

PROBLEM SET 1

Problem 1.1

Verify that each of the three forms of harmonic motion

$$\begin{aligned}q(t) &= B \sin(\tilde{\omega}t + \phi) \\ &= C \cos(\tilde{\omega}t + \psi) \\ &= D \cos \tilde{\omega}t + E \sin \tilde{\omega}t.\end{aligned}$$

satisfy the harmonic oscillator equation. Give formulas relating the constants B, C, D, E, ϕ, ψ in each case, *i.e.*, given B and ϕ in the first form of the motion, how to compute C, ψ and D, E ?

Problem 1.2

Consider a potential energy function $V(x) = ax^2 + bx^4$. Discuss the possible equilibrium positions for various choices of a and b . For $a > 0, b > 0$ show that the frequency for oscillations near equilibrium is independent of b .

Problem 1.3

Consider the motion of an oscillator which is started from rest from the initial position q_0 . Give the motion in each of the 3 trigonometric forms from the text. Do the same for the case where the initial velocity v_0 is non-zero but the initial position is zero.

Problem 1.4

Suppose that we changed the sign in the harmonic oscillator equation so that we consider the equation

$$\frac{d^2q}{dt^2} = \tilde{\omega}^2 q. \quad (1.40)$$

- What shape must the potential energy graph have in the neighborhood of an equilibrium point to lead to this (approximate) equation?
- Find the general solution to this equation. In particular, show that the solutions can grow exponentially with time (rather than oscillate), so that this equation permits solutions which depart from the initial values by arbitrarily large amounts.

- (c) For what initial conditions will the exponential growth found in (b) *not* occur.
- (d) Show that (1.40) and its solutions can be obtained from the general complex solution to the harmonic oscillator equation by letting the oscillator frequency become imaginary.

Problem 1.5

Find the absolute value and phase of the following complex numbers:

- (a) $3 + 5i$
(b) 10
(c) $10i$.

Problem 1.6

Using the relation between the (r, θ) and (x, y) parametrizations of a complex number show that

$$z^2 = r^2 e^{2i\theta} \quad \text{and} \quad zz^* = r^2$$

agree with the formulas for z^2 and zz^* obtained by using $z = x + iy$.

Problem 1.7

Prove that two complex numbers are equal if and only if the real parts are equal and the imaginary parts are equal.

Problem 1.8

Show that $q(t) = Ae^{i\tilde{\omega}t} + Be^{-i\tilde{\omega}t}$ is a real number for all values of t if *and only if* $A^* = B$.

Problem 1.9

In the text it is argued that one can find the general *real* solution to the harmonic oscillator equation by first finding the general *complex* solution and then taking its real part at the end of the day. How does the general real solution thus obtained compare to the real solution that is obtained if, instead, we take the *imaginary part* of the general complex solution?

Problem 1.10

With $q(t) = 2\text{Re}(\alpha e^{i\tilde{\omega}t})$, show that the initial position and velocities are given by $q(0) = 2\text{Re}(\alpha)$ and $v(0) = -2\tilde{\omega}\text{Im}(\alpha)$.

Problem 1.11

Using Euler's formula (1.28), prove the following trigonometric identities.

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta,$$

and

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta.$$

(*Hint: $e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}$.*)

Problem 1.12

Generalize equation (1.9) to allow for an arbitrary equilibrium point q_0 and an arbitrary reference point for the potential, $V(q_0) = V_0$. Show that (1.12) becomes the harmonic oscillator equation for the displacement from equilibrium ($q - q_0$).

Problem 1.13

Show that the set of complex numbers forms a complex vector space.

Problem 1.14

Using the complex form of a real solution to the harmonic oscillator equation,

$$q(t) = \alpha e^{i\tilde{\omega}t} + \alpha^* e^{-i\tilde{\omega}t},$$

show that the solution can be expressed in the forms (1.1) and (1.3). (*Hint: Make use of the polar representation of the complex number α .*)

Problem 1.15

The *energy* of a harmonic oscillator can be defined by

$$E = \gamma \frac{1}{2} \left\{ \left(\frac{dq}{dt} \right)^2 + \tilde{\omega}^2 q^2 \right\},$$

where γ is a constant (needed to get the units right). The energy E is conserved, that is, it doesn't depend upon time. Prove this in the following two distinct ways.

- (i) Substitute one of the general forms (1.1) – (1.3) of harmonic motion into E and show that the time dependence drops out.
- (ii) Take the time derivative of E and show that it vanishes provided (1.6) holds (without using the explicit form of the solutions).

Problem 1.16

It was pointed out in §1.2 that the functions $\cos \tilde{\omega}t$ and $\sin \tilde{\omega}t$ are orthogonal with respect to the scalar product (1.19). This implies they are linearly independent when viewed as elements of the vector space of solutions to the harmonic oscillator equation. Prove directly that these functions are linearly independent, *i.e.*, prove that if a and b are constants such that (for all values of t)

$$a \cos \tilde{\omega}t + b \sin \tilde{\omega}t = 0,$$

then $a = b = 0$.

(*Hint: This one is really easy!*)