PROBLEM SET 9

Problem 9.1

Compute the divergence and curl of the following vector fields:

- (a) $\mathbf{E}(\mathbf{r}) = \frac{\mathbf{r}}{r^3}, \quad r = \sqrt{x^2 + y^2 + z^2} > 0$ (Coulomb electric field)
- (b) $\mathbf{B}(\mathbf{r}) = -\frac{y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j}, \quad x^2 + y^2 > 0,$ (magnetic field outside a long wire)
- (c) $\mathbf{A}(\mathbf{r}) = -y\mathbf{i} + x\mathbf{j}$, (vector potential for a uniform magnetic field).

Problem 9.2

A common but erroneous claim, which stems from the notation $\nabla \times \mathbf{V}$ for the curl of a vector field \mathbf{V} , is that the curl of a vector field \mathbf{V} is everywhere orthogonal to \mathbf{V} . Disprove this claim by exhibiting a counter-example.

Problem 9.3

Show that

- (a) $\nabla \times (\nabla f) = 0$, where f is a function;
- (b) $\nabla \cdot (\nabla \times \mathbf{V}) = 0$, where **V** is a vector field;
- (c) $\nabla \times (\nabla \times \mathbf{V}) = \nabla (\nabla \cdot \mathbf{V}) \nabla^2 \mathbf{V}.$

Hints: Work in Cartesian coordinates. To prove equality between vectors (in (a) and (c)) it is enough to show that the identities hold for one component, e.g., the x component.

Problem 9.4

We derived the continuity equation for electric charge from the inhomogeneous Maxwell equations. Show that an analogous computation with the homogeneous Maxwell equations yields no new equations.

Problem 9.5

We used half the Maxwell equations to derive a wave equation for the magnetic field **B**. Using the other half of the equations, perform an analogous computation to derive a wave equation for **E**.

Problem 9.6

Suppose that

$$A\sin(\mathbf{k}\cdot\mathbf{r}+\alpha t) = B\sin(\mathbf{k}'\cdot\mathbf{r}+\beta t)$$

for all **r** and t. Show that A = B, $\alpha = \beta$, and $\mathbf{k} = \mathbf{k'}$.

Problem 9.7

For an electromagnetic plane wave show, from (18.9), that

$$\mathbf{B}_0 = \mathbf{n} \times E_0.$$

Problem 9.8

Verify (19.4) - (19.7)

Problem 9.9

Using the Maxwell equations (17.2)-(17.5), and charge continuity equation, (17.8), show that, when the Maxwell equations are satisfied,

$$\frac{\partial}{\partial t} \left(\nabla \cdot \mathbf{E} - 4\pi \rho \right) = 0,$$

and

$$\frac{\partial}{\partial t} \left(\nabla \cdot \mathbf{B} \right) = 0$$

This implies that if the constraint equations are satisfied at one time, then they are satisfied for all time by virtue of the evolution equations.

Problem 9.10

Show that the electric field in (20.14) traces out an ellipse.

Problem 9.11

Show that $\mathbf{V} = x \mathbf{i} - z \mathbf{k}$ has vanishing divergence. Find a vector whose curl is \mathbf{V} . Show that $\mathbf{W} = xy^2 \cos z \mathbf{i} + x^2y \cos z \mathbf{j} - \frac{1}{2}x^2y^2 \sin z \mathbf{k}$ has vanishing curl. Find a function whose gradient is \mathbf{W} .

Problem 9.12

Let S be a closed surface and let C be a closed curve. Prove that

- (a) $\oint_{S} (\nabla \times \mathbf{V}) \cdot d\mathbf{S} = 0$,
- (b) $\oint_C \nabla f \cdot d\mathbf{l} = 0.$

Problem 9.13

Let S be a closed surface enclosing a charge Q. Let A be a surface through which a current I passes; A has the (closed) boundary curve L. Using the divergence theorem and Stokes' theorem, derive the integral form of Maxwell's equations:

$$\begin{split} \oint_{S} \mathbf{E} \cdot d\mathbf{S} &= 4\pi Q, \quad \text{(Gauss)}, \\ \oint_{S} \mathbf{B} \cdot d\mathbf{S} &= 0, \\ \oint_{L} \mathbf{B} \cdot d\mathbf{l} - \frac{1}{c} \frac{\partial}{\partial t} \int_{A} \mathbf{E} \cdot d\mathbf{A} &= \frac{4\pi}{c} I, \quad \text{(Ampere - Maxwell)}, \\ \oint_{L} \mathbf{E} \cdot d\mathbf{l} + \frac{1}{c} \frac{\partial}{\partial t} \int_{A} \mathbf{B} \cdot d\mathbf{A} &= 0, \quad \text{(Faraday)}. \end{split}$$

Problem 9.14

Let C be a circle of unit radius in the x-y plane, enclosing the unit disk D, and let $\mathbf{V} = y\mathbf{i}$. Compute (i) the line integral of \mathbf{V} around C and (ii) the flux of $\nabla \times \mathbf{V}$ through D.

Hint: You should find that

$$\oint_C \mathbf{V} \cdot d\mathbf{l} = \int_D (\nabla \times \mathbf{V}) \cdot d\mathbf{S}.$$