## Utah State University DigitalCommons@USU

Foundations of Wave Phenomena

Physics, Department of

1-1-2004

## 03 How To Find Normal Modes

Charles G. Torre Department of Physics, Utah State University, Charles.Torre@usu.edu

## **Recommended** Citation

Torre, Charles G., "03 How To Find Normal Modes" (2004). *Foundations of Wave Phenomena*. Book 20. http://digitalcommons.usu.edu/foundation\_wave/20

This Book is brought to you for free and open access by the Physics, Department of at DigitalCommons@USU. It has been accepted for inclusion in Foundations of Wave Phenomena by an authorized administrator of DigitalCommons@USU. For more information, please contact digitalcommons@usu.edu.



## 3. How to find normal modes.

How do we find the normal modes and resonant frequencies without making a clever guess? Well, you can get a more complete explanation in an upper-level mechanics course, but the gist of the trick involves a little linear algebra. The idea is the same for any number of coupled oscillators, but let us stick to our example of two oscillators.

To begin, we again assemble the 2 coordinates,  $q_i$ , i = 1, 2, into a column vector  $\mathbf{q}$ ,

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}. \tag{3.1}$$

Let K be the  $2 \times 2$  symmetric matrix

$$K = \begin{pmatrix} \tilde{\omega}^2 + \tilde{\omega}'^2 & -\tilde{\omega}'^2 \\ -\tilde{\omega}'^2 & \tilde{\omega}^2 + \tilde{\omega}'^2 \end{pmatrix}.$$
 (3.2)

The coupled oscillator equations (2.3), (2.4) can be written in matrix form as (exercise)

$$\frac{d^2\mathbf{q}}{dt^2} = -K\mathbf{q}.\tag{3.3}$$

The fact that the matrix K is not diagonal corresponds to the fact that the equations for  $q_i(t)$  are coupled.

Exercise: Check that the matrix form of the uncoupled equations (2.1), (2.2) gives a diagonal matrix K.

Our strategy for solving (3.3) is to find the *eigenvalues*  $\lambda$  and *eigenvectors*  $\mathbf{e}$  of K. These are the solutions to the equation

$$K\mathbf{e} = \lambda \mathbf{e},\tag{3.4}$$

where  $\lambda$  is a scalar and **e** is a (column) vector. We will make two assumptions about the solutions to this well-known type of linear algebra problem. First, we assume that the two (possibly equal) eigenvalues,  $\lambda_1$  and  $\lambda_2$  are all positive. Second, we assume that the corresponding eigenvectors,  $\mathbf{e}_1$  and  $\mathbf{e}_2$ , are linearly independent.<sup>\*</sup> This means that any column vector **v** can be expressed as

$$\mathbf{v} = v_1 \mathbf{e}_2 + v_2 \mathbf{e}_2,$$

for some real numbers  $v_1$  and  $v_2$ , which are the *components* of **v**. We shall soon see why we need these assumptions and when they are satisfied.

<sup>\*</sup> In other words, the eigenvectors form a *basis* for the vector space of 2-component column vectors.

Given the solutions  $(\lambda_1, \mathbf{e}_1)$ ,  $(\lambda_2, \mathbf{e}_2)$  to (3.4), we can build a solution to (3.3) as follows. Write

$$\mathbf{q}(t) = \alpha_1(t)\mathbf{e}_1 + \alpha_2(t)\mathbf{e}_2. \tag{3.5}$$

Exercise: Why can we always do this?

Using

$$\frac{d^2\mathbf{q}}{dt^2} = \frac{d^2\alpha_1}{dt^2}\mathbf{e}_1 + \frac{d^2\alpha_2}{dt^2}\mathbf{e}_2,\tag{3.6}$$

and<sup>†</sup>

$$K\mathbf{q} = \alpha_1(t)K\mathbf{e}_1 + \alpha_2(t)K\mathbf{e}_2$$
  
=  $\lambda_1\alpha_1(t)\mathbf{e}_1 + \lambda_2\alpha_2(t)\mathbf{e}_2,$  (3.7)

you can easily check that (3.5) defines a solution to (3.3) if and only if

$$\left(\frac{d^2\alpha_1}{dt^2} + \lambda_1\alpha_1(t)\right)\mathbf{e}_1 + \left(\frac{d^2\alpha_2}{dt^2} + \lambda_2\alpha_2(t)\right)\mathbf{e}_2 = 0.$$
(3.8)

Using the linear independence of the eigenvectors, this means (exercise) that  $\alpha_1$  and  $\alpha_2$  each solves the harmonic oscillator equation with frequency  $\sqrt{\lambda_1}$  and  $\sqrt{\lambda_2}$ , respectively:

$$\frac{d^2\alpha_n}{dt^2} = -\lambda_n \alpha_n(t), \quad n = 1, 2.$$
(3.9)

The general solution to (3.3) can then be written as (exercise)

$$\mathbf{q}(t) = Re(A_1 e^{i\sqrt{\lambda_1}t} \mathbf{e}_1 + A_2 e^{i\sqrt{\lambda_2}t} \mathbf{e}_2), \qquad (3.10)$$

where  $A_1$  and  $A_2$  are any complex numbers. Thus, by finding the eigenvalues and eigenvectors we can reduce our problem to the harmonic oscillator equation, which we already know how to solve.

Now you can see why we made those assumptions about the eigenvalues and eigenvectors. Firstly, if the eigenvectors don't form a basis, we can't assume **q** takes the form (3.5) nor that (3.8) implies (3.9). It is an important theorem from linear algebra that for any symmetric matrix with real entries, such as (3.2), the eigenvectors will form a basis, so this assumption is satisfied in our current example. Secondly, the frequencies  $\sqrt{\lambda_n}$  will be real numbers if and only if the eigenvalues  $\lambda_n$  are always positive. While the aforementioned linear algebra theorem guarantees the eigenvalues of a symmetric matrix will be real, it doesn't guarantee that they will be positive. However, as we shall see, for the coupled oscillators the eigenvalues are positive definite, which one should expect on physical grounds. (Exercise: How would you interpret the situation in which the eigenvalues are negative?).

<sup>&</sup>lt;sup>†</sup> Note that here we use the fact that matrix multiplication is a linear operation.

Comparing our general solution (3.10) with (2.12) we see that the resonant frequencies ought to be related to the eigenvalues of K via

$$\Omega_i = \sqrt{\lambda_i}, \quad i = 1, 2$$

and the normal modes should correspond to the eigenvectors  $\mathbf{e}_i$ . Let us work this out in detail.

The eigenvalues of K are obtained by finding the two solutions  $\lambda$  to the equation (3.4). This equation is equivalent to

$$(K - \lambda I)\mathbf{e} = 0,$$

where I is the identity matrix. A standard result from linear algebra is that this equation has a non-trivial solution  $\dagger \mathbf{e}$  if and only  $\lambda$  is a solution of the *characteristic (or secular)* equation:

$$\det[K - \lambda I] = 0.$$

You can easily check that the secular equation for (3.2) is

$$\lambda^2 - 2(\tilde{\omega}^2 + \tilde{\omega}'^2)\lambda - \tilde{\omega}'^4 + (\tilde{\omega}^2 + \tilde{\omega}'^2)^2 = 0.$$

This quadratic equation is easily solved to get the two roots (exercise)

$$\lambda_1 = \tilde{\omega}^2 \lambda_2 = \tilde{\omega}^2 + 2\tilde{\omega}'^2.$$
(3.11)

Note that we have just recovered the (squares of the) resonant frequencies by finding the eigenvalues of K.

To find the eigenvectors  $\mathbf{e}_i$  of K we substitute each of the eigenvalues  $\lambda_i$ , i = 1, 2 into the eigenvalue equation (3.4) and solve for the components of the  $\mathbf{e}_i$  using standard techniques. As a very nice exercise you should check that the resulting eigenvectors are of the form

$$\mathbf{e}_{1} = a \begin{pmatrix} 1\\1 \end{pmatrix}$$

$$\mathbf{e}_{2} = b \begin{pmatrix} -1\\1 \end{pmatrix}$$
(3.12)

where a and b are any constants, which can be absorbed into the definition of  $A_1$  and  $A_2$  in (3.10) (exercise).

Exercise: Just from the form of (3.4), can you explain why the eigenvectors are only determined up to an overall multiplicative factor?

*<sup>†</sup>* Exercise: what is the trivial solution?

Using these eigenvectors in (3.10) we recover the expression (2.12) – you really should verify this yourself. In particular, it is the eigenvectors of K that determine the column vectors appearing in (2.16) (exercise).

Note that the eigenvectors are linearly independent as advertised (*exercise*). Indeed, using the usual scalar product on the vector space of column vectors  $\mathbf{v}$  and  $\mathbf{w}$ ,

$$(\mathbf{v}, \mathbf{w}) = \mathbf{v}^T \mathbf{w},$$

you can check that  $\mathbf{e}_1$  and  $\mathbf{e}_2$  are orthogonal (see Problems).

To summarize: The resonant frequencies of a system of coupled oscillators, described by the matrix differential equation

$$\frac{d^2}{dt^2}\mathbf{q} = -K\mathbf{q},$$

are determined by the eigenvalues of the matrix K. The normal modes of vibration are determined by the eigenvectors of K.