

PROBLEM SET 2

Problem 2.1

Consider a particle moving in two dimensions (x, y) under a force

$$\mathbf{F} = -k\mathbf{r} = -kx\mathbf{i} - ky\mathbf{j}.$$

- (a) Show that the equations of motion for $\mathbf{r}(t) = (x(t), y(t))$ are equivalent to those describing a pair of uncoupled oscillators.
- (b) Show that the equations of motion admit a solution in which the particle moves in a circle at constant speed.

Problem 2.2

Consider a general linear, homogeneous equation $M(v) = 0$, where M is a matrix and v is a column vector. Suppose that all matrix elements of M are real: $M = M^*$. Show that if v is a complex solution to the linear equation (*i.e.*, the elements of the column vector are complex numbers), then v^* , $Re(v)$, and $Im(v)$ are also solutions to the same equation. (If you like, you can just let M be a 2×2 matrix, but it is not necessary to assume this in order to prove this result.)

Problem 2.3

When finding the resonant frequencies Ω_1, Ω_2 for a pair of coupled oscillators we must solve the characteristic equation. Suppose that one of the frequencies, Ω_1 say, turns out to be a *complex* number $\Omega_1 = \alpha + i\beta$. Discuss the physical behavior of the (putative) normal modes.

Problem 2.4

When we analyzed the chain of coupled oscillators we set $A_j = a \sin(j\phi)$ and then found that ϕ must satisfy $\sin[(N+1)\phi] = 0$. Show that there is no loss in generality by assuming the solution for ϕ is $\phi = \frac{n\pi}{N+1}$, $n = 1, 2, \dots, N$. In particular, why can we choose the range $1 \leq n \leq N$? (*Hint*: Show that a solution A_j with $n > N$ is a constant multiple of a solution in which $n \leq N$.)

Problem 2.5

Show that the normal modes and resonant frequencies we obtained for the chain of n oscillators reduce to our previous results for a pair of coupled oscillators when $N = 2$.

Problem 2.6

Consider 3 coupled oscillators all with the same “spring constant”; use rigid wall boundary conditions.

- (i) Compute the equations of motion and derive the matrix K analogous to (3.2).
- (ii) Compute the eigenvectors and eigenvalues of K . Show that the eigenvectors are orthogonal.
- (iii) From (ii) find the normal modes and characteristic frequencies.
- (iv) Show that your result in (iii) agrees with the specialization of the chain of oscillator formulas in §4 to the case $N = 3$.

Problem 2.7

Show that (3.12) is equivalent to (2.12).

Problem 2.8

Derive equations (2.3) and (2.4).

Problem 2.9

Check that the eigenvectors appearing in (3.12) are linearly independent.

Problem 2.10

Show that equations (4.15) and (4.16) are equivalent representations of the general solution for the chain of oscillators.

Problem 2.11

Show that (2.16) is indeed a basis for the vector space of solutions to (2.3), (2.4). (*Hint:* Write the general solution (2.12) in terms of sines and cosines.)

Problem 2.12

Show that the equations (4.1) can be written in the matrix form

$$\ddot{\mathbf{q}} = K\mathbf{q},$$

where K is an $N \times N$ *symmetric* matrix and \mathbf{q} is a column vector.