Problem Set 4

Problem 4.1

Make a computer animation which depicts the solution (8.58) of the wave equation with the square pulse initial conditions. Describe the resulting motion.

Problem 4.2

In the text we solved the wave equation with initial data consisting of a Gaussian displacement with with vanishing initial velocity profile. Find the solution to the wave equation with initial data

$$q(x,0) \equiv a(x) = A\exp\left(-\frac{x^2}{a_0^2}\right)$$

$$\frac{\partial q(x,0)}{\partial t} = -2\frac{AV}{a_0^2}x\exp\left(-\frac{x^2}{a_0^2}\right),$$

where V is a constant with units of speed. What is the physical meaning of this initial data set?

Problem 4.3

Consider a wave propagating along the whole x-axis. Write the wave displacement in terms of its Fourier transform at each time:

$$q(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} a(k,t)e^{ikx}\,dk.$$

Substitute this expression for $q(x,t)$ into the wave equation and derive an equation for $a(k,t)$. Solve this equation for $a(k,t)$ and show that $q(x,t)$ can be put into the form (8.64).

Problem 4.4

Using a computer, graph the function D_ϵ in (8.42) and show that as ϵ gets smaller D_ϵ becomes a narrowly peaked function with ever-increasing maximimum value.

Problem 4.5

Recall the Gaussian wave packet discussed in §8.5. Consider the limit in which the Gaussian width vanishes, $a \to 0$, and the Gaussian height blows up, $A \to \infty$, such that the product of the width and height approaches a non-zero constant, $Aa \to const. \neq 0$. Show that the solution to the wave equation becomes a delta function displacement at rest at $t = 0$ which then splits into two delta function pulses traveling in opposite directions. *Hint: Consider the limit of the Fourier transforms.*

Problem 4.6

Prove Plancherel's identity (8.34) using the properties of the delta function.

Problem 4.7

Compute the most basic Gaussian integral,

$$I(a) = \int_{-\infty}^{\infty} e^{-a^2x^2}\, dx,$$

as follows. First, consider the square of the integral:

$$[I(a)]^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-a^2x^2} e^{-a^2y^2}\, dx\, dy.$$

Next, notice that this is simply the integral of a function over the whole x-y plane. Change integration variables to polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ (note: $dx\, dy = rdr\, d\theta$). You should now be able to easily compute the integral $[I(a)]^2$. Finally, note that $I(a)$ is positive and take the square root to determine the original integral.

Problem 4.8

In §8.5 we considered the Fourier representation of a wave with Gaussian initial displacement and vanishing initial velocity. We computed the Fourier transform and showed that $h(k) = w(k)$. Show that $h(k) = w(k)$ is always true (not just for a Gaussian) whenever we choose vanishing initial velocity profile.

Problem 4.9

In the text we computed the Fourier transform of a square pulse, viewing it as a function on the interval $[0, L]$ which vanishes at the endpoints. We can, of course, view the square pulse as a (square-integrable) function on the whole x axis. From this point of view, compute the Fourier transform $h(k)$ of the square pulse:

$$f(x) = \begin{cases} 1, & \text{if } a < x < b; \\ 0, & \text{otherwise.} \end{cases}$$

Plot the real part, imaginary part and absolute value of $h(k)$

Problem 4.10

In (8.73) we gave a formula for the Fourier transform $h(k)$ of a Gaussian function $f(x) = e^{-x^2/a^2}$. Use the general formulas we obtained to take the Fourier transform $w(x)$ of that Fourier transform ($h(k)$). Show that you get back the original Gaussian function, *i.e.*, $w(x) = f(x)$.

Problem 4.11

Show that $\int_{-\infty}^{\infty} e^{-ax^2} \sin bx \, dx = 0$. (Hint: this one is really easy.)

Problem 4.12

Find the Fourier series form of the solution $q(x,t)$, $0 < x < L$, to the wave equation with $q(0,t) = q(L,t) = 0$ such that

$$q(x,0) = x^2 - Lx, \qquad \frac{\partial q(x,0)}{\partial t} = 0.$$

Problem 4.13

Show that, for $a \neq 0$,

$$\delta(ax) = \frac{1}{|a|}\delta(x).$$

(*Hint:* this means you must show

$$\int_{-\infty}^{\infty} \delta(ax) f(x) \, dx = \frac{1}{|a|} f(0)$$

for any function $f(x)$.)

Problem 4.14

Verify that $q(x, t)$ defined by (8.47) –(8.54) does indeed satisfy the wave equation and match the initial conditions (8.48).

Problem 4.15

Verify that (7.15) leads to (6.4) as advertised.

Problem 4.16

A function f on an interval $[-L, L]$ is said to be *even* if $f(-x) = f(x)$ for all $x \in [-L, L]$. Similarly, a function is said to be *odd* if $f(-x) = -f(x)$. For even (odd) functions, show that the Fourier series representation (8.15) becomes a series involving only cosines (sines).

Problem 4.17

Show that for any two (real-valued, square-integrable) functions, f and g, on an interval (a, b) the following defines a scalar product on the vector space of such functions:

$$(f, g) = \int_a^b f(x)g(x)\, dx.$$

(See the Appendix for the definition of a scalar product.)