PROBLEM SET 5

Problem 5.1

Let **A** be a given constant vector field vector. Show that the equation $\mathbf{A} \cdot \mathbf{r} = c$, where c is a constant, defines a plane. If $\mathbf{A} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, and the c = 0, where is the plane?

Problem 5.2

Prove that the gradient of a function $f(\mathbf{r})$ is always orthogonal to the surfaces $f(\mathbf{r}) =$ constant. (Hint: This one is easy; think about the directional derivative of f along any direction tangent to the surface.)

Problem 5.3

Consider the sphere defined by $x^2 + y^2 + z^2 = 1$. Compute the gradient of the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

and check that it is everywhere orthogonal to the sphere. Consider a linear function

$$f(\mathbf{r}) = \mathbf{a} \cdot \mathbf{r},$$

where **a** is a fixed, constant vector field. Compute the gradient of f and check that the resulting vector field is perpendicular to the plane $f(\mathbf{r}) = 0$.

Problem 5.4

Compute the divergence of the following two vector fields:

(a) $\mathbf{E}(\mathbf{r}) = \frac{\mathbf{r}}{r^3}, \quad r = \sqrt{x^2 + y^2 + z^2} > 0,$ (Coulomb electric field)

(b) $\mathbf{B}(\mathbf{r}) = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}, \quad x^2+y^2 > 0,$ (magnetic field outside a long wire)

(c) $\mathbf{D}(\mathbf{r}) = \mathbf{r}$ (electric field inside a uniform ball of charge).

Problem 5.5

Derive (9.15) and (9.19).

Problem 5.6

Consider a spherically symmetric function f = f(r), $r = \sqrt{x^2 + y^2 + z^2}$. Show that its Fourier transform takes the following form:

$$\frac{1}{(2\pi)^{3/2}} \int_{\text{all space}} e^{-i\mathbf{k}\cdot\mathbf{r}} f(r) \, d^3x = \sqrt{\frac{2}{\pi}} \frac{1}{k} \int_0^\infty rf(r) \sin(kr) \, dr.$$

(*Hint*: Use spherical polar coordinates, choosing your z axis along \mathbf{k} .) Note that the transform is spherically symmetric also in \mathbf{k} space. Use this formula to compute the Fourier transform of a 3-dimensional Gaussian

$$f = e^{-a^{-2}(x^2 + y^2 + z^2)}.$$

Problem 5.7

Derive (9.24) from (9.20). In particular, express $c(\mathbf{k})$ in terms of the initial data.

Problem 5.8

Let f and g be two functions. We can take the gradient of g to get a vector field, which we denote by ∇g . We can multiply this vector field by the function f to get another vector field, $f \nabla g$. As with any vector field, we can form a function by taking the divergence of $f \nabla g$; using the definitions of gradient and divergence show that

$$\nabla \cdot (f\nabla g) = \nabla f \cdot \nabla g + f\nabla^2 g. \tag{10.3}$$