THE DIFFERENTIALGEOMETRY SOFTWARE PROJECT

How To Create A Jordan Algebra

Synopsis

- A Jordan algebra $\mathbb{J}(n, \mathbb{A})$ is defined as the set of $n \times n$ Hermitian matrices with entries which come from one of the algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$ (quaternions), or $\mathbb{O}$ (octonions). The diagonal entries are real. The product $\circ$ in the Jordan algebra is given by $x \circ y = \frac{1}{2}(x.y + y.x)$ where the right hand side is given by ordinary matrix multiplication and addition.
- We show how to create a Jordan algebra in Maple using the commands AlgebraLibraryData and AlgebraData.

Examples

Load in the required packages.

```
> with(DifferentialGeometry): with(LieAlgebras):
```

Example 1.

The structure equations for Jordan algebras of small dimensions are stored in the DifferentialGeometry libraries and can be accessed with the AlgebraLibraryData command. In this example we shall construct the Jordan algebra $\mathbb{J}(2, \mathbb{H})$ of 2×2 matrices over the quaternions.

Retrieve the structure equations and initialize the algebra.

```
> AD1 := AlgebraLibraryData("Jordan(2, Quaternions)", J2H);
```

$$AD1 := \left[e1^2 = e1, e1.e3 = \frac{1}{2} e3, e1.e4 = \frac{1}{2} e4, e1.e5 = \frac{1}{2} e5, e1.e6 = \frac{1}{2} e6, e2^2 = e2, e2.e3 = \frac{1}{2} e3, e2.e4 = \frac{1}{2} e4, e2.e5 = \frac{1}{2} e5, e2.e6 = \frac{1}{2} e6, e3.e1 = \frac{1}{2} e3, e3.e2 = \frac{1}{2} e3, e3^2 = e1 + e2, e4.e1 = \frac{1}{2} e4, e4.e2 = \frac{1}{2} e4, e4^2 = e1 + e2, e5.e1 = \frac{1}{2} e5, e5.e2 = \frac{1}{2} e5, e5^2 = e1 + e2, e6.e1 = \frac{1}{2} e6, e6.e2 = \frac{1}{2} e6, e6^2 = e1 + e2\right] \quad (1)$$

```
> DGsetup(AD1);
```

(2)

$$\textit{algebra name: J2H} \tag{2}$$

The command MultiplicationTable displays the product rules for the algebra $\mathbb{J}(2, \mathbb{H})$.

```
J2H > MultiplicationTable();
```

$$\begin{array}{c|cccccc} & e1 & e2 & e3 & e4 & e5 & e6 \\ \hline e1 & e1 & 0\,e1 & \frac{1}{2}\,e3 & \frac{1}{2}\,e4 & \frac{1}{2}\,e5 & \frac{1}{2}\,e6 \\ e2 & 0\,e1 & e2 & \frac{1}{2}\,e3 & \frac{1}{2}\,e4 & \frac{1}{2}\,e5 & \frac{1}{2}\,e6 \\ e3 & \frac{1}{2}\,e3 & \frac{1}{2}\,e3 & e1+e2 & 0\,e1 & 0\,e1 & 0\,e1 \\ e4 & \frac{1}{2}\,e4 & \frac{1}{2}\,e4 & 0\,e1 & e1+e2 & 0\,e1 & 0\,e1 \\ e5 & \frac{1}{2}\,e5 & \frac{1}{2}\,e5 & 0\,e1 & 0\,e1 & e1+e2 & 0\,e1 \\ e6 & \frac{1}{2}\,e6 & \frac{1}{2}\,e6 & 0\,e1 & 0\,e1 & 0\,e1 & e1+e2 \end{array} \tag{3}$$

At this point one can now perform operations on these objects such as adding and multiplying elements together.

```
J2H > X := evalDG(3*e1 - 2*e3 + 4*e4 + 2*e5);
```

$$X := 3\,e1 - 2\,e3 + 4\,e4 + 2\,e5 \tag{4}$$

```
J2H > Y := evalDG(-2*e2 + e6);
```

$$Y := -2\,e2 + e6 \tag{5}$$

```
J2H > evalDG(2*X + Y);
```

$$6\,e1 - 2\,e2 - 4\,e3 + 8\,e4 + 4\,e5 + e6 \tag{6}$$

```
J2H > evalDG(X.Y);
```

$$2\,e3 - 4\,e4 - 2\,e5 + \frac{3}{2}\,e6 \tag{7}$$

Example 2.

In this example we give an alternative construction of $\mathbb{J}(2, \mathbb{H})$; this construction can be used to construct higher-dimensional or split signature Jordan algebras. There are 3 steps. First, we use the AlgebraLibraryData command to initialize the algebra of quaternions. Second, we use the command JordanMatrices to generate the 2×2 Jordan matrices with entries in the quaternions. Then we use the command AlgebraData to calculate the structure equations for the algebra $\mathbb{J}(2, \mathbb{H})$.

Step 1. Use AlgebraLibraryData to retrieve the structure equations for the quaternions. Initialize with DGsetup. We use the standard labels (e, i, j, k) for the quaternions.

```
> AD2 := AlgebraLibraryData("Quaternions", H);
```

$$AD2 := \left[e1^2 = e1, e1.e2 = e2, e1.e3 = e3, e1.e4 = e4, e2.e1 = e2, e2^2 = -e1, e2.e3 = e4, e2.e4 = -e3, e3.e1 = e3, e3.e2 = -e4, e3^2 = -e1, e3.e4 = e2, e4.e1 = e4, e4.e2 = e3, e4.e3 = -e2, e4^2 = -e1\right] \quad \textbf{(8)}$$

```
> DGsetup(AD2, '[e, i, j ,k]', '[omega]');
```

algebra name: H **(9)**

Step 2. Use JordanMatrices to generate the 2 × 2 Jordan matrices with entries in the quaternions.

```
H > A := JordanMatrices(2, H);
```

$$A := \left[\begin{bmatrix} e & 0\,e \\ 0\,e & 0\,e \end{bmatrix}, \begin{bmatrix} 0\,e & 0\,e \\ 0\,e & e \end{bmatrix}, \begin{bmatrix} 0\,e & e \\ e & 0\,e \end{bmatrix}, \begin{bmatrix} 0\,e & i \\ -i & 0\,e \end{bmatrix}, \begin{bmatrix} 0\,e & j \\ -j & 0\,e \end{bmatrix}, \begin{bmatrix} 0\,e & k \\ -k & 0\,e \end{bmatrix}\right] \quad \textbf{(10)}$$

Define the multiplication procedure JordanProduct, which will be used to multiply these matrices together.

```
H > mu := JordanProduct;
```

μ := *DifferentialGeometry:-LieAlgebras:-JordanProduct* **(11)**

Step 3. Use AlgebraData to calculate the structure equations for the algebra $\mathbb{J}(2, \mathbb{H})$. The output of the command AlgebraData is a list of the non-zero product rules, where by default *e1*, *e2*, *e3*, *e4*, *e5*, *e6* denote the 6 matrices composing the basis matrices in **(10)**.

```
H > AD2 := AlgebraData(A, mu, J2Q);
```

$$AD2 := \left[e1^2 = e1, e1.e3 = \frac{1}{2}\,e3, e1.e4 = \frac{1}{2}\,e4, e1.e5 = \frac{1}{2}\,e5, e1.e6 = \frac{1}{2}\,e6, e2^2 = e2, e2.e3 = \frac{1}{2}\,e3, e2.e4 = \frac{1}{2}\,e4, e2.e5 = \frac{1}{2}\,e5, e2.e6 = \frac{1}{2}\,e6, e3.e1 = \frac{1}{2}\,e3, e3.e2 = \frac{1}{2}\,e3, e3^2 = e1 + e2, e4.e1 = \frac{1}{2}\,e4, e4.e2 = \frac{1}{2}\,e4, e4^2 = e1 + e2, e5.e1 = \frac{1}{2}\,e5, e5.e2 = \frac{1}{2}\,e5, e5^2 = e1 + e2, e6.e1 = \frac{1}{2}\,e6, e6.e2 = \frac{1}{2}\,e6, e6^2 = e1 + e2\right] \quad \textbf{(12)}$$

We have recovered the original structure equations.

```
H > AD1;
```

$$\left[e1^2 = e1, e1.e3 = \frac{1}{2}\,e3, e1.e4 = \frac{1}{2}\,e4, e1.e5 = \frac{1}{2}\,e5, e1.e6 = \frac{1}{2}\,e6, e2^2 = e2, e2.e3 = \frac{1}{2}\,e3, e2.e4 = \frac{1}{2}\,e4, e2.e5 = \frac{1}{2}\,e5, e2.e6 = \frac{1}{2}\,e6, e3.e1 = \frac{1}{2}\,e3, e3.e2 = \frac{1}{2}\,e3, e3^2 = e1 + e2, e4.e1 = \frac{1}{2}\,e4, e4.e2 = \frac{1}{2}\,e4, e4^2 = e1 + e2, e5.e1 = \frac{1}{2}\,e5, e5.e2 = \frac{1}{2}\,e5, e5^2 = e1 + e2, e6\right. \quad \textbf{(13)}$$

$$.e1 = \frac{1}{2}\, e6,\ e6.e2 = \frac{1}{2}\, e6,\ e6^2 = e1 + e2$$

Commands Illustrated

• AlgebraData, AlgebraLibraryData, DGsetup, JordanMatrices, JordanProduct, MultiplicationTable

Related Commands

• LieAlgebraData, SimpleLieAlgebraData, StandardRepresentation

References

• W. Fulton, J. Harris, *Representation Theory - A First Course*
• http://en.wikipedia.org/wiki/Jordan_algebra

Release Notes

• The illustrated commands are available in Maple 17 and subsequent releases.

Author

Ian M. Anderson and Thomas J Apedaile,
Department of Mathematics and Statistics, Utah State University
February 6, 2013