaseVectors")));
```
$$List := [0, 0, 0, 1, -1] \quad (10.12)$$

```
> evalDG(add(List[i]*COB[i],i=1..5)*(1)-op(IsoK));
```
$$0\ X1 \quad (10.13)$$

(F14, 1) = (33.14) (includes (33.18) when k < 0)

```
> C := DGTable[Indx5D[10]]["StructureConstants"]:
> LD := _DG([["LieAlgebra", alg, [5, table()]], C ]);
```
$$LD := [e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = e1, [e2, e3] = 0, [e2, e4] = e1, [e2, e5] = e2, [e3, e4] = e2, [e3, e5] = -\epsilon\, e1 + e3, [e4, e5] = 0 \quad (11.1)$$

```
> DGsetup(LD):
> Iso := DGTable[Indx5D[10]]["Isotropy"]:
> V := DGinformation("FrameBaseVectors"):
> H := DGzip(Iso, V, "plus");
```
$$H := [e4] \quad (11.2)$$

```
> DGsetup([x1,x2,x3,x4],P);
```
Manifold: P **(11.3)**

```
> KV:=DGzip([[0, 1, 0, 0], [0, 0, 1, 0], [0, x3, -exp(x1), 0], [1, x2, x3, 0], [exp(-x1), x1*_k-1/2*exp(-2*x4), 0, exp(-x1)]],DGinformation("FrameBaseVectors"));
```
$$KV := \left[\partial_{x2}, \partial_{x3}, x3\,\partial_{x2} - e^{x1}\,\partial_{x3}, \partial_{x1} + x2\,\partial_{x2} + x3\,\partial_{x3}, e^{-x1}\,\partial_{x1} + \left(x1\,\_k - \frac{e^{-2\,x4}}{2}\right)\partial_{x2} + e^{-x1}\,\partial_{x4}\right] \quad (11.4)$$

```
> LDK:=LieAlgebraData(KV,algK);
```
$$LDK := [e1, e2] = 0, [e1, e3] = 0, [e1, e4] = e1, [e1, e5] = 0, [e2, e3] = e1, [e2, e4] = e2, [e2, e5] = 0, [e3, e4] = 0, [e3, e5] = e2, [e4, e5] = \_k\, e1 - e5 \quad (11.5)$$

```
> DGsetup(LDK,[X],[O]);
```
Lie algebra: algK **(11.6)**

```
> IsoK:=IsotropySubalgebra(KV,[x1=0,x2=0,x3=0,x4=0],output=[algK]);
```
$$IsoK := [X2 + X3] \quad (11.7)$$

```
> Mat:=Matrix([[-abs(_k), 0, 0, 0, 0], [0, sqrt(abs(_k)), 0, 0, 0], [0, 0, 0, 0, 1], [0, -sqrt(abs(_k)), -sqrt(abs(_k)), 0, 0], [0, 0, 0, 1, -1]]):
> COB:=evalDG(convert(Mat.Matrix([[X1],[X2],[X3],[X4],[X5]]),list));
```
$$COB := \left[-|\_k|\, X1, \sqrt{|\_k|}\, X2, X5, -\sqrt{|\_k|}\, X2 - \sqrt{|\_k|}\, X3, X4 - X5\right] \quad (11.8)$$

```
> DGequal(eval(LieAlgebraData(COB),_k/abs(_k)=-epsilon),LD);
```
true **(11.9)**

```
> List:=op(GetComponents(H,DGinformation(alg,"FrameBaseVectors")));
```
$$List := [0, 0, 0, 1, 0] \quad (11.10)$$

```
> evalDG(add(List[i]*COB[i],i=1..5)*(-1/sqrt(abs(_k)))-op(IsoK));
```
$$0\ X1 \quad (11.11)$$

(F14, 2) = (33.16)

```
> C := DGTable[Indx5D[11]]["StructureConstants"]:
```

```
> LD := _DG([["LieAlgebra", alg, [5, table()]], C ]);
```

$$LD := [e1, e2] = 2\,e1, [e1, e3] = -e2, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = 2\,e3, [e2, e4] = 0, [e2, e5] = 0, [e3, e4] = 0, [e3, e5] = 0, [e4, e5] = 0 \tag{12.1}$$

```
> DGsetup(LD):
> Iso := DGTable[Indx5D[11]]["Isotropy"]:
> V := DGinformation("FrameBaseVectors"):
> H := DGzip(Iso, V, "plus");
```

$$H := [e3 + e4] \tag{12.2}$$

```
> DGsetup([x1,x2,x3,x4],P);
```

Manifold: P **(12.3)**

```
> KV:=DGzip([[1, 0, 0, 0], [0, exp(x3), 0, 0], [0, 0, 1, 0], [0, 0, 0, 1], [exp(-x3), -exp(-x3)*x2^2,
  -2*exp(-x3)*x2, 0]],DGinformation("FrameBaseVectors"));
```

$$KV := \left[\partial_{x1}, e^{x3}\,\partial_{x2}, \partial_{x3}, \partial_{x4}, e^{-x3}\,\partial_{x1} - e^{-x3}\,x2^2\,\partial_{x2} - 2\,e^{-x3}\,x2\,\partial_{x3}\right] \tag{12.4}$$

```
> LDK:=LieAlgebraData(KV,algK);
```

$$LDK := [e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = 0, [e2, e3] = -e2, [e2, e4] = 0, [e2, e5] = -2\,e3, [e3, e4] = 0, [e3, e5] = -e5, [e4, e5] = 0 \tag{12.5}$$

```
> DGsetup(LDK,[X],[O]);
```

Lie algebra: algK **(12.6)**

```
> IsoK:=IsotropySubalgebra(KV,[x1=0,x2=0,x3=0,x4=0],output=[algK]);
```

$$IsoK := [X1 - X5] \tag{12.7}$$

```
> Mat:=Matrix([[0, 1, 0, 0, 0], [0, 0, -2, 0, 0], [0, 0, 0, 0, -1], [1, 0, 0, 0, 0], [0, 0, 0, 1, 0]]):
> COB:=evalDG(convert(Mat.Matrix([[X1],[X2],[X3],[X4],[X5]]),list));
```

$$COB := [X2, -2\,X3, -X5, X1, X4] \tag{12.8}$$

```
> DGequal(eval(LieAlgebraData(COB)),LD);
```

true **(12.9)**

```
> List:=op(GetComponents(H,DGinformation(alg,"FrameBaseVectors")));
```

$$List := [0, 0, 1, 1, 0] \tag{12.10}$$

```
> evalDG(add(List[i]*COB[i],i=1..5)*(1)-op(IsoK));
```

$$0\,X1 \tag{12.11}$$

APPENDIX C

Maple Database

```
#####################################################################
### F8 ##############################################################
#####################################################################
#####################################################################
#         [5, F8, 0]
#####################################################################
DGTable[[5, F8, 0]]["StructureConstants"] := [[[1, 2, 1], 2], [[1, 3, 2], -1], [[2, 3, 3],
      2], [[4, 5, 4], 1]]:
DGTable[[5, F8, 0]]["Isotropy"] := [[0, 0, 1, 1, 0], [0, 1, 0, 0, -2]]:
DGTable[[5, F8, 0]]["Parameters"] := [[],[]]:

#####################################################################
#####################################################################
#         [5, F8, 1]
#####################################################################
DGTable[[5, F8, 1]]["StructureConstants"] := [[[1, 5, 1], -1], [[2, 4, 1], 1], [[3, 4, 2],
      1], [[3, 5, 3], 1], [[4, 5, 4], -1]]:
DGTable[[5, F8, 1]]["Isotropy"] := [[0, 0, 0, 1, 0], [0, 0, 0, 0, 1]]:
DGTable[[5, F8, 1]]["Parameters"] := [[],[]]:



#####################################################################
### F11 #############################################################
#####################################################################
#####################################################################
#         [5, F11, 0]
#####################################################################
DGTable[[5, F11, 0]]["StructureConstants"] := [[[1, 5, 1], tan(a)], [[2, 5, 2], -tan(a)],
     [[4, 5, 3], -1], [[3, 5, 4], 1]]:
DGTable[[5, F11, 0]]["Isotropy"] := [[0, 0, 0, 0, 1]]:
DGTable[[5, F11, 0]]["Parameters"] := [[a],[a > 0, a < Pi/2]]:



#####################################################################
### F12 #############################################################
#####################################################################
#####################################################################
#         [5, F12, 0]
#####################################################################
DGTable[[5, F12, 0]]["StructureConstants"] := [[[1, 3, 2], 1], [[2, 3, 1], -1], [[4, 5,
     4], -1]]:
DGTable[[5, F12, 0]]["Isotropy"] := [[0, 0, 1, 0, 0]]:
DGTable[[5, F12, 0]]["Parameters"] := [[],[]]:

#####################################################################
#####################################################################
#         [5, F12, 1]
#####################################################################
DGTable[[5, F12, 1]]["StructureConstants"] := [[[1, 2, 1], 2],[[1, 3, 2], -1], [[2, 3, 3],
      2], [[4, 5, 4], 1]]:
DGTable[[5, F12, 1]]["Isotropy"] := [[1, 0, -1, 0, 0]]:
DGTable[[5, F12, 1]]["Parameters"] := [[],[]]:

#####################################################################
#####################################################################
#         [5, F12, 2]
#####################################################################
DGTable[[5, F12, 2]]["StructureConstants"] := [[[1, 2, 3], 1], [[1, 3, 2], -1], [[2, 3,
     1], 1], [[4, 5, 4], 1]]:
DGTable[[5, F12, 2]]["Isotropy"] := [[1, 0, 0, 0, 0]]:
DGTable[[5, F12, 2]]["Parameters"] := [[],[]]:

#####################################################################
#####################################################################
#         [5, F12, 3]
#####################################################################
```

```
DGTable[[5, F12, 3]]["StructureConstants"] := [[[1, 2, 1], 2], [[1, 3, 2], -2], [[2, 3,
    3], 2]]:
DGTable[[5, F12, 3]]["Isotropy"] := [[1, 0, -1, 0, 0]]:
DGTable[[5, F12, 3]]["Parameters"] := [[],[]]:

#################################################################
#################################################################
#        [5, F12, 4]
#################################################################
DGTable[[5, F12, 4]]["StructureConstants"] := [[[1, 2, 1], 2], [[1, 3, 2], -2], [[2, 3,
    3], 2]]:
DGTable[[5, F12, 4]]["Isotropy"] := [[1, 0, -1, -2, 0]]:
DGTable[[5, F12, 4]]["Parameters"] := [[],[]]:

#################################################################
#################################################################
#        [5, F12, 5]
#################################################################
DGTable[[5, F12, 5]]["StructureConstants"] := [[[1, 2, 3], 1], [[1, 3, 2], -1], [[2, 3,
    1], 1]]:
DGTable[[5, F12, 5]]["Isotropy"] := [[1, 0, 0, 0, 0]]:
DGTable[[5, F12, 5]]["Parameters"] := [[],[]]:

#################################################################
#################################################################
#        [5, F12, 6]
#################################################################
DGTable[[5, F12, 6]]["StructureConstants"] := [[[1, 2, 3], 1], [[1, 3, 2], -1], [[2, 3,
    1], 1]]:
DGTable[[5, F12, 6]]["Isotropy"] := [[1, 0, 0, -1, 0]]:
DGTable[[5, F12, 6]]["Parameters"] := [[],[]]:

#################################################################
#################################################################
#        [5, F12, 7]
#################################################################
DGTable[[5, F12, 7]]["StructureConstants"] := [[[1, 3, 2], 1], [[2, 3, 1], -1]]:
DGTable[[5, F12, 7]]["Isotropy"] := [[0, 0, 1, 0, 0]]:
DGTable[[5, F12, 7]]["Parameters"] := [[],[]]:

#################################################################
#################################################################
#        [5, F12, 8]
#################################################################
DGTable[[5, F12, 8]]["StructureConstants"] := [[[2, 3, 1], 1], [[2, 4, 3], 1], [[3, 4, 2],
     -1]]:
DGTable[[5, F12, 8]]["Isotropy"] := [[0, 0, 0, 1, 0]]:
DGTable[[5, F12, 8]]["Parameters"] := [[],[]]:

#################################################################
#################################################################
#        [5, F12, 9]
#################################################################
DGTable[[5, F12, 9]]["StructureConstants"] := [[[1, 4, 1], -2], [[2, 3, 1], 1], [[2, 4,
    2], -1], [[2, 5, 3], -1], [[3, 4, 3], -1], [[3, 5, 2], 1]]:
DGTable[[5, F12, 9]]["Isotropy"] := [[0, 0, 0, 0, 1]]:
DGTable[[5, F12, 9]]["Parameters"] := [[],[]]:

#################################################################
#################################################################
#        [5, F12, 10]
#################################################################
DGTable[[5, F12, 10]]["StructureConstants"] := [[[1, 3, 1], -1], [[1, 4, 2], -1], [[2, 3,
    2], -1], [[2, 4, 1], 1]]:
DGTable[[5, F12, 10]]["Isotropy"] := [[0, 0, 0, 1, 0]]:
DGTable[[5, F12, 10]]["Parameters"] := [[],[]]:

#################################################################
```

```
######################################################################
#          [5, F12, 11]
######################################################################
DGTable[[5, F12, 11]]["StructureConstants"] := [[[1, 4, 1], beta], [[2, 4, 2], -1], [[2,
     5, 3], 1], [[3, 4, 3], -1], [[3, 5, 2], -1]]:
DGTable[[5, F12, 11]]["Isotropy"] := [[0, 0, 0, 0, 1]]:
DGTable[[5, F12, 11]]["Parameters"] := [[beta],[beta<>0]]:



######################################################################
### F13 ##############################################################
######################################################################
######################################################################
#          [5, F13, 0]
######################################################################
DGTable[[5, F13, 0]]["StructureConstants"] := [[[1, 3, 1], 1], [[2, 3, 2], -1], [[4, 5,
     4], 1]]:
DGTable[[5, F13, 0]]["Isotropy"] := [[0, 0, 1, 0, 0]]:
DGTable[[5, F13, 0]]["Parameters"] := [[],[]]:

######################################################################
######################################################################
#          [5, F13, 1]
######################################################################
DGTable[[5, F13, 1]]["StructureConstants"] := [[[1, 2, 1], 2], [[2, 3, 3], 2], [[1, 3, 2],
      -1], [[4, 5, 4], 1]]:
DGTable[[5, F13, 1]]["Isotropy"] := [[0, 1, 0, 0, 0]]:
DGTable[[5, F13, 1]]["Parameters"] := [[],[]]:

######################################################################
######################################################################
#          [5, F13, 2]
######################################################################
DGTable[[5, F13, 2]]["StructureConstants"] := [[[1, 3, 1], 1], [[2, 3, 2], -1]]:
DGTable[[5, F13, 2]]["Isotropy"] := [[0, 0, 1, 0, 0]]:
DGTable[[5, F13, 2]]["Parameters"] := [[],[]]:

######################################################################
######################################################################
#          [5, F13, 3]
######################################################################
DGTable[[5, F13, 3]]["StructureConstants"] := [[[2, 3, 1], -1], [[2, 4, 2], -1], [[3, 4,
     3], 1]]:
DGTable[[5, F13, 3]]["Isotropy"] := [[0, 0, 0, 1, 0]]:
DGTable[[5, F13, 3]]["Parameters"] := [[],[]]:

######################################################################
######################################################################
#          [5, F13, 4]
######################################################################
DGTable[[5, F13, 4]]["StructureConstants"] := [[[1, 2, 1], 2], [[1, 3, 2], 1], [[2, 3, 3],
      2]]:
DGTable[[5, F13, 4]]["Isotropy"] := [[0, 1, 0, 0, 0]]:
DGTable[[5, F13, 4]]["Parameters"] := [[],[]]:

######################################################################
######################################################################
#          [5, F13, 5]
######################################################################
DGTable[[5, F13, 5]]["StructureConstants"] := [[[1, 2, 1], 2], [[1, 3, 2], 1], [[2, 3, 3],
      2]]:
DGTable[[5, F13, 5]]["Isotropy"] := [[0, 1, 0, -2, 0]]:
DGTable[[5, F13, 5]]["Parameters"] := [[],[]]:

######################################################################
######################################################################
#          [5, F13, 6]
```

```
######################################################################
DGTable[[5, F13, 6]]["StructureConstants"] := [[[1, 4, 1], -1], [[2, 3, 1], 1], [[2, 5,
    2], 1], [[3, 4, 3], -1], [[3, 5, 3], -1]]:
DGTable[[5, F13, 6]]["Isotropy"] := [[0, 0, 0, 0, 1]]:
DGTable[[5, F13, 6]]["Parameters"] := [[],[]]:

######################################################################
######################################################################
#       [5, F13, 7]
######################################################################
DGTable[[5, F13, 7]]["StructureConstants"] := [[[1, 2, 1], 1], [[3, 4, 3], 1]]:
DGTable[[5, F13, 7]]["Isotropy"] := [[0, 1, 0, -1, 0]]:
DGTable[[5, F13, 7]]["Parameters"] := [[],[]]:

######################################################################
######################################################################
#       [5, F13, 8]
######################################################################
DGTable[[5, F13, 8]]["StructureConstants"] := [[[1, 5, 1], -1], [[2, 4, 2], -1], [[3, 4,
    3], -a], [[3, 5, 3], -a]]:
DGTable[[5, F13, 8]]["Isotropy"] := [[0, 0, 0, 1, -1]]:
DGTable[[5, F13, 8]]["Parameters"] := [[a],[a>0,a<=1]]:



######################################################################
### F14 ##############################################################
######################################################################
######################################################################
######################################################################
#       [5, F14, 0]
######################################################################
DGTable[[5, F14, 0]]["StructureConstants"] := [[[1, 5, 1], 1], [[2, 4, 1], 1], [[2, 5, 2],
    1], [[3, 4, 2], 1], [[3, 5, 3], 1]]:
DGTable[[5, F14, 0]]["Isotropy"] := [[0, 0, 0, 1, 0]]:
DGTable[[5, F14, 0]]["Parameters"] := [[],[]]:

######################################################################
######################################################################
#       [5, F14, 1]
######################################################################
DGTable[[5, F14, 1]]["StructureConstants"] := [[[1, 5, 1], 1], [[2, 4, 1], 1], [[2, 5, 2],
    1], [[3, 4, 2], 1], [[3, 5, 1], -epsilon], [[3, 5, 3], 1]]:
DGTable[[5, F14, 1]]["Isotropy"] := [[0, 0, 0, 1, 0]]:
DGTable[[5, F14, 1]]["Parameters"] := [[epsilon],[abs(epsilon)=1]]:

######################################################################
######################################################################
#       [5, F14, 2]
######################################################################
DGTable[[5, F14, 2]]["StructureConstants"] := [[[1, 2, 1], 2], [[1, 3, 2], -1], [[2, 3,
    3], 2]]:
DGTable[[5, F14, 2]]["Isotropy"] := [[0, 0, 1, 1, 0]]:
DGTable[[5, F14, 2]]["Parameters"] := [[],[]]:

######################################################################
######################################################################
#       [5, F14, 3]
######################################################################
DGTable[[5, F14, 3]]["StructureConstants"] := [[[1, 4, 1], 1], [[2, 3, 1], 1], [[2, 4, 2],
    1]]:
DGTable[[5, F14, 3]]["Isotropy"] := [[0, 1, 1, 0, 0]]:
DGTable[[5, F14, 3]]["Parameters"] := [[],[]]:

######################################################################
######################################################################
#       [5, F14, 4]
######################################################################
```

```
DGTable[[5, F14, 4]]["StructureConstants"] := [[[2, 5, 1], 1], [[3, 4, 1], 1], [[4, 5, 2],
    1]]:
DGTable[[5, F14, 4]]["Isotropy"] := [[0, 0, 0, 0, 1]]:
DGTable[[5, F14, 4]]["Parameters"] := [[],[]]:

#####################################################################
#####################################################################
#        [5, F14, 5]
#####################################################################
DGTable[[5, F14, 5]]["StructureConstants"] := [[[2, 5, 1], 1], [[3, 5, 2], 1]]:
DGTable[[5, F14, 5]]["Isotropy"] := [[0, 0, 0, 0, 1]]:
DGTable[[5, F14, 5]]["Parameters"] := [[],[]]:

#####################################################################
#####################################################################
#        [5, F14, 6]
#####################################################################
DGTable[[5, F14, 6]]["StructureConstants"] := [[[3, 4, 2], 1], [[3, 5, 1], 1], [[4, 5, 3],
    1]]:
DGTable[[5, F14, 6]]["Isotropy"] := [[0, 0, 0, 0, 1]]:
DGTable[[5, F14, 6]]["Parameters"] := [[],[]]:

#####################################################################
#####################################################################
#        [5, F14, 7]
#####################################################################
DGTable[[5, F14, 7]]["StructureConstants"] := [[[1, 5, 1], -1], [[2, 3, 1], 1], [[2, 5,
    2], -1], [[3, 5, 4], 1]]:
DGTable[[5, F14, 7]]["Isotropy"] := [[1, -1, -1, 0, 0]]:
DGTable[[5, F14, 7]]["Parameters"] := [[],[]]:

#####################################################################
#####################################################################
#        [5, F14, 8]
#####################################################################
DGTable[[5, F14, 8]]["StructureConstants"] := [[[2, 5, 1], 1], [[3, 4, 1], 1], [[3, 5, 2],
    -1], [[4, 5, 3], 1]]:
DGTable[[5, F14, 8]]["Isotropy"] := [[0, 0, 0, 1, 0]]:
DGTable[[5, F14, 8]]["Parameters"] := [[],[]]:

#####################################################################
#####################################################################
#        [5, F14, 9]
#####################################################################
DGTable[[5, F14, 9]]["StructureConstants"] := [[[2, 3, 1], -1], [[3, 5, 2], -1], [[4, 5,
    4], -1]]:
DGTable[[5, F14, 9]]["Isotropy"] := [[0, 1, 1, 1, 0]]:
DGTable[[5, F14, 9]]["Parameters"] := [[],[]]:

#####################################################################
#####################################################################
#        [5, F14, 10]
#####################################################################
DGTable[[5, F14, 10]]["StructureConstants"] := [[[2, 3, 1], -1], [[3, 5, 2], -1], [[4, 5,
    4], -1]]:
DGTable[[5, F14, 10]]["Isotropy"] := [[1, 0, -1, 0, 0]]:
DGTable[[5, F14, 10]]["Parameters"] := [[],[]]:

#####################################################################
#####################################################################
#        [5, F14, 11]
#####################################################################
DGTable[[5, F14, 11]]["StructureConstants"] := [[[1, 5, 1], -1], [[2, 3, 1], 1], [[2, 5,
    2], -1], [[4, 5, 4], -a]]:
DGTable[[5, F14, 11]]["Isotropy"] := [[0, 1, 1, 1, 0]]:
DGTable[[5, F14, 11]]["Parameters"] := [[a],[a<>1,a<>0]]:

#####################################################################
```

```
########################################################################
#          [5, F14, 12]
########################################################################
DGTable[[5, F14, 12]]["StructureConstants"] := [[[1, 5, 1], -1], [[2, 3, 1], 1], [[2, 5,
     2], -1], [[4, 5, 4], -a]]:
DGTable[[5, F14, 12]]["Isotropy"] := [[0, 0, 1, 1, 0]]:
DGTable[[5, F14, 12]]["Parameters"] := [[a],[a<>1,a<>0]]:

########################################################################
########################################################################
#          [5, F14, 13]
########################################################################
DGTable[[5, F14, 13]]["StructureConstants"] := [[[1, 5, 1], -1], [[2, 3, 1], 1], [[3, 5,
     3], -1], [[3, 5, 4], -1], [[4, 5, 1], -1], [[4, 5, 4], -1]]:
DGTable[[5, F14, 13]]["Isotropy"] := [[0, 1, 1, 0, 0]]:
DGTable[[5, F14, 13]]["Parameters"] := [[],[]]:

########################################################################
########################################################################
#          [5, F14, 14]
########################################################################
DGTable[[5, F14, 14]]["StructureConstants"] := [[[1, 5, 1], -1], [[2, 3, 1], 1], [[2, 5,
     2], -1], [[4, 5, 1], -1], [[4, 5, 4], -1]]:
DGTable[[5, F14, 14]]["Isotropy"] := [[0, 1, 1, 0, 0]]:
DGTable[[5, F14, 14]]["Parameters"] := [[],[]]:

########################################################################
#          [5, F14, 15]
########################################################################
DGTable[[5, F14, 15]]["StructureConstants"] := [[[1, 5, 1], -1], [[2, 3, 1], -1], [[3, 5,
     3], -1], [[3, 5, 4], -1], [[4, 5, 1], 0], [[4, 5, 4], -1]]:
DGTable[[5, F14, 15]]["Isotropy"] := [[0, 1, 1, 0, 0]]:
DGTable[[5, F14, 15]]["Parameters"] := [[],[]]:

########################################################################
#          [5, F14, 16]
########################################################################
DGTable[[5, F14, 16]]["StructureConstants"] := [[[1, 5, 1], -1], [[2, 3, 1], 1], [[2, 5,
     2], -1], [[4, 5, 4], -1]]:
DGTable[[5, F14, 16]]["Isotropy"] := [[0, 1, 1, 0, 0]]:
DGTable[[5, F14, 16]]["Parameters"] := [[],[]]:

########################################################################
#          [5, F14, 17]
########################################################################
DGTable[[5, F14, 17]]["StructureConstants"] := [[[2, 3, 1], 1], [[2, 4, 2], -1], [[3, 4,
     3], 1]]:
DGTable[[5, F14, 17]]["Isotropy"] := [[0, 1, -2, 0, 0]]:
DGTable[[5, F14, 17]]["Parameters"] := [[],[]]:

########################################################################
#          [5, F14, 18]
########################################################################
DGTable[[5, F14, 18]]["StructureConstants"] := [[[2, 3, 1], 1], [[2, 4, 2], -1], [[3, 4,
     3], 1]]:
DGTable[[5, F14, 18]]["Isotropy"] := [[0, 1, -2, 0, 1]]:
DGTable[[5, F14, 18]]["Parameters"] := [[],[]]:

########################################################################
#          [5, F14, 19]
########################################################################
DGTable[[5, F14, 19]]["StructureConstants"] := [[[2, 3, 1], 1], [[2, 4, 3], 1], [[3, 4,
     2], -1]]:
DGTable[[5, F14, 19]]["Isotropy"] := [[0, 1, 0, 0, 0]]:
DGTable[[5, F14, 19]]["Parameters"] := [[],[]]:

########################################################################
#          [5, F14, 20]
```

```
########################################################################
DGTable[[5, F14, 20]]["StructureConstants"] := [[[2, 3, 1], 1], [[2, 4, 3], 1], [[3, 4,
     2], -1]]:
DGTable[[5, F14, 20]]["Isotropy"] := [[0, 1, 0, 0, -1]]:
DGTable[[5, F14, 20]]["Parameters"] := [[],[]]:

########################################################################
#         [5, F14, 21]
########################################################################
DGTable[[5, F14, 21]]["StructureConstants"] := [[[2, 3, 1], 1], [[2, 5, 3], -1], [[3, 5,
     2], 1], [[4, 5, 1], -1]]:
DGTable[[5, F14, 21]]["Isotropy"] := [[0, 0, 1, 0, 0]]:
DGTable[[5, F14, 21]]["Parameters"] := [[],[]]:

########################################################################
#         [5, F14, 22]
########################################################################
DGTable[[5, F14, 22]]["StructureConstants"] := [[[2, 3, 1], 1], [[2, 5, 3], -1], [[3, 5,
     2], 1], [[4, 5, 1], -1]]:
DGTable[[5, F14, 22]]["Isotropy"] := [[0, 0, 1, 1, 0]]:
DGTable[[5, F14, 22]]["Parameters"] := [[],[]]:

########################################################################
#         [5, F14, 23]
########################################################################
DGTable[[5, F14, 23]]["StructureConstants"] := [[[2, 3, 1], 1], [[2, 5, 2], 1], [[3, 5,
     3], -1], [[4, 5, 1], -1]]:
DGTable[[5, F14, 23]]["Isotropy"] := [[0, 1, -1, 0, 0]]:
DGTable[[5, F14, 23]]["Parameters"] := [[],[]]:

########################################################################
#         [5, F14, 24]
########################################################################
DGTable[[5, F14, 24]]["StructureConstants"] := [[[2, 3, 1], 1], [[2, 5, 2], 1], [[3, 5,
     3], -1], [[4, 5, 1], -1]]:
DGTable[[5, F14, 24]]["Isotropy"] := [[0, 1, -1, -1, 0]]:
DGTable[[5, F14, 24]]["Parameters"] := [[],[]]:

########################################################################
#         [5, F14, 25]
########################################################################
DGTable[[5, F14, 25]]["StructureConstants"] := [[[1, 4, 1], -2], [[2, 3, 1], 1], [[2, 4,
     2], -1], [[3, 4, 2], -1], [[3, 4, 3], -1]]:
DGTable[[5, F14, 25]]["Isotropy"] := [[0, 0, 1, 0, 0]]:
DGTable[[5, F14, 25]]["Parameters"] := [[],[]]:

########################################################################
#         [5, F14, 26]
########################################################################
DGTable[[5, F14, 26]]["StructureConstants"] := [[[1, 4, 1], -2], [[2, 3, 1], 1], [[2, 4,
     2], -1], [[3, 4, 2], -1], [[3, 4, 3], -1]]:
DGTable[[5, F14, 26]]["Isotropy"] := [[0, 0, 1, 0, 1]]:
DGTable[[5, F14, 26]]["Parameters"] := [[],[]]:

########################################################################
#         [5, F14, 27]
########################################################################
DGTable[[5, F14, 27]]["StructureConstants"] := [[[1, 4, 1], -a-1], [[2, 3, 1], -1], [[2,
     4, 2], -1], [[3, 4, 3], -a]]:
DGTable[[5, F14, 27]]["Isotropy"] := [[0, 1, -1, 0, 0]]:
DGTable[[5, F14, 27]]["Parameters"] := [[a],[a<>0,a<=1,a>-1]]:

########################################################################
#         [5, F14, 28]
########################################################################
DGTable[[5, F14, 28]]["StructureConstants"] := [[[1, 4, 1], -a-1], [[2, 3, 1], -1], [[2,
     4, 2], -1], [[3, 4, 3], -a]]:
DGTable[[5, F14, 28]]["Isotropy"] := [[0, 1, -1, 0, 1]]:
```

```
DGTable[[5, F14, 28]]["Parameters"] := [[a],[a<>0,a<=1,a>-1]]:

#####################################################################
#         [5, F14, 29]
#####################################################################
DGTable[[5, F14, 29]]["StructureConstants"] := [[[1, 4, 1], -2*alpha], [[2, 3, 1], 1],
    [[2, 4, 2], -alpha], [[2, 4, 3], 1], [[3, 4, 2], -1], [[3, 4, 3], -alpha]]:
DGTable[[5, F14, 29]]["Isotropy"] := [[0, 1, 0, 0, 0]]:
DGTable[[5, F14, 29]]["Parameters"] := [[alpha],[alpha>0]]:

#####################################################################
#         [5, F14, 30]
#####################################################################
DGTable[[5, F14, 30]]["StructureConstants"] := [[[1, 4, 1], -2*alpha], [[2, 3, 1], 1],
    [[2, 4, 2], -alpha], [[2, 4, 3], 1], [[3, 4, 2], -1], [[3, 4, 3], -alpha]]:
DGTable[[5, F14, 30]]["Isotropy"] := [[0, 1, 0, 0, 1]]:
DGTable[[5, F14, 30]]["Parameters"] := [[alpha],[alpha>0]]:

#####################################################################
#         [5, F14, 31]
#####################################################################
DGTable[[5, F14, 31]]["StructureConstants"] := [[[2, 3, 1], 1], [[2, 5, 3], -1], [[3, 5,
    2], 1], [[4, 5, 4], -alpha]]:
DGTable[[5, F14, 31]]["Isotropy"] := [[0, 0, 1, 0, 0]]:
DGTable[[5, F14, 31]]["Parameters"] := [[alpha],[alpha<>1,alpha>0]]:

#####################################################################
#         [5, F14, 32]
#####################################################################
DGTable[[5, F14, 32]]["StructureConstants"] := [[[2, 3, 1], 1], [[2, 5, 3], -1], [[3, 5,
    2], 1], [[4, 5, 4], -alpha]]:
DGTable[[5, F14, 32]]["Isotropy"] := [[0, 0, 1, -1, 0]]:
DGTable[[5, F14, 32]]["Parameters"] := [[alpha],[alpha<>1,alpha>0]]:

#####################################################################
#         [5, F14, 33]
#####################################################################
DGTable[[5, F14, 33]]["StructureConstants"] := [[[2, 3, 1], 1], [[2, 5, 2], -1], [[3, 5,
    3], 1], [[4, 5, 4], -a]]:
DGTable[[5, F14, 33]]["Isotropy"] := [[0, 1, -1, 0, 0]]:
DGTable[[5, F14, 33]]["Parameters"] := [[a],[a<>1]]:

#####################################################################
#         [5, F14, 34]
#####################################################################
DGTable[[5, F14, 34]]["StructureConstants"] := [[[2, 3, 1], 1], [[2, 5, 2], -1], [[3, 5,
    3], 1], [[4, 5, 4], -a]]:
DGTable[[5, F14, 34]]["Isotropy"] := [[0, 1, -1, -1, 0]]:
DGTable[[5, F14, 34]]["Parameters"] := [[a],[a<>1]]:

#####################################################################
#         [5, F14, 35]
#####################################################################
DGTable[[5, F14, 35]]["StructureConstants"] := [[[2, 3, 1], 1], [[2, 5, 2], -1], [[3, 5,
    3], 1], [[4, 5, 4], -1]]:
DGTable[[5, F14, 35]]["Isotropy"] := [[0, 1, -1, 0, 0]]:
DGTable[[5, F14, 35]]["Parameters"] := [[],[]]:

#####################################################################
#         [5, F14, 36]
#####################################################################
DGTable[[5, F14, 36]]["StructureConstants"] := [[[2, 3, 1], 1], [[2, 5, 2], -1], [[3, 5,
    3], 1], [[4, 5, 4], -1]]:
DGTable[[5, F14, 36]]["Isotropy"] := [[0, 1, -1, -1/2, 0]]:
DGTable[[5, F14, 36]]["Parameters"] := [[],[]]:

#####################################################################
#         [5, F14, 37]
```

```
#########################################################################
DGTable[[5, F14, 37]]["StructureConstants"] := [[[1, 5, 1], -2*alpha], [[2, 3, 1], 1],
    [[2, 5, 2], -alpha], [[2, 5, 3], 1], [[3, 5, 2], -1], [[3, 5, 3], -alpha], [[4, 5, 4],
    -beta]]:
DGTable[[5, F14, 37]]["Isotropy"] := [[0, 1, 0, 1, 0]]:
DGTable[[5, F14, 37]]["Parameters"] := [[alpha,beta],[alpha<>0,beta>0]]:

#########################################################################
#        [5, F14, 38]
#########################################################################
DGTable[[5, F14, 38]]["StructureConstants"] := [[[1, 5, 1], -2*alpha], [[2, 3, 1], 1],
    [[2, 5, 2], -alpha], [[2, 5, 3], 1], [[3, 5, 2], -1], [[3, 5, 3], -alpha], [[4, 5, 4],
    -beta]]:
DGTable[[5, F14, 38]]["Isotropy"] := [[0, 1, 0, 0, 0]]:
DGTable[[5, F14, 38]]["Parameters"] := [[alpha,beta],[alpha<>0,beta>0]]:
#########################################################################

#########################################################################
#        [5, F14, 39]
#########################################################################
DGTable[[5, F14, 39]]["StructureConstants"] := [[[1, 5, 1], -1-a], [[2, 3, 1], 1], [[2, 5,
     2], -1], [[3, 5, 3], -a], [[4, 5, 4], -b]]:
DGTable[[5, F14, 39]]["Isotropy"] := [[0, 1, 1, 1, 0]]:
DGTable[[5, F14, 39]]["Parameters"] := [[a,b],[a<=1,a>-1,b<>0]]:
#########################################################################

#########################################################################
#        [5, F14, 40]
#########################################################################
DGTable[[5, F14, 40]]["StructureConstants"] := [[[1, 5, 1], -1-a], [[2, 3, 1], 1], [[2, 5,
     2], -1], [[3, 5, 3], -a], [[4, 5, 4], -b]]:
DGTable[[5, F14, 40]]["Isotropy"] := [[0, 1, 1, 0, 0]]:
DGTable[[5, F14, 40]]["Parameters"] := [[a,b],[a<=1,a>-1,b<>0]]:
#########################################################################

#########################################################################
#        [5, F14, 41]
#########################################################################
DGTable[[5, F14, 41]]["StructureConstants"] := [[[1, 5, 1], -2], [[2, 3, 1], 1], [[2, 5,
    2], -1], [[2, 5, 3], -1], [[3, 5, 3], -1], [[4, 5, 4], -a]]:
DGTable[[5, F14, 41]]["Isotropy"] := [[0, 1, 0, 1, 0]]:
DGTable[[5, F14, 41]]["Parameters"] := [[a],[a<>0]]:
#########################################################################

#########################################################################
#        [5, F14, 42]
#########################################################################
DGTable[[5, F14, 42]]["StructureConstants"] := [[[1, 5, 1], -2], [[2, 3, 1], 1], [[2, 5,
    2], -1], [[2, 5, 3], -1], [[3, 5, 3], -1], [[4, 5, 4], -a]]:
DGTable[[5, F14, 42]]["Isotropy"] := [[0, 1, 0, 0, 0]]:
DGTable[[5, F14, 42]]["Parameters"] := [[a,b],[a<>0]]:
#########################################################################

#########################################################################
#        [5, F14, 43]
#########################################################################
DGTable[[5, F14, 43]]["StructureConstants"] := [[[1, 5, 1], -1-a], [[2, 3, 1], 1], [[2, 5,
     2], -a], [[3, 5, 3], -1], [[3, 5, 4], -1], [[4, 5, 4], -1]]:
DGTable[[5, F14, 43]]["Isotropy"] := [[0, 1, 1, 0, 0]]:
DGTable[[5, F14, 43]]["Parameters"] := [[],[]]:
#########################################################################

#########################################################################
#        [5, F14, 44]
#########################################################################
DGTable[[5, F14, 44]]["StructureConstants"] := [[[1, 5, 1], -1-a], [[2, 3, 1], 1], [[2, 5,
     2], -a], [[3, 5, 3], -1], [[4, 5, 4], -1]]:
DGTable[[5, F14, 44]]["Isotropy"] := [[0, 1, 1, 0, 0]]:
```

```
DGTable[[5, F14, 44]]["Parameters"] := [[],[]]:
######################################################################

######################################################################
#          [5, F14, 45]
######################################################################
DGTable[[5, F14, 45]]["StructureConstants"] := [[[1, 5, 1], -2], [[2, 3, 1], 1], [[2, 5,
     2], -1], [[2, 5, 3], -1], [[3, 5, 3], -1], [[3, 5, 4], -1], [[4, 5, 4], -1]]:
DGTable[[5, F14, 45]]["Isotropy"] := [[0, 1, -1, 1, 0]]:
DGTable[[5, F14, 45]]["Parameters"] := [[],[]]:
######################################################################

######################################################################
#          [5, F14, 46]
######################################################################
DGTable[[5, F14, 46]]["StructureConstants"] := [[[1, 5, 1], -2], [[2, 3, 1], 1], [[2, 5,
     2], -1], [[2, 5, 3], -1], [[3, 5, 3], -1], [[4, 5, 4], -1]]:
DGTable[[5, F14, 46]]["Isotropy"] := [[0, 1, -1, 0, 0]]:
DGTable[[5, F14, 46]]["Parameters"] := [[],[]]:
######################################################################

######################################################################
#          [5, F14, 47]
######################################################################
DGTable[[5, F14, 47]]["StructureConstants"] := [[[1, 5, 1], -1-a], [[2, 3, 1], 1], [[2, 5,
     2], -1], [[3, 5, 3], -a], [[4, 5, 1], -1], [[4, 5, 4], -1-a]]:
DGTable[[5, F14, 47]]["Isotropy"] := [[0, 1, 1, 1, 0]]:
DGTable[[5, F14, 47]]["Parameters"] := [[a],[a<=1,a>-1]]:
######################################################################

######################################################################
#          [5, F14, 48]
######################################################################
DGTable[[5, F14, 48]]["StructureConstants"] := [[[1, 5, 1], -1-a], [[2, 3, 1], 1], [[2, 5,
     2], -1], [[3, 5, 3], -a], [[4, 5, 1], -1], [[4, 5, 4], -1-a]]:
DGTable[[5, F14, 48]]["Isotropy"] := [[0, 1, 1, 0, 0]]:
DGTable[[5, F14, 48]]["Parameters"] := [[a],[a<=1,a>-1]]:
######################################################################

######################################################################
#          [5, F14, 49]
######################################################################
DGTable[[5, F14, 49]]["StructureConstants"] := [[[1, 5, 1], -2*alpha], [[2, 3, 1], 1],
     [[2, 5, 2], -alpha], [[2, 5, 3], -1], [[3, 5, 2], 1], [[3, 5, 3], -alpha], [[4, 5, 1],
     -1], [[4, 5, 4], -2*alpha]]:
DGTable[[5, F14, 49]]["Isotropy"] := [[0, 0, 1, -1, 0]]:
DGTable[[5, F14, 49]]["Parameters"] := [[alpha],[alpha>0]]:
######################################################################

######################################################################
#          [5, F14, 50]
######################################################################
DGTable[[5, F14, 50]]["StructureConstants"] := [[[1, 5, 1], -2*alpha], [[2, 3, 1], 1],
     [[2, 5, 2], -alpha], [[2, 5, 3], -1], [[3, 5, 2], 1], [[3, 5, 3], -alpha], [[4, 5, 1],
     -1], [[4, 5, 4], -2*alpha]]:
DGTable[[5, F14, 50]]["Isotropy"] := [[0, 0, 1, 0, 0]]:
DGTable[[5, F14, 50]]["Parameters"] := [[alpha],[alpha>0]]:
######################################################################

######################################################################
#          [5, F14, 51]
######################################################################
DGTable[[5, F14, 51]]["StructureConstants"] := [[[1, 5, 1], -2], [[2, 3, 1], 1], [[2, 5,
     2], -1], [[2, 5, 3], -1], [[3, 5, 3], -1], [[4, 5, 1], -1], [[4, 5, 4], -2]]:
DGTable[[5, F14, 51]]["Isotropy"] := [[0, 1, 1, 1, 0]]:
DGTable[[5, F14, 51]]["Parameters"] := [[],[]]:
######################################################################
```

```
##########################################################################
#          [5, F14, 52]
##########################################################################
DGTable[[5, F14, 52]]["StructureConstants"] := [[[1, 5, 1], -2], [[2, 3, 1], 1], [[2, 5,
     2], -1], [[2, 5, 3], -1], [[3, 5, 3], -1], [[4, 5, 1], -1], [[4, 5, 4], -2]]:
DGTable[[5, F14, 52]]["Isotropy"] := [[0, 1, 1, 0, 0]]:
DGTable[[5, F14, 52]]["Parameters"] := [[],[]]:
##########################################################################

##########################################################################
#          [5, F14, 53]
##########################################################################
DGTable[[5, F14, 53]]["StructureConstants"] := [[[1, 5, 1], -1-a], [[2, 3, 1], 1], [[2, 5,
      2], -1], [[3, 5, 3], -a], [[4, 5, 4], -1-a]]:
DGTable[[5, F14, 53]]["Isotropy"] := [[0, 1, 1, 1, 0]]:
DGTable[[5, F14, 53]]["Parameters"] := [[a],[a<=1,a>-1]]:
##########################################################################

##########################################################################
#          [5, F14, 54]
##########################################################################
DGTable[[5, F14, 54]]["StructureConstants"] := [[[1, 5, 1], -1-a], [[2, 3, 1], 1], [[2, 5,
      2], -1], [[3, 5, 3], -a], [[4, 5, 4], -1-a]]:
DGTable[[5, F14, 54]]["Isotropy"] := [[0, 1, 1, 0, 0]]:
DGTable[[5, F14, 54]]["Parameters"] := [[a],[a<=1,a>-1]]:
##########################################################################

##########################################################################
#          [5, F14, 55]
##########################################################################
DGTable[[5, F14, 55]]["StructureConstants"] := [[[1, 5, 1], -2*alpha], [[2, 3, 1], 1],
     [[2, 5, 2], -alpha], [[2, 5, 3], -1], [[3, 5, 2], 1], [[3, 5, 3], -alpha], [[4, 5, 4],
      -2*alpha]]:
DGTable[[5, F14, 55]]["Isotropy"] := [[0, 0, 1, -1, 0]]:
DGTable[[5, F14, 55]]["Parameters"] := [[alpha],[alpha>0]]:
##########################################################################

##########################################################################
#          [5, F14, 56]
##########################################################################
DGTable[[5, F14, 56]]["StructureConstants"] := [[[1, 5, 1], -2*alpha], [[2, 3, 1], 1],
     [[2, 5, 2], -alpha], [[2, 5, 3], -1], [[3, 5, 2], 1], [[3, 5, 3], -alpha], [[4, 5, 4],
      -2*alpha]]:
DGTable[[5, F14, 56]]["Isotropy"] := [[0, 0, 1, 0, 0]]:
DGTable[[5, F14, 56]]["Parameters"] := [[alpha],[alpha>0]]:
##########################################################################

##########################################################################
#          [5, F14, 57]
##########################################################################
DGTable[[5, F14, 57]]["StructureConstants"] := [[[1, 5, 1], -2], [[2, 3, 1], 1], [[2, 5,
     2], -1], [[2, 5, 3], -1], [[3, 5, 3], -1], [[4, 5, 4], -2]]:
DGTable[[5, F14, 57]]["Isotropy"] := [[0, 1, 1, 1, 0]]:
DGTable[[5, F14, 57]]["Parameters"] := [[],[]]:
##########################################################################

##########################################################################
#          [5, F14, 58]
##########################################################################
DGTable[[5, F14, 58]]["StructureConstants"] := [[[1, 5, 1], -2], [[2, 3, 1], 1], [[2, 5,
     2], -1], [[2, 5, 3], -1], [[3, 5, 3], -1], [[4, 5, 4], -2]]:
DGTable[[5, F14, 58]]["Isotropy"] := [[0, 1, 1, 0, 0]]:
DGTable[[5, F14, 58]]["Parameters"] := [[],[]]:
##########################################################################
```